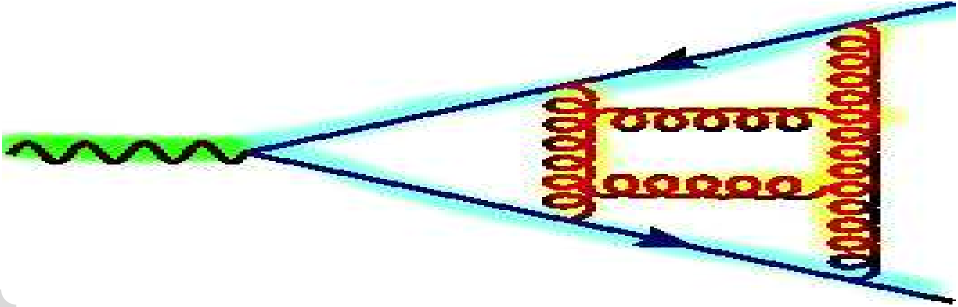
Karlsruhe Institute of Technology

Form factors in QCD

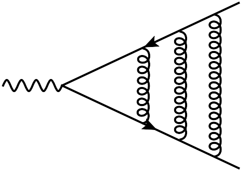
Loops and Legs 2018, St. Goar, Germany, April 29, 2018 to May 4, 2018

Matthias Steinhauser | TTP Karlsruhe

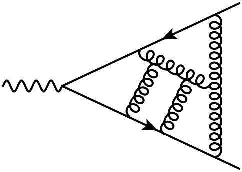


Photon-quark-anti-quark form factor

massive quarks



massless quarks



R. N. Lee, A. V. Smirnov, V. A. Smirnov, MS:

"Three-loop massive form factors: complete light-fermion and large-N_c corrections for vector, axial-vector, scalar and pseudo-scalar currents", 2018

R. N. Lee, A. V. Smirnov, V. A. Smirnov, MS:

"Three-loop massive form factors: complete light-fermion corrections for the vector current", 2018

R. N. Lee, A. V. Smirnov, V. A. Smirnov, MS:

"The n_f^2 contributions to fermionic four-loop form factors", 2017

J. Henn, R. Lee, A. Smirnov, V. Smirnov, MS:

"Four-loop photon quark form factor and cusp anomalous dimension in the large-N_c limit of QCD", 2016

J. Henn, A. Smirnov, V. Smirnov, MS:

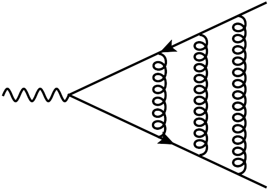
"A planar four-loop form factor and cusp anomalous dimension in QCD", 2016

J. Henn, A. Smirnov, V. Smirnov, MS:

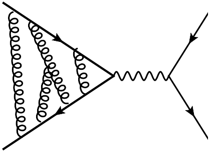
"Massive three-loop form factor in the planar limit", 2016

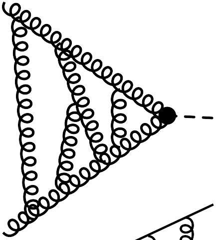
Form factors for physical processes

- $e^+e^- \to Q\bar{Q}$
- Forward-backward asymmetry
- static properties: $(g-2)_{\rm quark}$

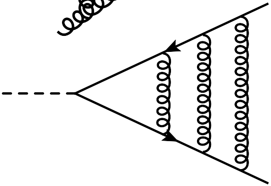


- $\Gamma_{\rm cusp}$
- Drell-Yan
- Higgs production

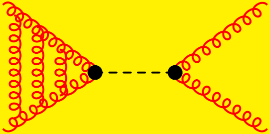




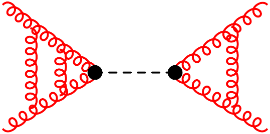
- Higgs decay: $H \to Q\bar{Q}$, $A \to Q\bar{Q}$

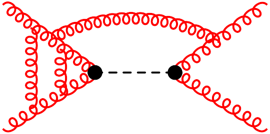


Example: Higgs production at the LHC

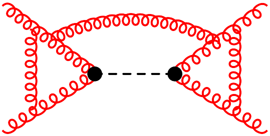


[Baikov,Chetyrkin,Smirnov,Smirnov,
Steinhauser'09],
[Gehrmann,Glover,Huber,Ikizlerli,
Studerus'10]; [Lee,Smirnov'10]

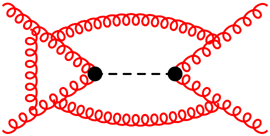




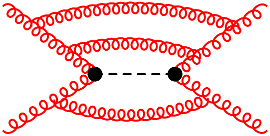
[Duhr,Gehrmann'13], [Li,Zhu'13],
[Dulat,Mistlberger'14],
[Duhr,Gehrmann,Jaquier'14]



[Anastasiou,Duhr,Dulat,Herzog,
Mistlberger'13], [Kilgore'13]



[Anastasiou,Duhr,Dulat,Furlan,Gehrmann,
Herzog,Mistlberger'14],
[Li,von Manteuffel,Schabinger,Zhu'14]



[Anastasiou,Duhr,Dulat,Mistlberger'13]

N^{3}LO: [Anastasiou,Duhr,Dulat,Herzog,Mistlberger'15]

[Anastasiou,Duhr,Dulat,Furlan,Gehrmann,Herzog,Lazopoulos,Mistlberger'16; Mistlberger'18]

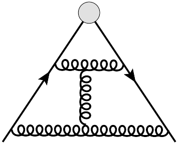
Currents and form factors

$$j_\mu^v = \bar{\psi}\gamma_\mu\psi$$ vector

$$j_\mu^a = \bar{\psi}\gamma_\mu\gamma_5\psi$$ axial-vector

$$j^s = m\bar{\psi}\psi$$ scalar

$$j^p = im\bar{\psi}\gamma_5\psi$$ pseudo-scalar

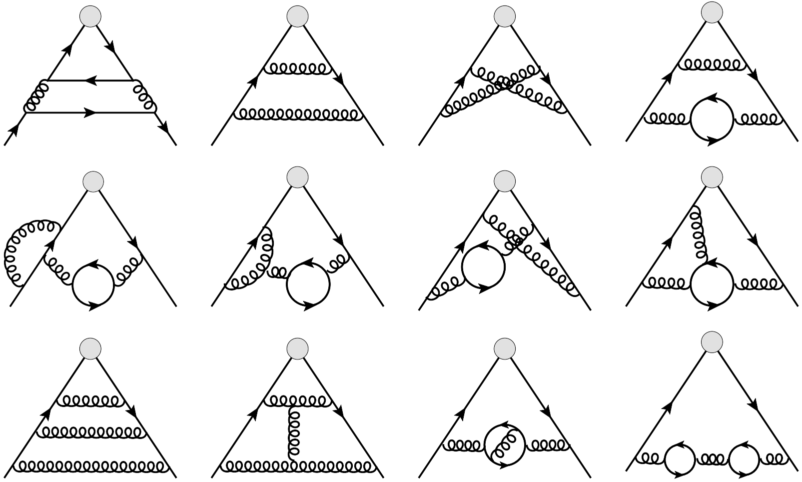


$$\Gamma_\mu^v = F_1^v(q^2)\gamma_\mu - \frac{i}{2m}F_2^v(q^2)\sigma_{\mu\nu}q^\nu$$

$$\Gamma_\mu^a = F_1^a(q^2)\gamma_\mu\gamma_5 - \frac{1}{2m}F_2^a(q^2)q_\mu\gamma_5$$

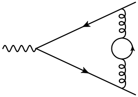
$$\Gamma^s = mF^s(q^2)$$

$$\Gamma^p = imF^p(q^2)\gamma_5\,,$$



Massive form factors at 2 loops

[Hoang,Teubner'97]



fermionic corrections

Massive form factors at 2 loops

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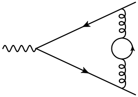
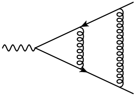


fermionic corrections

[Bernreuther,Bonciani,Gehrmann,Heinesch,Leineweber,Mastrolia,Remiddi'04-'06]



Massive form factors at 2 loops

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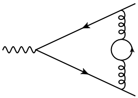
[Hoang,Teubner'97]



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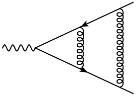

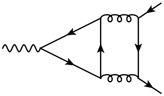

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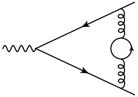

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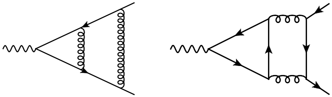

vector, axial-vector, scalar, pseudo-scalar current

Massive form factors at 2 loops

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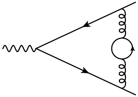

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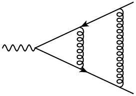
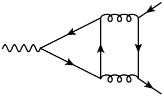




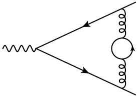
vector, axial-vector, scalar, pseudo-scalar current

[Gluza,Mitov,Moch,Riemann'09]

$+\mathcal{O}(\epsilon)$

Massive form factors at 2 loops

[Hoang,Teubner'97]



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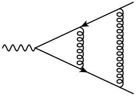

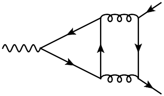

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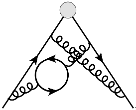
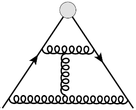
[Gluza,Mitov,Moch,Riemann'09] $+\mathcal{O}(\epsilon)$

[Ahmed,Henn,Steinhauser'17; Ablinger,Behring,Blümlein,Falcioni,De Freitas,Marquard,Rana,Schneider'17] $+\mathcal{O}(\epsilon^2)$

this talk: steps towards 3 loops: large-N_c + all-n_l

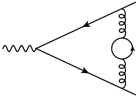
[Henn,Smirnov,Smirnov,Steinhauer'16, Lee,Smirnov,Smirnov,Steinhauer'18]





Massive form factors at 2 loops

[Hoang,Teubner'97]



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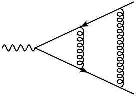

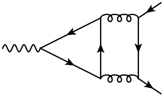

vector, axial-vector, scalar, pseudo-scalar current

[Gluza,Mitov,Moch,Riemann'09] $+\mathcal{O}(\epsilon)$

[Ahmed,Henn,Steinhauser'17; Ablinger,Behring,Blümlein,Falcioni,De Freitas,Marquard,Rana,Schneider'17] $+\mathcal{O}(\epsilon^2)$

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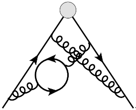
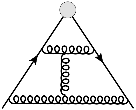


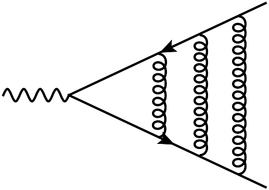


3-loop results agree with [Ablinger,Blümlein,Marquard,Rana,Schneider'18]

– independent calculation

– different method

IR structure of massive form factors: $\Gamma_{\rm cusp}$



- F: UV-renormalized massive form factor
- $F = Z\, F^{\rm finite}$

$$
\begin{aligned}
Z &= 1 + \frac{\alpha_s}{\pi}\left(-\frac{1}{2\epsilon}\Gamma^{(1)}_{\rm cusp}\right) \\
&\quad + \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\#}{\epsilon^2} - \frac{1}{4\epsilon}\Gamma^{(2)}_{\rm cusp}\right) \\
&\quad + \left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} - \frac{1}{6\epsilon}\Gamma^{(3)}_{\rm cusp}\right) + \ldots
\end{aligned}
$$

$$\Gamma^{(i)}_{\rm cusp} = \Gamma^{(i)}_{\rm cusp}(x) \qquad \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

➪ extract $\Gamma_{\rm cusp}$ from pole of F

- Note: $\Gamma_{\rm cusp}$ is universal

  ➪ same result for vector, axial-vector, scalar, pseudo-scalar current

Reduction and MIs

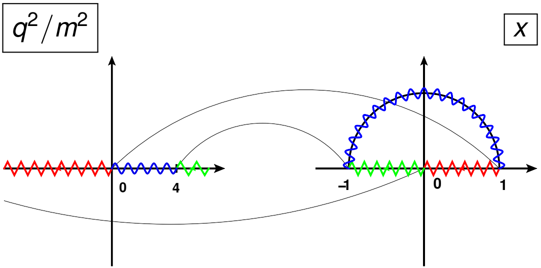
Reduction to MIs: `FIRE` [A. Smirnov]

Reduction and MIs

[Henn,Smirnov,Smirnov'16, . . . , Lee,Smirnov,Smirnov,Steinhauser'18]

$$\frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

$x \to 0$	high energy:	$q^2 \to -\infty$
$x \to -1$	threshold:	$q^2 \to 4m^2$
$x \to 1$	low energy:	$q^2 \to 0$



Reduction and MIs

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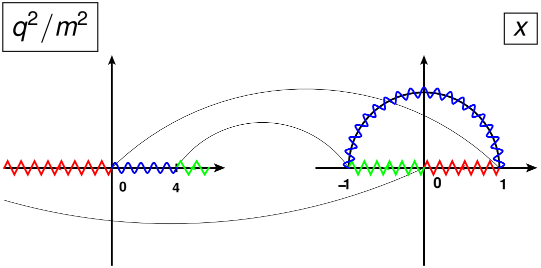
[Henn,Smirnov,Smirnov'16, . . . , Lee,Smirnov,Smirnov,Steinhauser'18]



- canonical basis: [Henn'13][Lee'14]

$$\text{MI}'(x,\epsilon) = \epsilon\, A(x) \cdot \text{MI}(x,\epsilon)$$

$$A(x) = \frac{a_1}{x} + \frac{a_2}{1+x} + \frac{a_3}{1-x} + \frac{a_4}{1-x+x^2}$$

$$q^2 \to -\infty,\ \ q^2 \to 4m^2,\ \ q^2 \to 0,\ \ q^2 = m^2$$

new at 3 loops

Reduction and MIs

[Henn,Smirnov,Smirnov'16, ..., Lee,Smirnov,Smirnov,Steinhauser'18]

- canonical basis: $$\mathrm{MI}'(x,\epsilon) = \epsilon\, A(x) \cdot \mathrm{MI}(x,\epsilon)$$ [Henn'13][Lee'14]

$$A(x) = \frac{a_1}{x} + \frac{a_2}{1+x} + \frac{a_3}{1-x} + \frac{a_4}{1-x+x^2}$$

$$q^2 \to -\infty,\;\; q^2 \to 4m^2,\;\; q^2 \to 0,\;\; q^2 = m^2$$

new at 3 loops

- $1 - x + x^2 = (x - r_1)(x - r_2)$

$r_{1/2} = (1 \pm \sqrt{3}i)/2 = e^{\pm i\pi/3}$ $\qquad$ 6^{th} root of unity

➪ Goncharov polylogarithms [Goncharov'98]

$$G(\alpha_1,\ldots,\alpha_n;z) \;=\; \int_0^z \frac{\mathrm{d}t}{t-\alpha_1} G(\alpha_2,\ldots,\alpha_n;z)$$

- BC: $q^2 \to 0$ $(x \to 1)$: 3-loop on-shell integrals [Laporta,Remiddi'96,...,Lee,Smirnov'11]

Analytic results

... in terms of Goncharov polylogarithms up to transcendelity weight 6.
[Henn,Smirnov,Smirnov,Steinhauser'16; Lee,Smirnov,Smirnov,Steinhauser'18]

Analytic results

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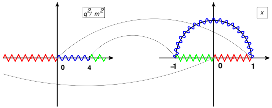

Analytic expansions in kinematical limits:

- $q^2 \to -\infty$, $x \to 0$
- $q^2 \to 0$, $x \to 1$
- $q^2 \to 4m^2$, $x \to -1$

Threshold: $q^2 \to 4m^2$, $x \to -1$

$\beta = \sqrt{1 - \frac{4m^2}{q^2}}$

- form factors ➪ cross sections and decay rates
- real radiation is suppressed by a relative order β^3

$$\sigma(e^+e^- \to Q\bar{Q}) = \sigma_0 R^v + \ldots$$

$$\Gamma(H \to Q\bar{Q}) = \frac{3G_F M_H M_Q^2}{4\sqrt{2}\pi} R^s + \ldots$$

$$\Gamma(A \to Q\bar{Q}) = \frac{3G_F M_A M_Q^2}{4\sqrt{2}\pi} R^p + \ldots$$

$$R^v = \beta \left(|F_1^v + F_2^v|^2 + \frac{|(1-\beta^2)F_1^v + F_2^v|^2}{2(1-\beta^2)} \right)$$

$$R^a = \beta^3 |F_1^a|^2 \qquad R^s = \beta^3 |F^s|^2 \qquad R^p = \beta |F^p|^2$$

Threshold: $q^2 \to 4m^2$, $x \to -1$

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- form factors ➪ cross sections and decay rates
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$$R^v = \beta \left(|F_1^v + F_2^v|^2 + \frac{|(1-\beta^2)F_1^v + F_2^v|^2}{2(1-\beta^2)} \right)$$

$$R^a = \beta^3 |F_1^a|^2 \qquad R^s = \beta^3 |F^s|^2 \qquad R^p = \beta |F^p|^2$$

$$R^v = \frac{3\beta}{2} \left[\left(1 - \frac{\beta^2}{3}\right) + \sum_{i\geq 1} \left(\frac{\alpha_s}{4\pi}\right)^i \Delta^{(i),v} \right]$$

$$R^{a/s} = \beta^3 \left[1 + \sum_{i\geq 1} \left(\frac{\alpha_s}{4\pi}\right)^i \Delta^{(i),a/s} \right]$$

$$R^p = \beta \left[1 + \sum_{i\geq 1} \left(\frac{\alpha_s}{4\pi}\right)^i \Delta^{(i),p} \right]$$

Threshold: $q^2 \to 4m^2$, $x \to -1$ (cont.)

$$\Delta^{(1),s} = C_F \left[\frac{2\pi^2}{\beta} - 4 + 2\pi^2 \beta \right]$$

Threshold: $q^2 \to 4m^2$, $x \to -1$ (cont.)

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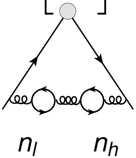
$$\Delta^{(2),s} = \ldots$$

Threshold: $q^2 \to 4m^2$, $x \to -1$ (cont.)

$$
\begin{aligned}
\Delta^{(3),s} = \\
& N_c^3\Bigg[\frac{\pi^4}{\beta^3} + \frac{1}{\beta^2}\Bigg(-\frac{44}{9}\pi^4\log(2\beta) - \frac{44}{3}\pi^2\log(2\beta) - \frac{88\pi^2\zeta(3)}{3} \\
& + \frac{176\pi^4}{27} + \frac{374\pi^2}{9}\Bigg) + \frac{1}{\beta}\Bigg(\frac{484}{9}\pi^2\log^2(2\beta) - 8\pi^4\log(2\beta) - 4\pi^4 l_2 \\
& - \frac{4484}{27}\pi^2\log(2\beta) - \frac{104\pi^2\zeta(3)}{3} - \frac{\pi^6}{4} + \frac{1375\pi^4}{54} + \frac{11434\pi^2}{81}\Bigg)\Bigg] \\
& + C_F^2 T_F n_l\Big[\ldots\Big] + C_A C_F T_F n_l\Big[\ldots\Big] + C_F T_F^2 n_l^2\Big[\ldots\Big] \\
& + C_F T_F^2 n_h n_l\left(\frac{640\pi^2}{27} - \frac{20096}{81}\right) \longleftarrow
\end{aligned}
$$



Threshold: $q^2 \to 4m^2$, $x \to -1$ (cont.)

$$\Delta^{(3),s} =$$

$$N_c^3 \left[\frac{\pi^4}{\beta^3} + \frac{1}{\beta^2} \left(-\frac{44}{9}\pi^4 \log(2\beta) - \frac{44}{3}\pi^2 \log(2\beta) - \frac{88\pi^2\zeta(3)}{3} \right. \right.$$

[Landau,Lifschitz: "Quantum mechnics"; Messiah: "Quantum mechnics 1", Fadin,Khoze'91]

$$\begin{aligned} \Delta^\delta &= \frac{y}{1-e^{-y}}\left(1 + P^\delta \frac{y^2}{4\pi^2}\right) + \ldots \\ &= 1 + \frac{\alpha_s}{4\pi} C_F \frac{2\pi^2}{\beta} + \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{C_F^2}{\beta^2}\left(\frac{4\pi^4}{3} + P^\delta 4\pi^2\right) \\ &\quad + \left(\frac{\alpha_s}{4\pi}\right)^3 \frac{C_F^3}{\beta^3} P^\delta 8\pi^4 + \ldots . \end{aligned}$$

$y = C_F\alpha_s\pi/\beta$ $\quad$ $P^\delta = 0$ for S-wave , $\quad$ $P^\delta = 1$ for P-wave

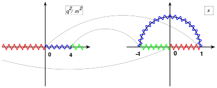
$P^v = P^p = 0$, $P^a = P^s = 1$

n_l $\quad$ n_h

Numerical example 1: F^s for $x \in [-1, 1]$

- GPLs ➪ `ginac` [Bauer,Frink,Kreckel'00; Vollinga,Weinzierl'04]
- subtract threshold and high-energy singularities

$$\tilde{F}_s(q^2) = (1+x)^4 \left[F_s(q^2) - F_s(q^2)\Big|_{q^2\to\infty} \right]$$



Numerical example 1: F^s for $x \in [-1, 1]$

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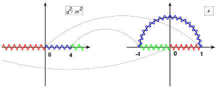

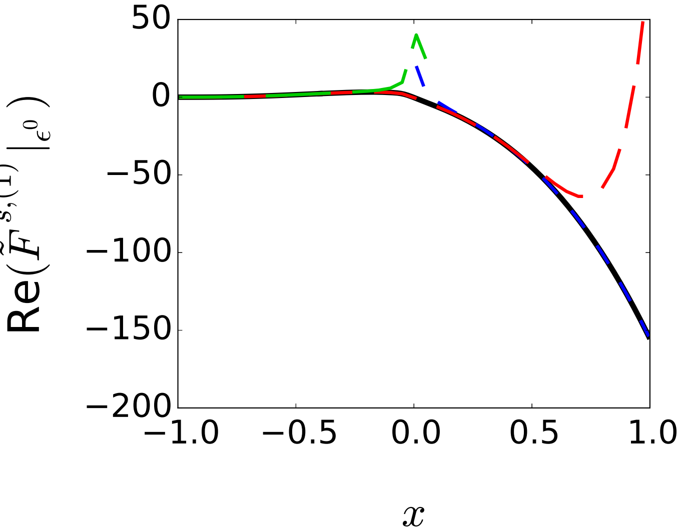

Numerical example 1: F^s for $x \in [-1, 1]$

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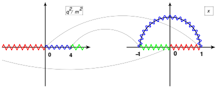

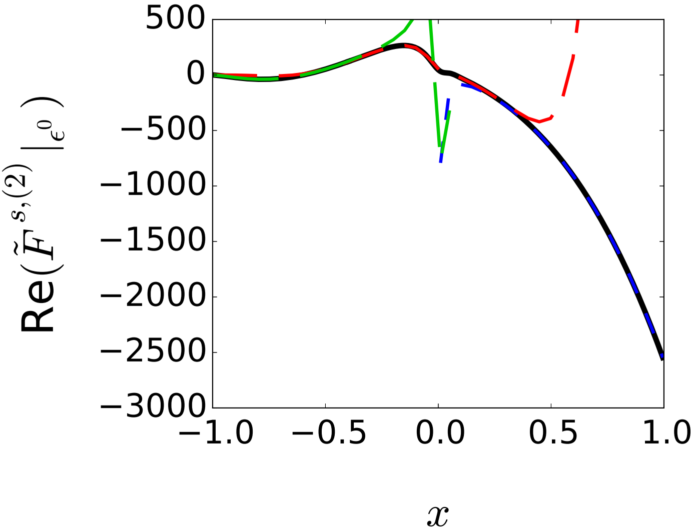

Numerical example 1: F^s for $x \in [-1, 1]$

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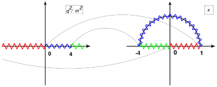
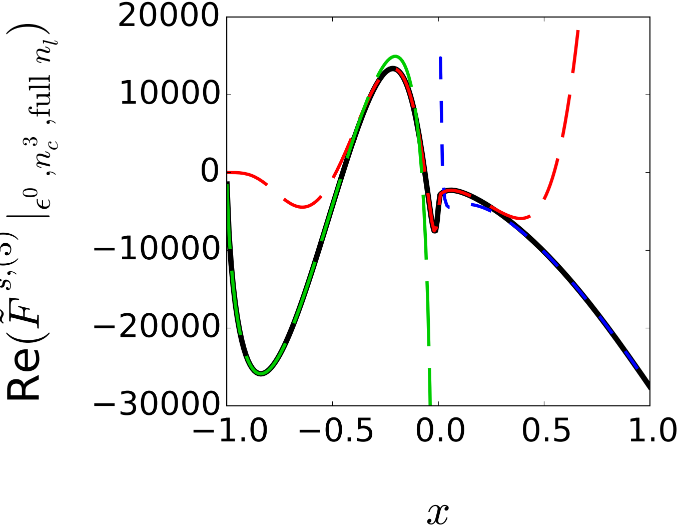
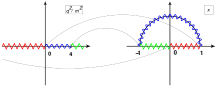
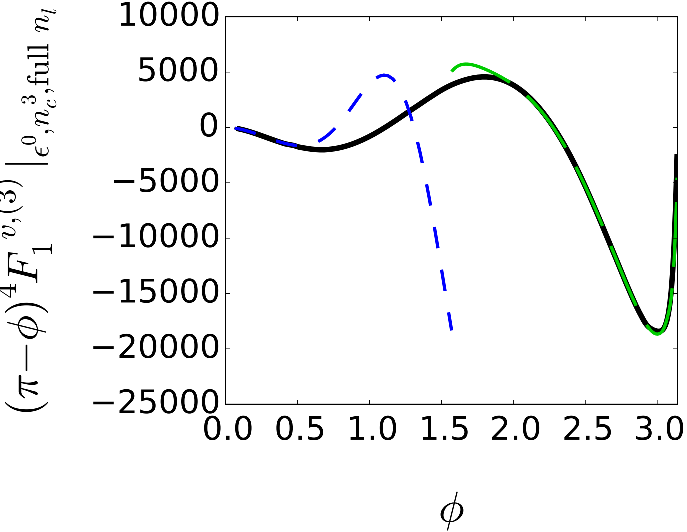




Numerical example 2: F_1^v for $\phi \in [0, \pi]$



$(\pi - \phi)^4 F_1^v$

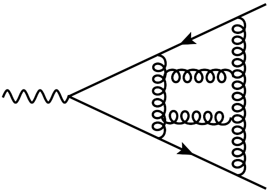


Checks

- $\Gamma_{\rm cusp}$ [Korchemsky,Radyushkin'87], [Grozin,Henn,Korchemsky,Marquard'14'15], [Grozin,Henn,Stahlhofen'17]
- $F_1^v = F_1^a$ and $F^s = F^p$ for $q^2 \to \infty$
- arbitray QCD gauge paramter ξ ➪ drops out for renormalized F
- numerical cross check of MIs with FIESTA [Smirnov'15]
- agrement for $F_2^v(0)$ [Grozin,Marquard,Piclum,Steinhauser] and $F_1^a(0)$ [Archambault,Czarnecki'04]
- Ward identity satisfied:

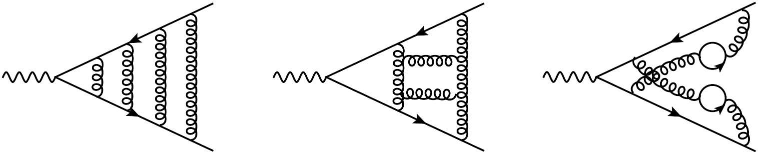
$$q^\mu \Gamma_\mu^a = 2i\Gamma^p \quad ➪ \quad F_1^a + \frac{q^2}{4m^2} F_2^a = F^p$$

II. Massless form factor



F_q

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2} \mathrm{Tr}\left(\not{q}_2\, \Gamma^{\mu}_{q} \not{q}_1\, \gamma_\mu\right)$$



- All planar diagrams ⇨ large-N_c

[Henn,Smirnov,Smirnov,Steinhauser'16; Henn,Lee,Smirnov,Smirnov,Steinhauser'16]

- All n_f^2 terms (planar and non-planar) [Lee,Smirnov,Smirnov,Steinhauser'17]

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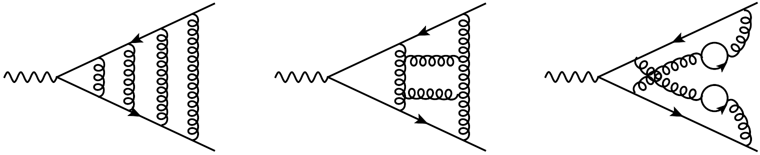

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1. Reduction to master integrals
2. Compute master integrals

F_q

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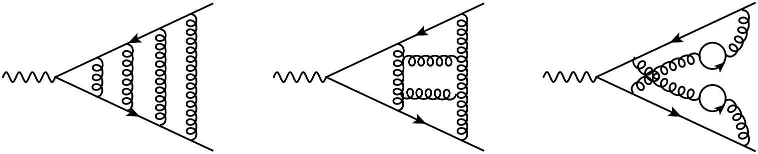


- All planar diagrams ➪ large-N_c

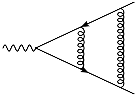
[Henn,Smirnov,Smirnov,Steinhauser'16; Henn,Lee,Smirnov,Smirnov,Steinhauser'16]

- All n_f^2 terms (planar and non-planar) [Lee,Smirnov,Smirnov,Steinhauser'17]

1. Reduction to master integrals ➪ bottleneck
2. Compute master integrals

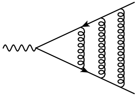
Status

- massless form factor



[Kramer,Lampe'87; Matsuura,van der Marck,van Neerven'88;

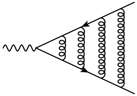
Harlander'00; Ravindran,Smith,van Neerven'05; Gehrmann,Huber,Maitre'05]



pole part: [Moch,Vermaseren,Vogt'05]

fermionic part: [Moch,Vermaseren,Vogt'05]

full: [Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09]
[Gehrmann,Glover,Huber,Ikizlerli,Studerus'10]; [Lee,Smirnov'10]



all n_f terms, large-N_c: [Henn,Smirnov,Smirnov,Steinhauser'16]

n_f^3 terms Higgs-gluon and photon-quark FF: [von Manteuffel,Schabinger'16]

full large-N_c: [Henn,Lee,Smirnov,Smirnov,Steinhauser'16]

complete n_f^2 terms: [Lee,Smirnov,Smirnov,Steinhauser'17]

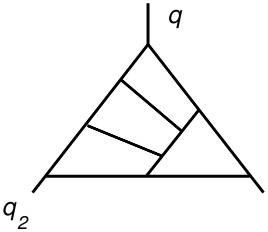
$\mathcal{N} = 4$ SYM [Boels,Kniehl,Tarasov,Yang'12'16] numerical methods for MIs

- γ_{cusp}

[Boels,Huber,Yang'17]: $\mathcal{N} = 4$ SYM, numerically

[Moch,Ruijl,Ueda,Vermaseren,Vogt'17] analytic and numerical results

Computation of MIs

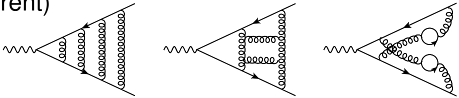


- reduction: FIRE [A. Smirnov]
- MIs: $q^2 \neq 0$ and
  $q_2^2 = (q_2 + q)^2 = 0$
- integrals simple if $q_2^2 = q^2$

- idea: introduce arbitray q_2^2 ➪ differential equations [Henn,Smirnov,Smirnov'14]
- BCs: $q_2^2 = q^2$
- use canonical basis [Henn'13'14] [Lee'14] [Gituliar,Magerya'16; Meyer'16; Prausa'17]
  solution: iterated integrals ➪ harmonic polylogarithms (HPLs)
  [Remiddi,Vermaseren'99][Maitre'05]
- $q_2^2 \to 0$: $\left(q_2^2/q^2\right)^{a\,\epsilon}$ extract term with $a = 0$

Results

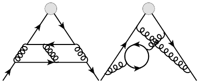
- analytic results (incl. ϵ^0) for:
  - full n_f^2 (vector and scalar current)
  - rest: large-N_c



- n_f^3 agrees with [von Manteuffel,Schabinger'16]
- pole part: $\gamma_{\rm cusp}$ [Moch,Ruijl,Ueda,Vermaseren,Vogt'17]

Conclusions/Outlook

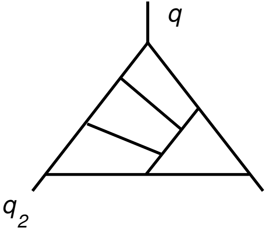
- massive 3-loop form factor
  - all n_l and N_c^3
  - vector, axial-vector, scalar, pseudo-scalar
  - next steps: massive closed quark loops, singlet contribution, . . . ➪ elliptic integrals



- massless 4-loop form factor
  - large-N_c limit for n_f^0 and n_f^1
  - planar and non-planar n_f^2 (also for Higgs-quark FF)
  - next steps: all n_f terms, singlet contribution, . . .
    ➪ effective reduction needed

BACKUP

Computation of MIs



$$q^2 \neq 0$$
$$q_2^2 = (q_2 + q)^2 = 0$$

idea 1:

- $q_2^2 \neq 0$ ➪ $x = \frac{q_2^2}{q^2}$ [Henn,Smirnov,Smirnov'14]

  consider system of differential equations in x
- boundary conditions for $x = 1$ ➪ integrals "simple"
- get result for $x = 0$

idea 2: use canonical basis where differential equations have the form

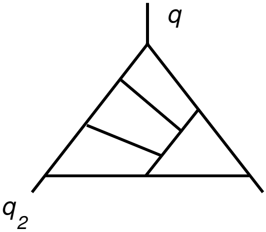
$$g'(x, \epsilon) = \epsilon\, A(x) \cdot g(x, \epsilon)$$ [Henn'13'14]

$$A(x) = \frac{a}{x} + \frac{b}{x-1}$$ [Lee'14] [Gituliar,Magerya'16; Meyer'16; Prausa'17]

solution: iterated integrals ➪ harmonic polylogarithms (HPLs)

[Remiddi,Vermaseren'99][Maitre'05]

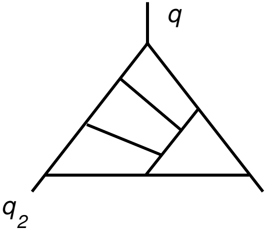
Computation of MIs: more details



➪ 76 MIs

$q_2^2 \neq 0$ ➪ 332 MIs

Computation of MIs: more details



primary basis

➪ 76 MIs

$q_2^2 \neq 0$ ➪ 332 MIs

canonical basis

$f(x,\epsilon)$ $\xrightarrow{f=T\cdot g \text{ [Lee'14]}}$

$$g(x,\epsilon) = \sum_{k=0}^{8} g_k(x)\epsilon^k$$
$$g'(x,\epsilon) = \epsilon\, A(x)\cdot g(x,\epsilon)$$
$$A(x) = \frac{a}{x} + \frac{b}{x-1}$$

Computation of MIs: more details

primary basis

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$\downarrow$

solve in terms of HPLs

Computation of MIs: more details

primary basis

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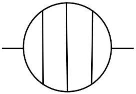
solve in terms of HPLs

boundary conditions for $x = 1$:



4-loop

2-point functions

[Baikov,Chetyrkin,Kühn'05'08;...; Lee,Smirnov,Smirnov'11]

Computation of MIs: more details

primary basis

canonical basis

$f(x,\epsilon)$ $\xrightarrow{f=T\cdot g \text{ [Lee'14]}}$

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$$g'(x,\epsilon) = \epsilon\, A(x)\cdot g(x,\epsilon)$$
$$A(x) = \frac{a}{x} + \frac{b}{x-1}$$

↓

solve in terms of HPLs
boundary conditions for $x = 1$:

↓

get $x = 0$ result (“naive”):

1. $g_{x\to 0} = x^{\epsilon a} h(\epsilon)$ a: 332 × 332 matrix

➪ $x^{\epsilon a}$ is 332 × 332 matrix; each element is linear combination of $x^{k\epsilon}$ terms with $k \leq 0$

Computation of MIs: more details

Karlsruhe Institute of Technology

primary basis

canonical basis

$f(x,\epsilon)$ $\xrightarrow{f=T\cdot g \text{ [Lee'14]}}$ $g(x,\epsilon)$

$\downarrow$

solve in terms of HPLs
boundary conditions for $x = 1$:

$\downarrow$

get $x = 0$ result ("naive"):

1. $g_{x\to 0} = x^{\epsilon a} h(\epsilon)$
2. expand HPLs for $x \to 0$

match 1. and 2.

➪ $h(\epsilon)$

➪ get $x^{0\epsilon}$ terms $\widehat{=}$ "naive"

Computation of MIs: more details

primary basis | canonical basis

$f(x,\epsilon)$ $\xrightarrow{f=T\cdot g \text{ [Lee'14]}}$ $g(x,\epsilon)$

$\downarrow$

solve in terms of HPLs
boundary conditions for $x = 1$:

$\downarrow$

get $x = 0$ result ("naive"):

1. $g_{x\to 0} = x^{\epsilon a} h(\epsilon)$
2. expand HPLs for $x \to 0$

match 1. and 2.

➪ $h(\epsilon)$

➪ get $x^{0\epsilon}$ terms $\widehat{=}$ "naive"

$f(0,\epsilon)$ $\xleftarrow{f=T\cdot g}$

$\downarrow$

332 MIs

$\downarrow$

76 MIs