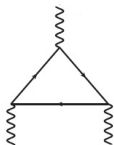


*On the gauge and scheme-dependence of the  
generalized Crewther relation in QCD*

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$$G_{\mu\nu\rho}^{abc}(p, q) = i \int \langle 0 | T A_{\mu}^a(x) V_{\nu}^b(y) V_{\rho}^c(0) | 0 \rangle e^{i(px+qy)} dx dy,$$

$V_{\mu}^a(x) = \bar{\psi}(x) \gamma_{\mu} t^a \psi(x)$  — vector non-singlet (NS) current,

$A_{\mu}^a(x) = \bar{\psi}(x) \gamma_{\mu} \gamma_5 t^a \psi(x)$  — axial NS current.

$$\int T V_{\mu}^a(x) V_{\nu}^b(0) e^{iqx} dx \Big|_{q^2 \rightarrow \infty} \simeq d^{abc} \varepsilon_{\mu\nu\rho\lambda} \frac{q^{\lambda}}{Q^2} C_{Bjp} A_{\rho}^c(0) + \dots,$$

$$i \int T A_{\mu}^a(x) A_{\nu}^b(0) e^{iqx} dx = \delta^{ab} (g_{\mu\nu} q^2 - q_{\mu} q_{\nu}) \Pi(q^2).$$

## The polarized Bjorken sum rule

The Bjorken function  $C_{Bjp}(a_s)$  is a characteristic of deep inelastic scattering of charged leptons on polarized nucleons and is related to the corresponding Bjorken polarized sum rule:

$$\int_0^1 \left( g_1^{lp}(x, Q^2) - g_1^{ln}(x, Q^2) \right) dx = \frac{1}{6} \left| \frac{g_A}{g_V} \right| C_{Bjp}(a_s(Q^2)) ,$$

where  $g_1^{lp}$  and  $g_1^{ln}$  are the structure functions of polarized lepton-proton and lepton-neutron deep inelastic scattering, characterizing the spin distribution of quarks inside nucleons. The ratio of axial and vector charge of the neutron  $\beta$ -decay is  $g_A/g_V \approx -1.2723 \pm 0.0023$ .

Neglecting the mass dependence in the conditions of a large momentum transfer  $Q^2$  in the  $\overline{\text{MS}}$  renormalization scheme in the  $\mathcal{O}(a_s^4)$  approximation of PT the Bjorken function can be represented in the form of two terms:

$$C_{Bjp}(a_s) = C_{Bjp}^{NS}(a_s) + \sum_f Q_f C_{Bjp}^{SI}(a_s) ,$$

where  $C_{Bjp}^{NS}(a_s)$  and  $C_{Bjp}^{SI}(a_s)$  — the flavor NS and SI contributions to the Bjorken function,  $a_s = \alpha_s/\pi$ ,  $Q_f$  is the electric charge of the  $f$ -th quark.

$$C_{Bjp}^{NS}(a_s) = 1 + \sum_{k \geq 1} c_k a_s^k .$$

## Non-singlet contribution

The coefficients of this series are known up to the fourth order:

$$c_1^{\overline{\text{MS}}} = -\frac{3}{4}C_F ,$$

$$c_2^{\overline{\text{MS}}} = \frac{21}{32}C_F^2 - \frac{23}{16}C_F C_A + \frac{1}{2}C_F T_F n_f , \quad (\text{Gorishny, Larin, 1986}) ,$$

$$c_3^{\overline{\text{MS}}} = -\frac{3}{128}C_F^3 + \left( \frac{1241}{576} - \frac{11}{12}\zeta_3 \right) C_F^2 C_A + \left( -\frac{5437}{864} + \frac{55}{24}\zeta_5 \right) C_F C_A^2 \\ - \left( \frac{133}{576} + \frac{5}{12}\zeta_3 \right) C_F^2 T_F n_f + \left( \frac{3535}{864} + \frac{3}{4}\zeta_3 - \frac{5}{6}\zeta_5 \right) C_F C_A T_F n_f \\ - \frac{115}{216}C_F T_F^2 n_f^2 , \quad (\text{Larin, Vermaseren, 1991}) ,$$

where  $C_F$  and  $C_A$  are the Casimir operators,  $(T^a T^a)_{ij} = C_F \delta_{ij}$ ,  $f^{acd} f^{bcd} = C_A \delta^{ab}$ ,  $T_F = 1/2$ . For special case  $SU_c(3)$  QCD  $C_F = 4/3$ ,  $C_A = 3$ .  $\zeta_n$  is the Riemann zeta-function. The coefficient  $c_4^{\overline{\text{MS}}}$  was calculated in the work (*Baikov, Chetyrkin, Kühn, Phys.Rev.Lett. 104 (2010) 132004*) and contains additional color configurations  $d_F^{abcd} d_A^{abcd}$  and  $d_F^{abcd} d_F^{abcd}$ .

# The Adler function

The Adler function is related to the experimentally measured characteristic of the electron-positron annihilation process into hadrons, called the  $R$ -relation

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \gamma \rightarrow \text{hadrons})}{\sigma_{\text{Born}}(e^+e^- \rightarrow \gamma \rightarrow \mu^+\mu^-)},$$

by means of the dispersion relation:

$$D(Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{(s + Q^2)^2} = -12\pi^2 Q^2 \frac{d}{dQ^2} \Pi(Q^2).$$

By analogy with the Bjorken function the following representation holds:

$$D(Q^2) = N_c \left( \sum_f Q_f^2 D^{NS}(a_s) + \left( \sum_f Q_f \right)^2 D^{SI}(a_s) \right).$$

# NS contribution to the Adler function

$$d_1^{\overline{\text{MS}}} = \frac{3}{4}C_F ,$$

$$d_2^{\overline{\text{MS}}} = -\frac{3}{32}C_F^2 + \left( \frac{123}{32} - \frac{11}{4}\zeta_3 \right) C_F C_A + \left( -\frac{11}{8} + \zeta_3 \right) C_F T_F n_f ,$$

(Chetyrkin, Kataev, Tkachov, 1979)

$$\begin{aligned} d_3^{\overline{\text{MS}}} = & -\frac{69}{128}C_F^3 + \left( -\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5 \right) C_F^2 C_A + \\ & + \left( \frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5 \right) C_F C_A^2 + \left( -\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5 \right) C_F^2 T_F n_f \\ & + \left( -\frac{485}{27} + \frac{112}{9}\zeta_3 + \frac{5}{6}\zeta_5 \right) C_F C_A T_F n_f + \left( \frac{151}{54} - \frac{19}{9}\zeta_3 \right) C_F T_F^2 n_f^2 \end{aligned}$$

(Gorishny, Kataev, Larin, 1991).

The coefficient  $d_4^{\overline{\text{MS}}}$  was computed in work of (Baikov, Chetyrkin, Kühn, *Phys.Rev.Lett.* 104 (2010) 132004) and confirmed later in work (Herzog, Ruijl, Ueda, Vermaseren, Vogt, *JHEP* 1708 (2017) 113).

On the other hand, in the conformally invariant limit this AVV three-point Green's function is proportional to the triangular one-loop fermionic loop that determines the  $\pi^0 \rightarrow \gamma\gamma$  decay:

$$G_{\mu\nu\rho}^{abc}(p, q) \Big|_{\text{conf-inv}} = d^{abc} \Delta_{\mu\nu\rho}^{1\text{-loop}}(p, q) .$$

In this limit in the massless QCD the Crewther relation is performed (*Crewther, 1972*):

$$D^{NS} C_{Bjp}^{NS} \Big|_{c-i \text{ limit}} = 1 .$$



# The generalized Crewther relation

However, when the charge renormalization is taken into account, the conformal symmetry is violated. This circumstance leads to a modification of the Crewther relation:

$$D^{NS}(a_s)C_{Bjp}^{NS}(a_s) = 1 + \Delta_{csb}(a_s) ,$$

where term  $\Delta_{csb}(a_s)$  can be presented in the **gauge-invariant**  $\overline{\text{MS}}$ -scheme in the following factorized form (*Broadhurst, Kataev, Phys.Lett. B315 (1993)*):

$$\Delta_{csb} = \left( \frac{\beta(a_s)}{a_s} \right) \sum_{i \geq 1} K_i a_s^i , \quad \beta(a_s) = \mu^2 \frac{da_s}{d\mu^2} = - \sum_{i \geq 0} \beta_i a_s^{i+2} .$$

This statement was confirmed explicitly at the  $\mathcal{O}(a_s^4)$  level by (*Baikov, Chetyrkin, Kühn, Phys.Rev.Lett. 104 (2010) 132004*). Theoretical indications on validity in all orders of PT were given in works of (*Crewther (1997)*; *Gabadadze, Kataev (1995)*) and others.

# Consideration in the mMOM-scheme

Is the  $\beta(a_s)$ -factorization possible in the **gauge-dependent renormalization schemes** such as the mMOM-scheme?

$$A_{0,\mu}^a = \sqrt{Z_A} A_\mu^a, \quad c_0^a = \sqrt{Z_c} c^a, \quad g_0 = \mu^\epsilon Z_g g, \quad \xi_0 = Z_A Z_\xi^{-1} \xi,$$

where  $A_\mu^a, c^a$  are fields of gluons and ghosts correspondingly,  $\xi$  is the gauge parameter, included in the Lagrangian in the form  $(\partial_\mu A_\mu^a)^2/2\xi$ . The gauge-dependent mMOM-scheme is determined by the requirement of equality of the renormalization constant of the gluon-ghost-antighost vertex  $Z_{cg} = Z_g Z_A^{1/2} Z_c$  to its analogue, defined in the  $\overline{\text{MS}}$ -scheme (*Smekal, Maltman, Sternbeck, Phys.Lett. B681 (2009)*):

$$Z_{cg}^{\text{mMOM}}(a_s^{\text{mMOM}}) = Z_{cg}^{\overline{\text{MS}}}(a_s^{\overline{\text{MS}}})$$

In this case the relation between coupling constants in these two schemes will look like as

$$a_s^{\text{mMOM}}(\mu^2) = \frac{Z_A^{\text{mMOM}}}{Z_A^{\overline{\text{MS}}}} \left( \frac{Z_c^{\text{mMOM}}}{Z_c^{\overline{\text{MS}}}} \right)^2 a_s^{\overline{\text{MS}}}(\mu^2).$$

Taking into account the renormalization conditions on the polarization operators of gluons and ghosts, we obtain the following relations between the coupling constants and the gauge parameters

(Gracey, *J.Phys. A*46 (2013) 225403),

(Ruijl, Ueda, Vermaseren, Vogt, *JHEP* 1706 (2017) 040):

$$a_s^{\text{mMOM}}(\mu^2) = \left(1 + \Pi^{\overline{\text{MS}}}(\mu^2)\right)^{-1} \left(1 + \tilde{\Pi}^{\overline{\text{MS}}}(\mu^2)\right)^{-2} a_s^{\overline{\text{MS}}}(\mu^2),$$
$$\xi^{\text{mMOM}}(\mu^2) = \left(1 + \Pi^{\overline{\text{MS}}}(\mu^2)\right) \xi^{\overline{\text{MS}}},$$

where  $\Pi^{\overline{\text{MS}}}$  and  $\tilde{\Pi}^{\overline{\text{MS}}}$  are the self-energy operators of the gluons and ghosts correspondingly, which depend on  $\xi^{\overline{\text{MS}}}$ .

Further we find the following expansions of the quantities defined in the  $\overline{\text{MS}}$ -scheme in terms of the quantities computed in the mMOM-scheme:

$$\begin{aligned} \xi^{\overline{\text{MS}}} &= \xi \left( 1 + \left[ \left( \frac{97}{144} + \frac{1}{8}\xi + \frac{1}{16}\xi^2 \right) C_A - \frac{5}{9} T_{F^n f} \right] a_s + \right. \\ &+ \left[ \left( \frac{5591}{4608} - \frac{3}{16}\zeta_3 + \left( -\frac{121}{1536} + \frac{1}{8}\zeta_3 \right) \xi + \frac{7}{256}\xi^2 + \frac{7}{256}\xi^3 + \frac{1}{256}\xi^4 \right) C_A^2 + \right. \\ &\left. \left. + \left( -\frac{371}{576} - \frac{1}{2}\zeta_3 \right) C_A T_{F^n f} + \left( -\frac{55}{48} + \zeta_3 \right) C_F T_{F^n f} \right] a_s^2 + \dots \right), \end{aligned}$$

$$a_s^{\overline{\text{MS}}} = a_s + b_1 a_s^2 + b_2 a_s^3 + \dots,$$

$$b_1 = \left( -\frac{169}{144} - \frac{1}{8}\xi - \frac{1}{16}\xi^2 \right) C_A + \frac{5}{9} T_{F^n f},$$

$$\begin{aligned} b_2 &= \left( -\frac{18941}{20736} + \frac{39}{128}\zeta_3 + \left( \frac{889}{2304} - \frac{11}{64}\zeta_3 \right) \xi + \left( \frac{203}{2304} + \frac{3}{128}\zeta_3 \right) \xi^2 - \frac{3}{256}\xi^3 \right) C_A^2 \\ &+ \left( -\frac{107}{648} + \frac{\zeta_3}{2} - \frac{5}{36}\xi - \frac{5}{72}\xi^2 \right) C_A T_{F^n f} + \left( \frac{55}{48} - \zeta_3 \right) C_F T_{F^n f} + \frac{25}{81} T_{F^n f}^2. \end{aligned}$$

where  $a_s^{\text{mMOM}} = a_s$  и  $\xi^{\text{mMOM}} = \xi$ .

# RG $\beta$ -function in the mMOM-scheme

$$\beta^{\text{mMOM}} = \beta^{\overline{\text{MS}}} \frac{\partial a_s^{\text{mMOM}}}{a_s^{\overline{\text{MS}}}} + \xi^{\overline{\text{MS}}} \gamma_\xi^{\overline{\text{MS}}} \frac{\partial a_s^{\text{mMOM}}}{\partial \xi^{\overline{\text{MS}}}},$$

$$\beta_0^{\text{mMOM}} = \frac{11}{12} C_A - \frac{1}{3} T_F n_f,$$

$$\beta_1^{\text{mMOM}} = \left[ \frac{17}{24} - \frac{13}{192} \xi - \frac{5}{96} \xi^2 + \frac{1}{64} \xi^3 \right] C_A^2 + \left[ -\frac{5}{12} + \frac{1}{24} \xi + \frac{1}{24} \xi^2 \right] C_A T_F n_f - \frac{1}{4} C_F T_F n_f,$$

$$\begin{aligned} \beta_2^{\text{mMOM}} = & \left[ \frac{9655}{4608} - \frac{143}{512} \zeta_3 + \left( -\frac{1097}{6144} + \frac{33}{512} \zeta_3 \right) \xi + \left( -\frac{725}{6144} + \frac{13}{512} \zeta_3 \right) \xi^2 + \right. \\ & + \left. \left( \frac{21}{2048} - \frac{3}{512} \right) \xi^3 + \frac{55}{6144} \xi^4 \right] C_A^3 + \left[ -\frac{2009}{1152} - \frac{137}{384} \zeta_3 + \frac{37}{384} \xi + \left( \frac{23}{256} - \frac{\zeta_3}{128} \right) \xi^2 + \right. \\ & + \left. \frac{1}{128} \xi^3 \right] C_A^2 T_F n_f + \left[ -\frac{641}{576} + \frac{11}{12} \zeta_3 + \frac{1}{16} \xi + \frac{3}{64} \xi^2 \right] C_A C_F T_F n_f + \frac{1}{32} C_F^2 T_F n_f + \\ & \left[ \frac{23}{96} + \frac{\zeta_3}{6} \right] C_A T_F^2 n_f^2 + \left[ \frac{23}{72} - \frac{\zeta_3}{3} \right] C_F T_F^2 n_f^2 \end{aligned}$$

# *NS contribution to the Bjorken function in the mMOM-scheme*

Using the renormalization invariance of the  $C_{Bjp}^{NS}(a_s)$  function and the obtained relation  $a_s^{\overline{\text{MS}}}(\xi^{\text{mMOM}}, a_s^{\text{mMOM}})$  we find:

$$c_1^{\text{mMOM}} = -\frac{3}{4}C_F,$$
$$c_2^{\text{mMOM}} = \frac{21}{32}C_F^2 + \left(-\frac{107}{192} + \frac{3}{32}\xi + \frac{3}{64}\xi^2\right)C_F C_A + \frac{1}{12}C_F T_F n_f,$$
$$c_3^{\text{mMOM}} = -\frac{3}{128}C_F^3 + \left[\frac{13}{9} + \frac{3}{8}\zeta_3 - \frac{5}{6}\zeta_5 - \frac{1}{48}\xi - \frac{1}{96}\xi^2\right]C_F C_A T_F n_f +$$
$$+ \left[-\frac{13}{36} + \frac{\zeta_3}{3}\right]C_F^2 T_F n_f + \left[\frac{1415}{2304} - \frac{11}{12}\zeta_3 - \frac{21}{128}\xi - \frac{21}{256}\xi^2\right]C_F^2 C_A -$$
$$-\frac{5}{24}C_F T_F^2 n_f^2 + \left[-\frac{20585}{9216} - \frac{117}{512}\zeta_3 + \frac{55}{24}\zeta_5 + \left(\frac{215}{3072} + \frac{33}{256}\zeta_3\right)\xi +$$
$$+ \left(\frac{349}{3072} - \frac{9}{512}\zeta_3\right)\xi^2 + \frac{9}{1024}\xi^3\right]C_F C_A^2.$$

# *NS contribution to the Adler function in the mMOM-scheme*

$$\begin{aligned}
 d_1^{\text{mMOM}} &= \frac{3}{4} C_F, \\
 d_2^{\text{mMOM}} &= -\frac{3}{32} C_F^2 + \left[ \frac{569}{192} - \frac{11}{4} \zeta_3 - \frac{3}{32} \xi - \frac{3}{64} \xi^2 \right] C_F C_A + \left[ \zeta_3 - \frac{23}{24} \right] C_F T_F n_f, \\
 d_3^{\text{mMOM}} &= -\frac{69}{128} C_F^3 + \left[ -\frac{1355}{768} - \frac{143}{16} \zeta_3 + \frac{55}{4} \zeta_5 + \frac{3}{128} \xi + \frac{3}{256} \xi^2 \right] C_F^2 C_A + \\
 &+ \left[ -\frac{2033}{192} + \frac{89}{12} \zeta_3 + \frac{5}{6} \zeta_5 + \left( \frac{23}{96} - \frac{\zeta_3}{4} \right) \xi + \left( \frac{23}{192} - \frac{\zeta_3}{8} \right) \xi^2 \right] C_F C_A T_F n_f + \\
 &+ \left[ \frac{50575}{3072} - \frac{18929}{1536} \zeta_3 - \frac{55}{24} \zeta_5 + \left( -\frac{2063}{3072} + \frac{143}{256} \zeta_3 \right) \xi + \right. \\
 &\quad \left. + \left( -\frac{1273}{3072} + \frac{185}{512} \zeta_3 \right) \xi^2 - \frac{9}{1024} \xi^3 \right] C_F C_A^2 + \\
 &+ \left[ \frac{29}{96} + 4\zeta_3 - 5\zeta_5 \right] C_F^2 T_F n_f + \left[ \frac{3}{2} - \zeta_3 \right] C_F T_F^2 n_f^2.
 \end{aligned}$$

# *Is the factorization of RG $\beta$ -function possible in the generalized Crewther relation in the gauge-dependent schemes?*

Using the obtained expressions for the Bjorken and Adler functions, we find that in the  $\mathcal{O}(a_s^2)$  approximation the factorization of the  $\beta$ -function is possible for any value of gauge parameter  $\xi$ :

$$K_1^{\text{mMOM}} = K_1^{\overline{\text{MS}}} = \left( -\frac{21}{8} + 3\zeta_3 \right) C_F ,$$

In the  $\mathcal{O}(a_s^3)$  order of PT this property holds for three values of the gauge parameter only, namely

$$\xi = -3, \quad -1, \quad 0 .$$



# The $\beta$ -factorization in the mMOM-scheme in the $\mathcal{O}(a_s^3)$ approximation

*Landau gauge*  $\xi = 0$  :

$$K_2^{\text{mMOM}} = \left( \frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5 \right) C_F^2 + \left( -\frac{2591}{192} + \frac{91}{8}\zeta_3 \right) C_F C_A + \\ + \left( \frac{31}{8} - 3\zeta_3 \right) C_F T_F n_f ,$$

*anti-Feynman gauge*  $\xi = -1$  :

$$K_2^{\text{mMOM}} = \left( \frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5 \right) C_F^2 + \left( -\frac{1327}{96} + \frac{47}{4}\zeta_3 \right) C_F C_A + \\ + \left( \frac{31}{8} - 3\zeta_3 \right) C_F T_F n_f ,$$

*Stefanis-Mikhailov gauge*  $\xi = -3$  :

$$K_2^{\text{mMOM}} = \left( \frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5 \right) C_F^2 + \left( -\frac{695}{48} + \frac{25}{2}\zeta_3 \right) C_F C_A + \\ + \left( \frac{31}{8} - 3\zeta_3 \right) C_F T_F n_f .$$

# The $\beta$ -factorization in the mMOM-scheme in the $\mathcal{O}(a_s^4)$ approximation

In the  $\mathcal{O}(a_s^4)$  order of PT the factorization property of the RG  $\beta$ -function in the generalized Crewther relation remains valid for the Landau gauge  $\xi = 0$  only, namely

$$\begin{aligned} K_{3, \xi=0}^{\text{mMOM}} = & \left( \frac{2471}{768} + \frac{61}{8}\zeta_3 - \frac{715}{8}\zeta_5 + \frac{315}{4}\zeta_7 \right) C_F^3 + \left( \frac{132421}{4608} + \frac{451}{8}\zeta_3 - \right. \\ & \left. - \frac{3685}{48}\zeta_5 - \frac{105}{8}\zeta_7 \right) C_F^2 C_A + \\ & + \left( -\frac{1840145}{18432} + \frac{152329}{3072}\zeta_3 + \frac{2975}{48}\zeta_5 - \frac{2113}{128}\zeta_3^2 \right) C_F C_A^2 + \\ & + \left( -\frac{1273}{144} - \frac{599}{24}\zeta_3 + \frac{75}{2}\zeta_5 \right) C_F^2 T_F n_f + \left( -\frac{49}{6} + \frac{7}{2}\zeta_3 + 5\zeta_5 \right) C_F T_F^2 n_f^2 + \\ & \left( \frac{71251}{1152} - \frac{539}{24}\zeta_3 - \frac{125}{3}\zeta_5 + \frac{5}{2}\zeta_3^2 \right) C_F C_A T_F n_f . \end{aligned}$$

## Consideration in the $MOM_{gggg}$ -scheme

It is interesting to find out whether there are other MOM-schemes in QCD, which respect the property of the  $\beta$ -function factorization in the GCR for concrete choice of the gauge parameter. We consider the  $MOM_{gggg}$ -scheme, determined by renormalization of the quartic gluon vertex through subtractions of UV divergences in the symmetric point (*Gracey, Phys. Rev. D 90 (2014) 025011*). For Landau gauge in QCD with  $SU(3)$  color group we find:

$$\beta_1^{MOM_{gggg}} \Big|_{\xi=0}^{N_c=3} = \frac{51}{8} - \frac{19}{24} n_f ,$$

$$K_2^{MOM_{gggg}} \Big|_{\xi=0}^{N_c=3} = -\frac{280073}{8640} + \frac{3017}{100} \log\left(\frac{4}{3}\right) - \frac{595}{256} \Phi_1 - \frac{50533}{51200} \Phi_2 \\ + \zeta_3 \left( \frac{15973}{360} - \frac{862}{25} \log\left(\frac{4}{3}\right) + \frac{85}{32} \Phi_1 + \frac{7219}{6400} \Phi_2 \right) - \frac{80}{3} \zeta_5 +$$

$$\left[ \frac{65}{36} - \frac{49}{24} \log\left(\frac{4}{3}\right) - \frac{49}{96} \Phi_1 + \frac{7}{96} \Phi_2 + \zeta_3 \left( -\frac{10}{9} + \frac{7}{3} \log\left(\frac{4}{3}\right) + \frac{7}{12} \Phi_1 + \frac{1}{12} \Phi_2 \right) \right] n_f$$

The special functions  $\Phi_1$  and  $\Phi_2$  are expressed through the Clausen function  $\text{Cl}_2(\theta)$  and have the following form

$$\Phi_1 = \sqrt{2} \left[ 2\text{Cl}_2 \left( 2 \arccos \left( \frac{1}{\sqrt{3}} \right) \right) + \text{Cl}_2 \left( 2 \arccos \left( \frac{1}{3} \right) \right) \right],$$

$$\Phi_2 = \frac{4}{\sqrt{5}} \left[ 2\text{Cl}_2 \left( 2 \arccos \left( \frac{2}{3} \right) \right) + \text{Cl}_2 \left( 2 \arccos \left( \frac{1}{9} \right) \right) \right],$$

$$\text{Cl}_2(\theta) = - \int_0^\theta dx \log \left| 2 \sin \frac{x}{2} \right|,$$

and numerically  $\Phi_1 \approx 2.832045$  and  $\Phi_2 \approx 3.403614$  correspondingly.

# Consideration in the $MOMgggg$ -scheme

At  $\xi = -3$  we have:

$$\begin{aligned} \frac{\beta_1^{\overline{MS}} - \beta_{1, \xi=-3}^{MOMgggg}}{\beta_0} &= -\frac{333}{20} - \frac{3537}{200} \log\left(\frac{4}{3}\right) + \frac{9}{4}\Phi_1 + \frac{33993}{6400}\Phi_2, \\ K_2^{MOMgggg} \Big|_{\xi=-3}^{N_c=3} &= -\frac{9337}{270} + \frac{13769}{400} \log\left(\frac{4}{3}\right) - \frac{35}{32}\Phi_1 - \frac{2191}{12800}\Phi_2 \\ &+ \zeta_3 \left( \frac{2108}{45} - \frac{1967}{50} \log\left(\frac{4}{3}\right) + \frac{5}{4}\Phi_1 + \frac{313}{1600}\Phi_2 \right) - \frac{80}{3}\zeta_5 \\ &+ \left[ \frac{65}{36} - \frac{49}{24} \log\left(\frac{4}{3}\right) - \frac{49}{96}\Phi_1 + \frac{7}{96}\Phi_2 \right. \\ &\left. + \zeta_3 \left( -\frac{10}{9} + \frac{7}{3} \log\left(\frac{4}{3}\right) + \frac{7}{12}\Phi_1 + \frac{1}{12}\Phi_2 \right) \right] n_f. \end{aligned}$$

Thus, we come to conclusion that in the  $\mathcal{O}(a_s^3)$  order of PT in the  $MOMgggg$ -scheme the factorization property holds in Landau and Stefanis–Mikhailov gauges and is not satisfied in anti-Feynman gauge (the gauge  $\xi = -1$  is the feature of the mMOM-scheme since  $\beta_1^{mMOM} = \beta_1^{\overline{MS}}$  in this gauge).

# Are the values of $\xi = -3$ and $\xi = 0$ distinguished in all gauge-dependent schemes?

Using relation  $a_s^{\overline{\text{MS}}} = a_s^{\text{AS}} + \sum_{k=1} b_k^{\text{AS}} (a_s^{\text{AS}})^{k+1}$  and explicit form of the term which breaks the conformal symmetry, we arrive at equation (AS denotes any scheme with linear covariant gauge):

$$\frac{\beta^{\overline{\text{MS}}}(a_s^{\overline{\text{MS}}}(a_s^{\text{AS}}))}{a_s^{\overline{\text{MS}}}(a_s^{\text{AS}})} K^{\overline{\text{MS}}}(a_s^{\overline{\text{MS}}}(a_s^{\text{AS}})) = \frac{\beta^{\text{AS}}(a_s^{\text{AS}})}{a_s^{\text{AS}}} K^{\text{AS}}(a_s^{\text{AS}}),$$

which allows us to obtain the following relations:

$$\begin{aligned} K_1^{\text{AS}} &= K_1^{\overline{\text{MS}}}, \\ K_2^{\text{AS}} &= K_2^{\overline{\text{MS}}} + \left( \frac{\beta_1^{\overline{\text{MS}}} - \beta_1^{\text{AS}}}{\beta_0} + 2b_1^{\text{AS}} \right) K_1^{\overline{\text{MS}}}, \\ K_3^{\text{AS}} &= K_3^{\overline{\text{MS}}} + \left( \frac{\beta_1^{\overline{\text{MS}}} - \beta_1^{\text{AS}}}{\beta_0} + 3b_1^{\text{AS}} \right) K_2^{\overline{\text{MS}}} + \left( 2b_2^{\text{AS}} + (b_1^{\text{AS}})^2 \right. \\ &\quad \left. + \frac{\beta_2^{\overline{\text{MS}}} - \beta_2^{\text{AS}}}{\beta_0} + \frac{(3\beta_1^{\overline{\text{MS}}} - 2\beta_1^{\text{AS}})b_1^{\text{AS}}}{\beta_0} + \frac{\beta_1^{\text{AS}}(\beta_1^{\text{AS}} - \beta_1^{\overline{\text{MS}}})}{\beta_0^2} \right) K_1^{\overline{\text{MS}}}. \end{aligned}$$

Thus, we come to the conclusion that the question of the factorization of the  $\beta$ -function reduces to the conditions of division without remainder of terms of type  $(\beta_1^{\overline{\text{MS}}} - \beta_1^{\text{AS}})/\beta_0$ ,  $(\beta_2^{\overline{\text{MS}}} - \beta_2^{\text{AS}})/\beta_0$ ,  $\beta_1^{\text{AS}}(\beta_1^{\text{AS}} - \beta_1^{\overline{\text{MS}}})/\beta_0^2$  etc. (hence we conclude that there is the factorization in QED in all popular schemes, such as MS-like, MOM and OS-schemes).

Further we find the relation of the  $\mathcal{O}(a_s^2)$  coefficients of the  $\beta$ -functions:

$$\beta_1^{\overline{\text{MS}}} - \beta_1^{\text{AS}} = \xi \gamma_0^{\overline{\text{MS}}}(\xi) \frac{\partial b_1(\xi)}{\partial \xi},$$

where  $\gamma_0^{\overline{\text{MS}}} = (-13/24 + \xi^{\overline{\text{MS}}}/8)C_A + T_F n_f/3$ . From this relation we obtain, that at  $\xi = 0$   $\beta_1^{\text{AS}} = \beta_1^{\overline{\text{MS}}}$ , and division by  $\beta_0$  is performed in the  $\mathcal{O}(a_s^3)$  approximation.

At  $\xi = -3$ :  $\gamma_0^{\overline{\text{MS}}} = -\beta_0$  and division without remainder also carried out.

Similarly, we obtain the following formulas at  $\xi = 0$ :

$$K_2^{AS} = K_2^{\overline{MS}} + 2b_1^{AS}(0)K_1^{\overline{MS}},$$

$$K_3^{AS} = K_3^{\overline{MS}} + 3b_1(0)K_2^{\overline{MS}} + 3b_2(0)K_1^{\overline{MS}},$$

$$K_4^{AS} = K_4^{\overline{MS}} + 4b_1(0)K_3^{\overline{MS}} + \left(4b_2(0) + 2b_1^2(0)\right)K_2^{\overline{MS}} + 4b_3(0)K_1^{\overline{MS}},$$

$$K_5^{AS} = K_5^{\overline{MS}} + 5b_1(0)K_4^{\overline{MS}} + 5\left(b_2(0) + b_1^2(0)\right)K_3^{\overline{MS}} \\ + 5\left(b_3(0) + b_1(0)b_2(0)\right)K_2^{\overline{MS}} + 5b_4(0)K_1^{\overline{MS}} \dots$$

Thus, we come to the conclusion that the gauge invariance of the renormalization schemes is a sufficient condition for the factorization of the RG  $\beta$ -function in the GCR, but is not a necessary condition.



- Initially, it was assumed that the true cause of the  $\beta$ -factorization lies in the gauge invariance of the renormalization schemes. In a more detailed analysis, it was unexpectedly found that a similar property is possible in gauge-dependent schemes.
- On the example of the mMOM-scheme we explained that gauges  $\xi = 0, -3$  are highlighted among the remaining values of  $\xi$  (the gauge  $\xi = -1$  is a specific of the mMOM-scheme).
- It is shown that for  $\xi = -3$  the factorization of the  $\beta$ -function takes place in all schemes with linear covariant gauge in the  $\mathcal{O}(a_s^3)$  approximation.
- The factorization in the Landau gauge occurs in all orders of PT (if such is observed in the  $\overline{\text{MS}}$ -scheme; there are theoretical grounds for believing this).
- The gauge invariance of renormalization schemes is a sufficient but not necessary condition for factorization.
- The question of the theoretical reason for the factorization of the  $\beta$ -function still remains open.

Thank you for your attention!

$$\beta_{2, \xi=0}^{AS} = \beta_2^{\overline{MS}} + \beta_1^{\overline{MS}} b_1(0) + \beta_0 \left( b_1^2(0) - b_2(0) \right) ,$$

$$\beta_{3, \xi=0}^{AS} = \beta_3^{\overline{MS}} + 2\beta_2^{\overline{MS}} b_1(0) + \beta_1^{\overline{MS}} b_1^2(0) + 2\beta_0 \left( 2b_1(0)b_2(0) - b_1^3(0) - b_3(0) \right) ,$$

$$\beta_{4, \xi=0}^{AS} = \beta_4^{\overline{MS}} + \beta_2^{\overline{MS}} \left( b_2(0) + 2b_1^2(0) \right) + \beta_1^{\overline{MS}} \left( 3b_1(0)b_2(0) - b_1^3(0) - b_3(0) \right) \\ + 3\beta_3^{\overline{MS}} b_1(0) + \beta_0 \left( 4b_1^4(0) - 11b_2(0)b_1^2(0) + 6b_3(0)b_1(0) + 4b_2^2(0) - 3b_4(0) \right) .$$