Numerical evaluation of Mellin-Barnes integrals in Minkowskian regions

In collaboration with Tord Riemann and Ievgen Dubovyk

One-loop Mellin-Barnes integrals

In collaboration with Tord Riemann

News about integral reduction program — Kira

In collaboration with Philipp Maierhöfer and Peter Uwer

—Loops and Legs 2018—

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Outline

1. Introduction
2. Construction of Mellin-Barnes integrals
3. Problems with Minkowskian regions
4. Some solutions
5. Applications
6. MBOneLoop
7. Kira News
8. Conclusions

Overview

- *The two-loop electroweak bosonic corrections to* $\sin^2\theta_{\rm eff}^{\rm b}$,
  I. Dubovyk, A. Freitas, J. Gluza, T. Riemann, and J. Usovitsch,
  Phys. Lett. **B762** (2016) 184–189, `arXiv:1607.08375 [hep-ph]`.
- *Numerical integration of massive two-loop Mellin-Barnes integrals in Minkowskian regions*,
  I. Dubovyk, J. Gluza, T. Riemann, and J. Usovitsch,
  PoS **LL2016** (2016) 034, `arXiv:1607.07538 [hep-ph]`.
- *30 years, some 700 integrals, and 1 dessert, or: Electroweak two-loop corrections to the* $Z\bar{b}b$ *vertex*,
  I. Dubovyk, A. Freitas, J. Gluza, T. Riemann, and J. Usovitsch,
  PoS **LL2016** (2016) 075, `arXiv:1610.07059 [hep-ph]`.
- *Complete electroweak two-loop corrections to Z boson production and decay*,
  I. Dubovyk, A. Freitas, J. Gluza, T. Riemann, and J. Usovitsch,
  `arXiv:1804.10236 [hep-ph]`.

Asymmetries measured at the Z pole

We study the process $e^+e^- \to (Z) \to b\bar{b}$

Pseudo-observables, unfolded at the Z peak

forward-backward asymmetry $A_{\rm FB}^{\rm b\bar{b},0} = \frac{3}{4} A_{\rm e} A_{\rm b}$

f-b left-right asymmetry $A_{\rm FB,LR}^{\rm b\bar{b},0} = \frac{3}{4} P_{\rm e} A_{\rm b}$, $P_{\rm e}$ is the electron polarization

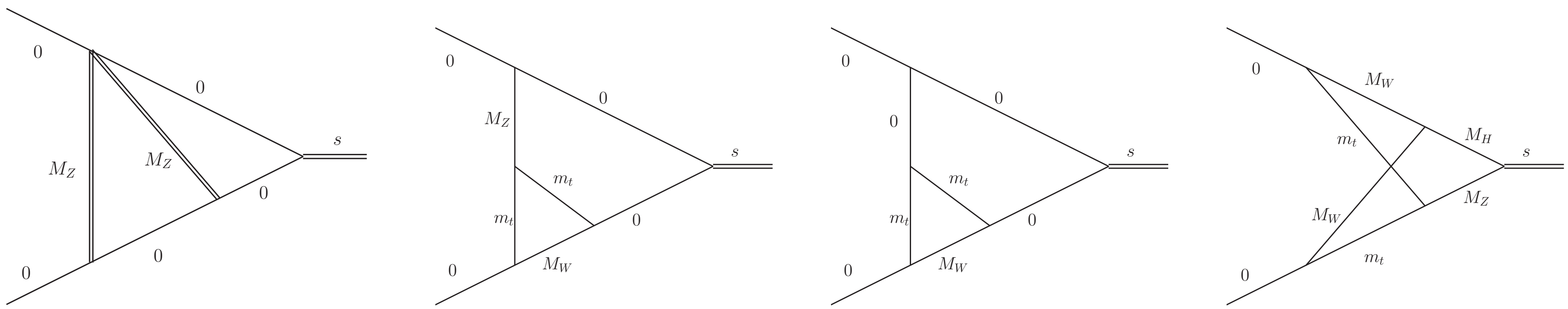
$$A_{\rm b} = \frac{2\Re e \frac{v_b}{a_b}}{1 + \left(\Re e \frac{v_b}{a_b}\right)^2} = \frac{1 - 4|Q_b| \sin^2\theta_{\rm eff}^{\rm b}}{1 - 4|Q_b| \sin^2\theta_{\rm eff}^{\rm b} + 8Q_b^2 (\sin^2\theta_{\rm eff}^{\rm b})^2} \quad (1)$$

Definition of the effective weak mixing angle

$$\sin^2\theta_{\rm eff}^{\rm b} = \frac{1}{4|Q_b|} \left(1 - \Re e \frac{v_b}{a_b}\right) \quad (2)$$

v_b and a_b are effective vector coupling and axial-vector coupling of the $Zb\bar{b}$ vertex (contain the electroweak corrections)

Samples of Feynman integrals for the $Z\bar{b}b$ vertex



- We project all Feynman integrals to scalar integrals
- No IBP-reduction is involved
- We need to compute all Feynman integrals only up to the finite order in $\epsilon = (4 - D)/2$
- We can compute all two loop Feynman integrals appearing in the $Z\bar{b}b$ vertex either with sector decomposition or Mellin-Barnes integral approach numerically with eight significant digits of accuracy in physical kinematic regions

Samples of Feynman integrals for the $Z\bar{b}b$ vertex

- The integrals contain up to three dimensionless parameters

$$\left\{ \frac{M_H^2}{M_Z^2}, \frac{M_W^2}{M_Z^2}, \frac{m_t^2}{M_Z^2}, \frac{(s+i\delta)}{M_Z^2} \right\} \Big|_{s=M_Z^2} \tag{3}$$

- Many of them contain ultraviolet and infrared singularities, even though the divergences cancel in the final result
- In general, it is not possible to compute all integrals analytically with available methods and tools, but instead one has to resort to numerical integration strategies
- The Feynman prescription $i\delta$ can be savely attached to s

Numerical Methods

Sector decomposition

FIESTA 3 [A.V.Smirnov, 2014], FIESTA 4 [A.V.Smirnov, 2015], SecDec 3 [Borowka, et. al., 2015] and pySecDec [Borowka, et. al., 2017]

Mellin-Barnes integral approach

- With AMBRE 2 [Gluza, et. al., 2011] (AMBRE 3 [Dubovyk, et. al., 2015]) we derived Mellin-Barnes representations for planar (non-planar) topologies. One may use PlanarityTest [Bielas, et. al, 2013] for automatic identification.
- Expansion in terms of $\epsilon = (4 - D)/2$ is done with MB [Czakon, 2006], MBresolve [A. Smirnov, V. Smirnov, 2009], barnesroutines (D. Kosower).
- For the numerical treatment of massive Mellin-Barnes integrals with Minkowskian kinematics, the package MBnumerics [Dubovyk, Riemann, Usovitsch] is being developed since 2015.

Feynman integral

Loop integral with loop momenta k_i and propagator internal momenta q:

$$G_L[T(k)] = \frac{1}{(i\pi^{D/2})^L} \int \frac{\mathrm{d}^D k_1 \ldots \mathrm{d}^D k_L \;\; T(k)}{(q_1^2 - m_1^2)^{\nu_1} \ldots (q_i^2 - m_i^2)^{\nu_j} \ldots (q_N^2 - m_N^2)^{\nu_N}} \tag{4}$$

Introduce Feynman parameter integrals:

$$= \frac{(-1)^\nu \Gamma\left(\nu - \frac{DL}{2}\right)}{\prod_{i=1}^{N} \Gamma(\nu_i)} \int_0^1 \prod_{j=1}^{N} \mathrm{d}x_j \; x_j^{\nu_j - 1} \delta(1-h) \frac{U(x)^{\nu - \frac{D(L+1)}{2}}}{F(x)^{\nu - \frac{DL}{2}}} R_L(x)[T] \tag{5}$$

- T(k) is a non-trivial numerator
- $h = \sum_{i=1}^{N} x_i$, $\nu = \sum_{i=1} \nu_j$
- $R_L(x)[T]$ is rational in the Feynman parameters
- $U(x)$ and $F(x)$ are Symanzik polynomials of degree L and $L+1$
- Everything is general in dimension D and the propagator exponents ν_i

Construction of Mellin-Barnes integrals with AMBRE

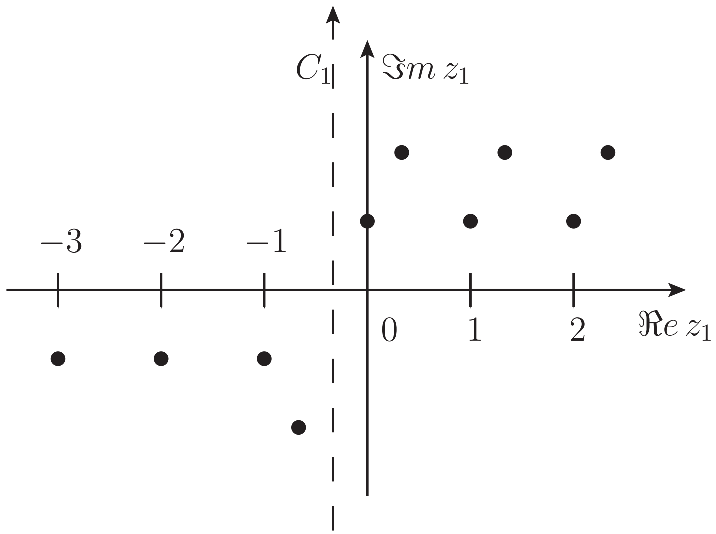
Mellin-Barnes integral master formula

$$\frac{1}{(A+B)^\nu} = \int\limits_{-i\infty}^{i\infty} \frac{dz}{2\pi i} \frac{A^z B^{-z-\nu}\Gamma(-z)\Gamma(\nu+z)}{\Gamma(\nu)}, \quad |\arg A - \arg B| < \pi \tag{6}$$

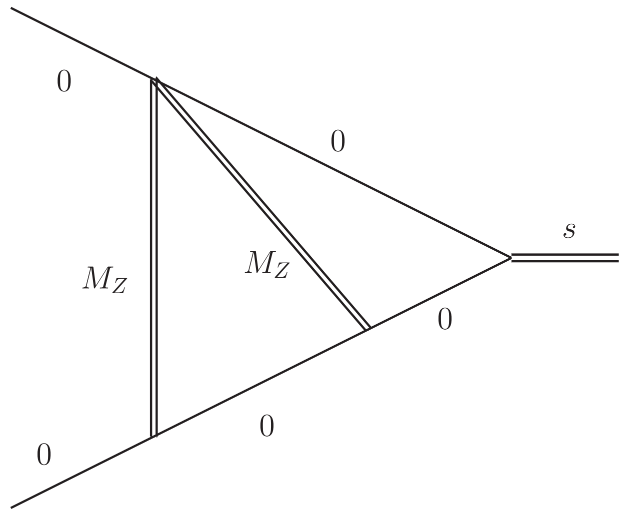
Integral representation of the Euler's Beta-function

$$B(x,y) = \int\limits_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Re e\, x > 0, \; \Re e\, y > 0 \tag{7}$$

Satisfy all relations with a straight contour in the complex plane $z_i = x_i + it_i$, where x_i is a constant generated with MB.m or MBresolve.m $t_i \in [-\infty, +\infty]$ [Bas Tausk,1999]



Example



$$I_{\mathsf{0h0w14r}} = \int \frac{\mathrm{d}^D k_1}{i\pi^{D/2}} \frac{\mathrm{d}^D k_2}{i\pi^{D/2}} \quad \frac{k_1 p_1}{k_1^2((k_1-k_2)^2-M_Z^2)k_2^2((k_2+p_1)^2-M_Z^2)(k_1+p_1+p_2)^2}$$

$$= \int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} \frac{\mathrm{d}z_1}{2\pi i} \int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} \frac{\mathrm{d}z_2}{2\pi i} \frac{\Gamma(-z_1)\Gamma(-z_2)\Gamma(z_2+1)\Gamma(-\epsilon-z_1)\Gamma(2\epsilon+z_1+1)\left(-\frac{M_Z^2}{s}\right)^{z_1}}{2\Gamma(1-z_2)\Gamma(-3\epsilon-z_1+2)\Gamma(-2\epsilon-2z_1-z_2)}$$

$$\times (-s)^{-2\epsilon}\,\Gamma(-2\epsilon - z_1 - z_2)^2\Gamma(-\epsilon - z_1 - z_2)\Gamma(\epsilon + z_1 + z_2 + 1)$$

Expansion in ϵ

$$
-\int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty}\frac{\mathrm{d}z_1}{2\pi i}\int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty}\frac{\mathrm{d}z_2}{2\pi i}\,\frac{\Gamma(-z_1)^2\Gamma(z_1+1)\Gamma(-z_2)\Gamma(z_2+1)\left(-\frac{M_Z^2}{s}\right)^{z_1}\Gamma(-z_1-z_2)^3}{2\Gamma(2-z_1)\Gamma(1-z_2)\Gamma(-2z_1-z_2)}
$$

$$
\times\Gamma(z_1+z_2+1)+\mathcal{O}(\epsilon)
$$

- Minkowskian regions: $s > 0$
- Euclidean regions: $s < 0$
- With MBnumerics we prepare the coefficients in the ϵ expansion for the numerical evaluation in the Minkowskian regions

Asymptotic behavior in Minkowskian regions

$$\lim_{|z|\to\infty} \frac{\Gamma(z)}{z^{z-1/2}e^{-z}} = \sqrt{2\pi}, \quad |\arg z| < \pi \tag{8}$$

Thus the following approximation is valid:

$$\Gamma(z) \underset{|z|\to\infty}{\approx} z^{z-1/2}e^{-z}\sqrt{2\pi}, \quad |\arg z| < \pi \tag{9}$$

For our example integrand: $z_1 = -\frac{1}{3} + it_1$ and $z_2 = -\frac{1}{3} + it_2$
$t_1 \to -t$ and $t_2 \to t$

$$\mathcal{I}_{\mathsf{0h0w14r}} \underset{t\to\infty}{\approx} t^{-2+2x_1+2x_2} \tag{10}$$

everywhere else exponential damping factor.

Linear transformation

$$z_2 \to z_2 - z_1 \tag{11}$$

$$I_{\text{0h0w14}} = -\int\limits_{-\frac{1}{3}-i\infty}^{-\frac{1}{3}+i\infty} \frac{\mathrm{d}z_1}{2\pi i} \int\limits_{-\frac{2}{3}-i\infty}^{-\frac{2}{3}+i\infty} \frac{\mathrm{d}z_2}{2\pi i} \frac{(-\frac{M_Z^2}{s})^{z_1}\Gamma(-z_1)^2\Gamma(1+z_1)\Gamma(z_1-z_2)\Gamma(-z_2)^3}{2\Gamma(2-z_1)\Gamma(-z_1-z_2)\Gamma(1+z_1-z_2)} \times\Gamma(1+z_2)\Gamma(1-z_1+z_2) \tag{12}$$

$z_1 = -\frac{1}{3} + it_1$ and $z_2 = -\frac{2}{3} + it_2$
$t_1 \to -t$ and $t_2 \to 0$

$$\mathcal{I}_{\text{0h0w14}} \underset{t\to\infty}{\approx} t^{-2+2x_2} \tag{13}$$

Numerical integration is more stable

Mappings of integration regions to finite intervals

- Logarithmic mapping $t = \ln[\frac{d}{1-d}]$ yields:

$$\frac{1}{t^a} = \frac{1}{\ln[\frac{d}{1-d}]^a}, \text{ Jacobian: } \frac{1}{d(1-d)} \tag{14}$$

$$\lim_{d\to 0, d\to 1} \frac{1}{d(1-d)} \frac{1}{\ln(\frac{d}{1-d})^a} = \frac{1}{0}, \ \forall\, a. \tag{15}$$

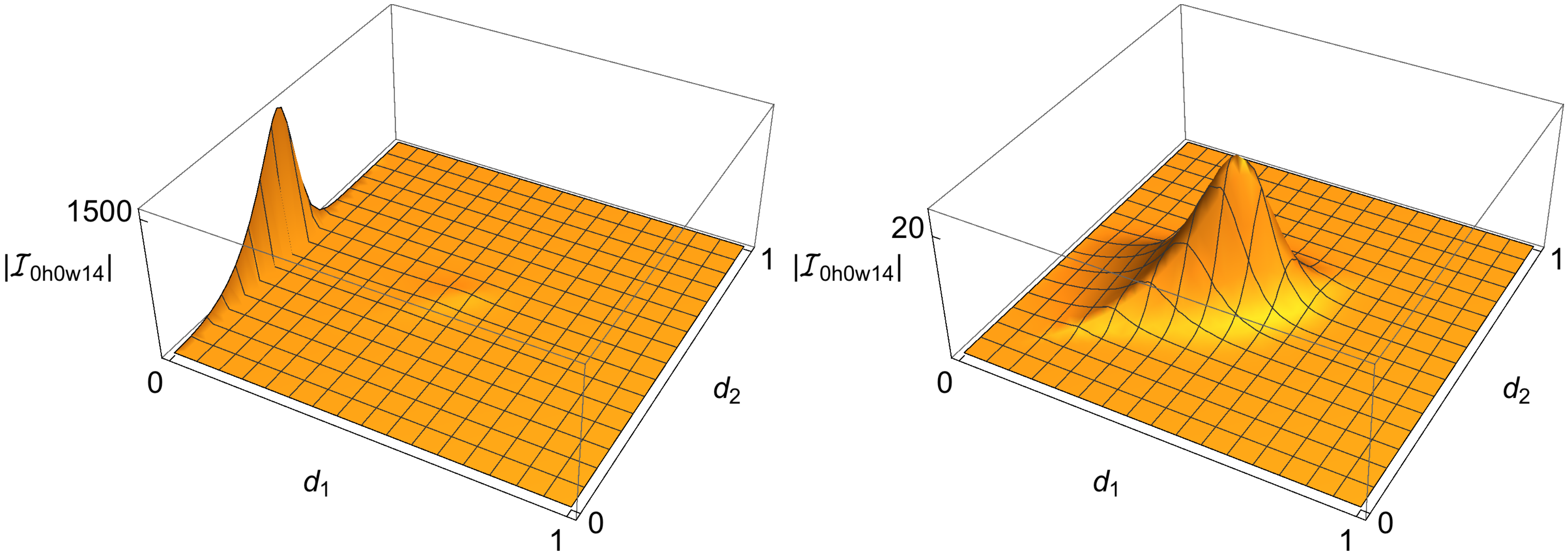
- Cotangent mapping $t = \frac{1}{\tan(-\pi d)}$ yields (thanks to Ievgen Dubovyk):

$$\frac{1}{t^a} = \tan(-\pi d)^a, \text{ Jacobian: } \frac{\pi}{\sin(\pi d)^2} \tag{16}$$

$$\lim_{d\to 0, d\to 1} \frac{\pi \tan(-\pi d)^a}{\sin(\pi d)^2} = \begin{cases} \frac{1}{0}, & a < 2, \\ \pi, & a = 2, \\ 0, & a > 2, \end{cases} \tag{17}$$

$$\prod_i \Gamma_i \to \exp\left(\sum_i \log \Gamma_i\right) \tag{18}$$

Example: Mappings of integration intervals to finite intervals



$$\mathcal{I}_{\mathrm{0h0w14}} = -\frac{(-\frac{M_Z^2}{s})^{z_1}\Gamma(-z_1)^2\Gamma(1+z_1)\Gamma(z_1-z_2)\Gamma(-z_2)^3}{2\Gamma(2-z_1)\Gamma(-z_1-z_2)\Gamma(1+z_1-z_2)} \times\Gamma(1+z_2)\Gamma(1-z_1+z_2) \quad (19)$$

Left figure logarithmic mapping, right figure cotangent mapping

Shifts may improve asymptotic behavior

$$z_i = x_i + it_i + n_i \tag{20}$$

x_i is a constant generated with MB.m or MBresolve.m
$t_i \in [-\infty, \infty]$

$$n_i \in \mathbb{R} \text{ is a shift} \tag{21}$$

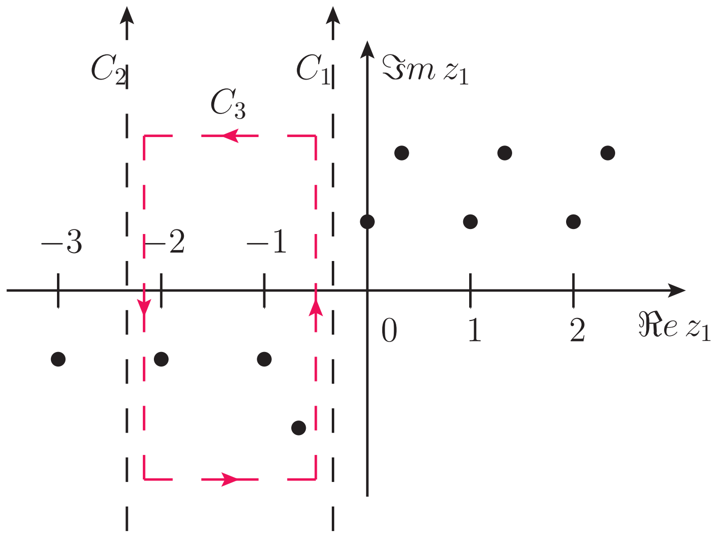
First observation: Shifts may change the asymptotic behavior:

$$\mathcal{I}_{\text{0h0w14r}} \underset{t\to\infty}{\approx} t^{-2+2x_1+2x_2+2n_1+2n_2} \tag{22}$$

Shifts may change order of magnitude

$$z_i = x_i + it_i + n_i \tag{23}$$

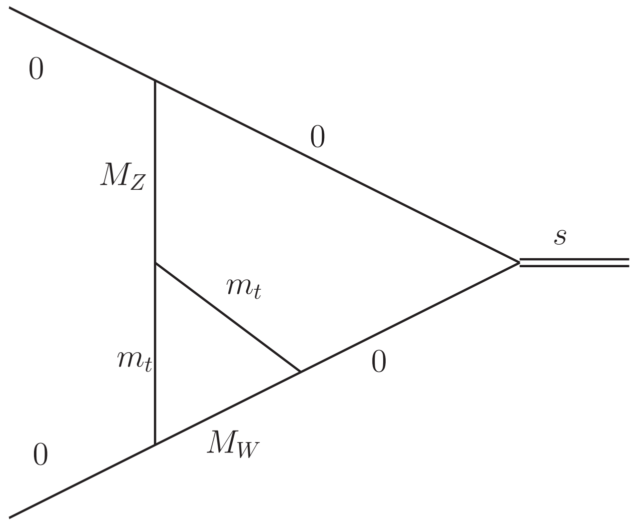
Second observation: Integrals with the shifted contour may have a value of lower order of magnitude than the original integral



$$\int_{C_1} \mathcal{I}_{\texttt{0h0w14}} \mathrm{d}z_1 \mathrm{d}z_2 = \int_{C_2} \mathcal{I}_{\texttt{0h0w14}} \mathrm{d}z_1 \mathrm{d}z_2 + \int_{C_3} \mathcal{I}_{\texttt{0h0w14}} \mathrm{d}z_1 \mathrm{d}z_2 \tag{24}$$

- Original integral at C_1: $(3.92382858885\boldsymbol{7} + 7.45638853661\boldsymbol{3}i) * 10^{-1}$
- Shifted integral at C_2 with $n_1 = -2$: $-9.74965823202\boldsymbol{2} * 10^{-3}$
- Three 1-dim integrals due to C_3: $0.40213251711780\boldsymbol{7} + 0.74563885366131\boldsymbol{8}i$

0hxwxtxz96 [MB - 4 dim] [SD - 5 dim]

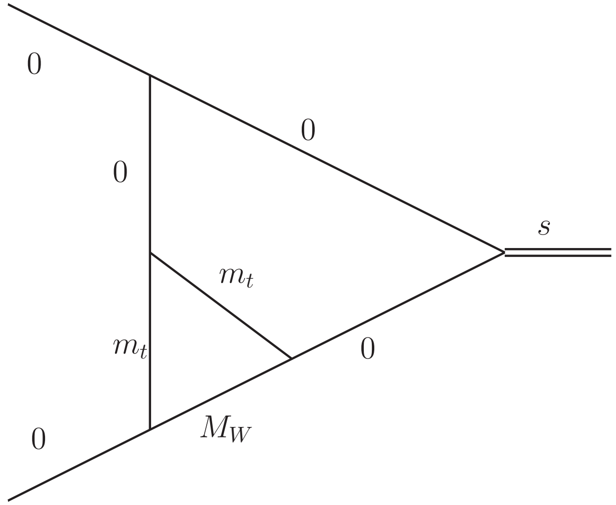


$$\int \frac{\mathrm{d}^D k_1}{i\pi^{D/2}} \frac{\mathrm{d}^D k_2}{i\pi^{D/2}} \quad \frac{\exp(2\epsilon\gamma_E)(k_2 p_2)}{(k_1)^2((k_1-k_2)^2-m_t^2)((k_2)^2-M_W^2)((k_1+p_1)^2-M_Z^2)}$$

$$\times \frac{1}{((k_2+p_1)^2-m_t^2)(k_1+p_1+p_2)^2}$$

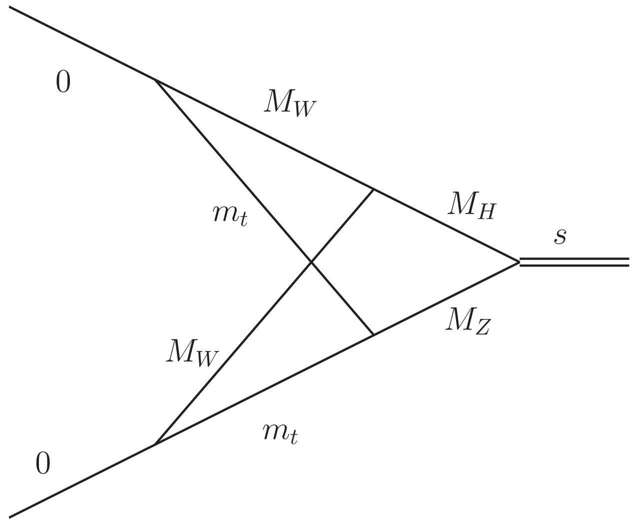
MB	(-0.0348955***3***	-0.1010186***6*** $i) + \mathcal{O}(\epsilon)$
SD - 540 Mio	(-0.034895529***3***	-0.101018681***3*** $i) + \mathcal{O}(\epsilon)$
SD - 90 Mio	(-0.0348954***2***	-0.1010188***2*** $i) + \mathcal{O}(\epsilon)$
SD - 15 Mio	(-0.034890***5***	-0.101025***3*** $i) + \mathcal{O}(\epsilon)$

soft7 ϵ^0:[MB - 3 dim] [SD - 5 dim], ϵ^{-1}:[MB - 2 dim] [SD - 4 dim], ϵ^{-2}:[MB - 1 dim] [SD - 3 dim]



MB	$0.06026648655769999\,\epsilon^{-2}$	
SD - 90 Mio	$0.06026648655\,\epsilon^{-2}$	
MB	$(-0.0315124890$	$+0.1893327514i)\,\epsilon^{-1}$
SD - 90 Mio	$(-0.031512481$	$+0.189332716i)\,\epsilon^{-1}$
MB	$(-0.22823186751$	$-0.08824794569i) + \mathcal{O}(\epsilon)$
SD - 90 Mio	$(-0.2282265$	$-0.0882459i) + \mathcal{O}(\epsilon)$
SD - 15 Mio	$(-0.22816$	$-0.08820i) + \mathcal{O}(\epsilon)$

xhxwxz63 [MB - 8 dim] [SD - 5 dim]

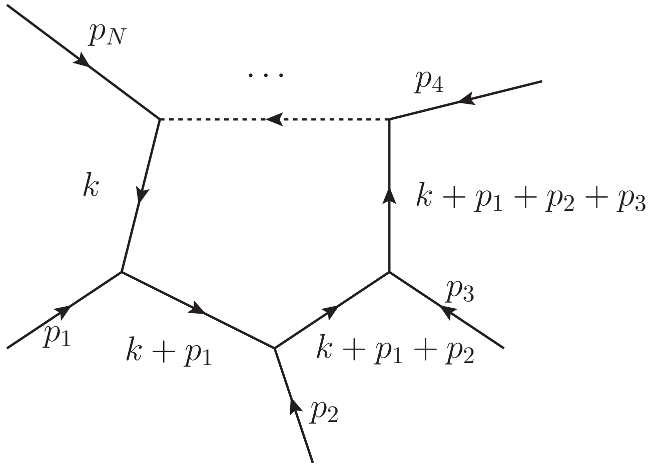


$$\int \frac{\mathrm{d}^D k_1}{i\pi^{D/2}} \frac{\mathrm{d}^D k_2}{i\pi^{D/2}} \quad \frac{\exp(2\epsilon\gamma_E)(k_1 p_1)^2}{(k_1^2-M_Z^2)((k_1-k_2)^2-m_t^2)(k_2^2-m_t^2)((k_1-k_2+p_1)^2-M_W^2)}$$

$$\times \frac{1}{((k_2+p_2)^2-M_W^2)((k_1+p_1+p_2)^2-M_H^2)}$$

MB	(0.0029 ± 0.0008)
SD - 15 Mio	$(0.0031338336\boldsymbol{3}) + \mathcal{O}(\epsilon)$

MBOneLoop



$$I_N(D,\{p_i^2,s_{ij},s_{ijk},\dots,m_i^2\}) = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}} \frac{1}{(q_1^2-m_1^2)\dots(q_N^2-m_N^2)} \quad (25)$$

where

$q_i = k + \sum_{n=1}^{i} p_n$, $s_{ij} = (p_i+p_j)^2$, $s_{ijk} = (p_i+p_j+p_k)^2$

$r_N = -\frac{\lambda_N}{G_N}$, G_N is the gram determinant and λ_N is the Cayley determinant

MBOneLoop

[J. Bluemlein, K. Phan, T. Riemann, 2017]

$$I_N(D,\{p_i^2,s_{ij},s_{ijk},\ldots,m_i^2\}) = -\int_{-i\infty}^{i\infty}\frac{\mathrm{d}z}{2\pi i}\frac{\Gamma[-z]\Gamma[\frac{D-N+1}{2}+z]\Gamma[z+1]}{2\Gamma[\frac{D-N+1}{2}]}\left(\frac{1}{r_N}\right)^z \times\sum_{k=1}^{N}\left(\frac{1}{r_N}\frac{\partial r_N}{\partial m_k^2}\right)\mathbf{k}^- I_N(D+2z,\{p_i^2,s_{ij},s_{ijk},\ldots,m_i^2\}) \quad (26)$$

$\mathbf{k}^- I_N = \int \frac{\mathrm{d}^D k}{i\pi^{D/2}}\frac{1}{\prod\limits_{j\neq k,j=1}^{N}(q_j^2-m_j^2)}$, where the k's propagator is shrinked

- The integrands are exponentially damped for any kinematical regions.
- The Mellin-Barnes integrals are at most $N-1$ dimensional.
- See Tords talk "Scalar one-loop Feynman integrals in arbitrary space-time dimension" on Friday

MBOneLoop

$$J_4(D,n_1,n_2,n_3,n_4) = I_4(D,p_1^2,p_2^2,p_3^2,p_4^2,s,t,m_1^2,m_2^2,m_3^2,m_4^2)[n_1,n_2,n_3,n_4]$$

$p_1^2 = p_2^2 = m_1^2 = m_3^2 = m_4^2 = 0$, $p_3^2 = 10000$, $p_4^2 = -60000(1+x)$, $s = (p_1+p_2)^2 = -40000$, $t = (p_2+p_3)^2 = 20000$, $m_2^2 = (911876/10000)^2$

x	value for $J_4(12-2\epsilon,1,5,1,1)4!$	
10^{-8}	$(2.05969289342 + 1.55594909187i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-8}	$(2.05969289363 + 1.55594909187i)10^{-10}$	MBOneLoop + Kira + MBnumerics
10^{-4}	$(2.05965609497 + 1.55585605343i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-4}	$(2.05965609489 + 1.55585605343i)10^{-10}$	MBOneLoop + Kira + MBnumerics

x	value for $J_4(10-2\epsilon,1,4,1,1)3!$	
10^{-8}	$-(3.15407250057 + 3.31837790700i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-8}	$-(3.15407250055 + 3.31837790700i)10^{-10}$	MBOneLoop + Kira + MBnumerics
10^{-4}	$-(3.15403282194 + 3.31818461838i)10^{-10}$	[J. Fleischer, T. Riemann, 2010]
10^{-4}	$-(3.15403282195 + 3.31818461838i)10^{-10}$	MBOneLoop + Kira + MBnumerics

Kira version 1.1 - News

Kira is an implementation of the Laporta algorithm

Get Kira and the release notes (26.04.2018) at:
https://www.physik.hu-berlin.de/de/pep/tools

- Anchor points — reduction can be stopped and resumed at any time during the back substitution
- Performance issues resolved (memory allocation improved)
- Number of Fermat workers can now be freely chosen
- Cut propagators are introduced
- Start a reduction with a preferred list of master integrals
- Focus the reduction only to a subset of master integrals — set all other coefficients to zero

Kira bench mark

`Kira` is an implementation of the Laporta algorithm

Get `Kira` and the release notes (26.04.2018) at:
https://www.physik.hu-berlin.de/de/pep/tools

NNLO pp$\rightarrow$ Hj — IBP reduction

- Full mass dependence
- 11 topologies, of which 3 are nonplanar
- Reduction was done for the amplitude and the DEQs at the same time
- Run time 3 weeks on a single computer node (Dual Intel E5-2630, 2x10 cores 2.2 GHz)

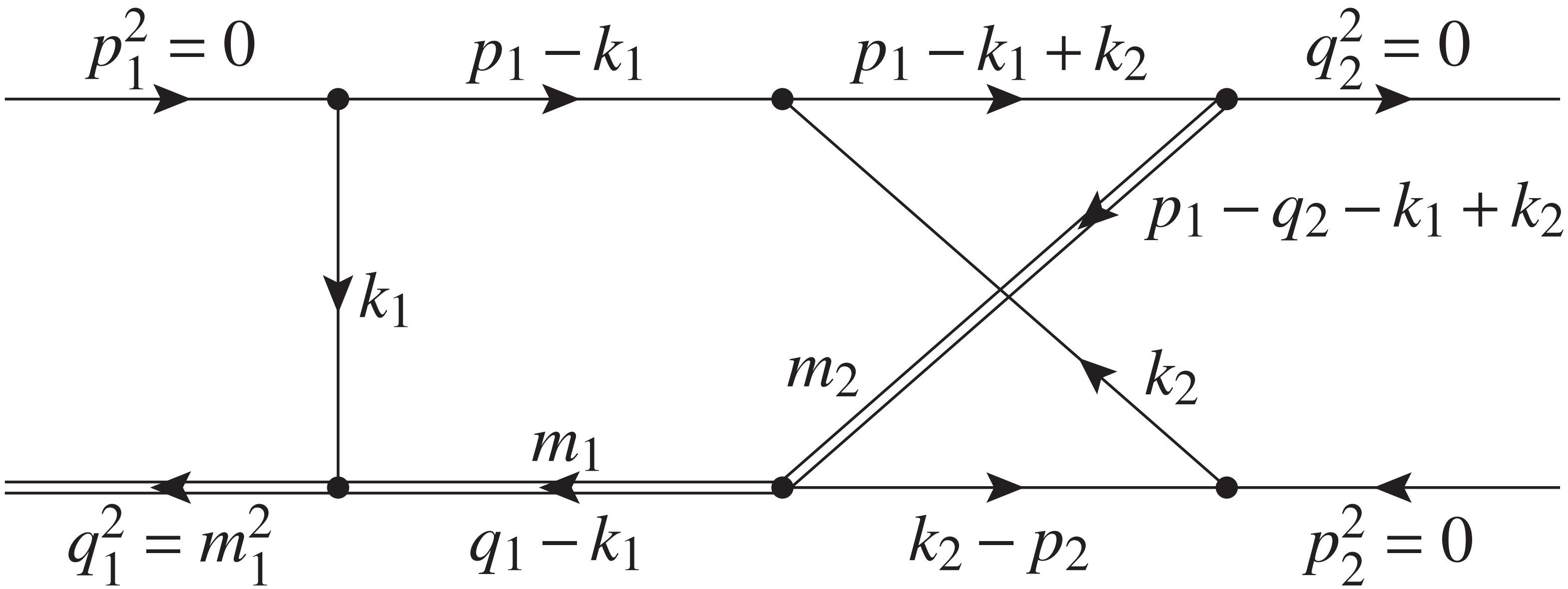
Conclusions and Outlook

- The main challenge so far in the electroweak Z-boson resonance physics was the calculation of massive two-loop vertex Feynman diagrams
- Completion of the electroweak two-loop corrections to Z boson production and decay
- We used automatized tools AMBRE 3 and MBnumerics for the evaluation of the Mellin-Barnes integrals in Minkowskian kinematics (which were developed for this project) and used together with sector decomposition programs SecDec 3 and Fiesta 3
- One can derive Mellin-Barnes integrals for one-loop integrals, which are at most $N-1$ dimensional and very good convergent in any kinemtical region.
- The new Kira (IBP-reduction) version provides new important features and improvements

Outlook

- We start our focus now to the leading unknown three loop contributions to the Z boson production and decay
- Generalization of MBOneLoop to MBTwoLoop
- Kira - new equation generator is upcoming, where the run time is improved by $\sim 10^L$, L is the number of loops
- Interface for feeding Kira an arbitrary linear system of equations

Kira performance topo4



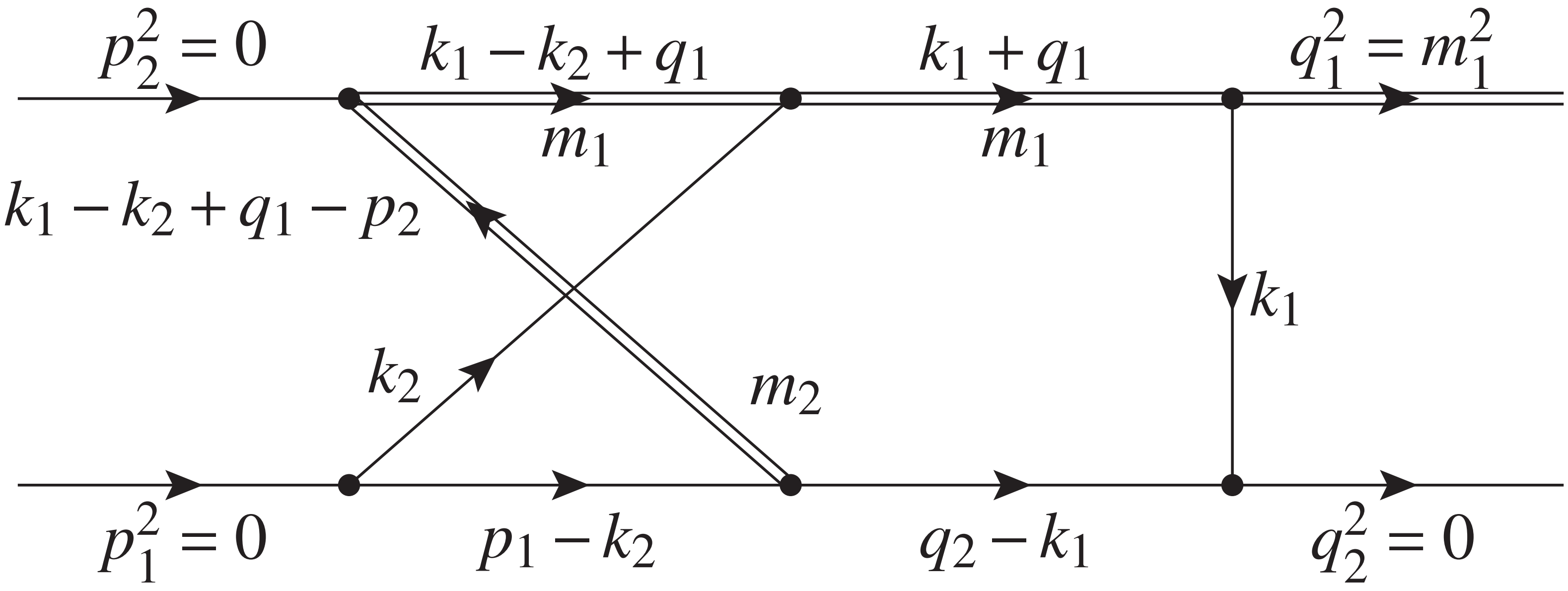
topo4 is a non planar double box with two massive propagators and one massive external momentum. Momentum conservation reads $q_1 = p_1 + p_2 - q_2$.

Kira performance topo4

Type	$T_{\texttt{pyRed}}$	$T_{\texttt{Kira}}$	$T_{\texttt{Reduze 2}}$	$T_{\texttt{FIRE 5}}$	$\frac{T_{\texttt{pyRed}}}{T_{\texttt{Kira}}}$	$\frac{T_{\texttt{Reduze 2}}}{T_{\texttt{Kira}}}$	$\frac{T_{\texttt{FIRE 5}}}{T_{\texttt{Kira}}}$
default	67 s	2.4 h	2.7 d	23.5 h	0.007	26	9.8
A	70.2 s	35.3 min	-	22.4 h	0.03	-	38
B	69.5 s	24 min	-	22.4 h	0.05	-	56

- $s_{\max} = 4$ is the sum of negative propagator exponents and $r_{\max}$ is the sum of positve propagator exponents.
- Reduce all Feynamn integrals for up to $s_{\max} = 4$ and $r_{\max} = 7$.
- A: Select and reduce all Feynman integrals with up to 4 scalar products and zero additional dots.
- B: Select and reduce all Feynman integrals in the top level sector 127 involving 7 propagators (with exactly 4 scalar products and zero additional dots).
- A mass ratio of is used $m_2^2 = \frac{3}{14} m_1^2$.
- Reduze 2 A. von Manteuffel and C. Studerus (2012), FIRE 5 A. V. Smirnov (2014)

Kira performance topo5



topo5 is a non planar double box with three massive propagators and one massive external momentum. The momentum conservation reads $q_1 = p_1 + p_2 - q_2$.

Kira performance topo5

Topology	r_{max}	s_{max}	$T_{\texttt{pyRed}}$	$T_{\texttt{Kira}}$	$\frac{T_{\texttt{pyRed}}}{T_{\texttt{Kira}}}$
topo4	8	3	41 s	14 h	0.0008
	7	4	130 s	10 h	0.003
topo5	8	2	94 s	3 d	0.0003
	8	3	125 s	8 d	0.0002
	7	4	237 s	7 d	0.0004

- r_{max} is the sum of positive propagator exponents.
- We keep full mass dependence.

Numerical integration in MB.m and MBnumerics.m

$$z_i^{\mathrm{L}} = x_i + i \, \ln\left(\frac{t_i}{1-t_i}\right), \; t_i \in (0,1), \text{ Jacobians: } J_i^{\mathrm{L}} = \frac{i}{t_i(1-t_i)} \tag{27}$$

$$I^{\mathrm{MB.m}} = \frac{1}{(2\pi i)^2} \int_0^1 \int_0^1 J_1^{\mathrm{L}} J_2^{\mathrm{L}} \mathcal{I}\left(z_1^{\mathrm{L}}, z_2^{\mathrm{L}}\right) \mathrm{d}t_1 \mathrm{d}t_2 \tag{28}$$

$$z_i^{\mathrm{T}} = x_i + n_i + \frac{(i+\theta)}{\tan(-\pi \, t_i)}, \; t_i \in (0,1), \text{ Jacobians: } J_i^{\mathrm{T}} = \frac{(i+\theta)\pi}{\sin^2(\pi \, t_i)} \tag{29}$$

$$I^{\mathrm{MBnumerics.m}}(n_1, n_2) = \frac{1}{(2\pi i)^2} \int_0^1 \int_0^1 J_1^{\mathrm{T}} J_2^{\mathrm{T}} \hat{\mathcal{I}}\left(z_1^{\mathrm{T}}, z_2^{\mathrm{T}}\right) \mathrm{d}t_1 \mathrm{d}t_2 \tag{30}$$

- $n_i \in \mathrm{Integers}$, shifts [Anastasiou, Daleo, 2006]
- $\theta \in \mathrm{Reals}$, rotations [Freitas, Huang, 2010]
- tangent mapping imposes $\mathcal{I}[\prod_i \Gamma_i] \to \hat{\mathcal{I}}[e^{\sum_i \ln(\Gamma_i)}]$
- $I^{\mathrm{MBnumerics.m}}(n_1, n_2)$ is now a discrete function in n_i

Pole scheme, accuracy calculations

- The ill defined Z-propagator is replaced by the Breit-Wigner function

$$\frac{1}{s - M_Z^2 + i\varepsilon}\Big|_{s=M_Z^2} \to \frac{1}{s - \overline{M}_Z^2 + i\overline{M}_Z\overline{\Gamma}_Z}\Big|_{s=M_Z^2} \tag{31}$$

- The width $\overline{\Gamma}_Z$ is now a prediction of the theory and mixes the diagrams
- In the pole scheme, near the Z pole, the amplitude is written as

$$\mathcal{A}^{e^+e^-\to b\bar{b}} = \frac{R}{s - s_0} + S + (s - s_0)S' + \dots, \quad s_0 = \overline{M}_Z^2 - i\overline{M}_Z\overline{\Gamma}_Z \tag{32}$$

- Due to the analyticity of the S-matrix, $R, S, S', \dots, s_0$ are individually gauge-invariant and UV-finite. IR-finite, when soft and collinear real photon emission is added. [Willenbrock, Valencia,1991] [Sirlin,1991] [Stuart,1991] [Riemann, 1991, 1992] [H. Veltman,1994] [Passera, Sirlin, 1998] [Gambino, Grassi, 2000] [Bohm, Harshman, 2000].
- Experiment: $\mathcal{A} \propto \frac{1}{s - M_Z^2 + is\Gamma_Z/M_Z}$
  $\overline{M}_Z^2 = M_Z^2/(1 + \Gamma_Z^2/M_Z^2)$, $\overline{M}_Z \approx M_Z - 34$ MeV.
  $\overline{\Gamma}_Z^2 = \Gamma_Z^2/(1 + \Gamma_Z^2/M_Z^2)$, $\overline{\Gamma}_Z \approx \Gamma_Z - 1$ MeV [Bardin, Leike, Riemann, Sachwitz, 1988]

Breit-Wigner parametrization of the Z line shape

- The first term in $\mathcal{A}^{e^+e^-\to b\bar{b}}$ corresponds to a Breit-Wigner parametrization of the Z line shape with a constant decay width.
- Experimentally, however, the gauge-boson mass is determined based on a Breit-Wigner function with a running (energy-dependent) width, $\mathcal{A} \propto \frac{1}{s-M_Z^2+is\Gamma_Z/M_Z}$
- As a consequence of these different parametrizations, there is a shift between the experimental mass parameter, M_Z, and the mass parameter of the pole scheme, $\overline{M}_Z$ [Bardin, Leike, Riemann, Sachwitz, 1988], $\overline{M}_Z^2 = M_Z^2/(1+\Gamma_Z^2/M_Z^2)$, amounting to $\overline{M}_Z \approx M_Z - 34\ \mathrm{MeV}$. $\overline{\Gamma}_Z^2 = \Gamma_Z^2/(1+\Gamma_Z^2/M_Z^2)$, amounting to $\overline{\Gamma}_Z \approx \Gamma_Z - 1\ \mathrm{MeV}$.

Higher order expansion in pole scheme

- Near the Z pole α, $\overline{\Gamma}_Z$ and $(s-s_0)$ are of the same order
- For a next-to-next-to-leading order calculation R needs to be determined to $\mathcal{O}(\alpha^2)$, S only to $\mathcal{O}(\alpha)$ and S' at tree level.
- The residue R in $\mathcal{A}^{[e^+e^-\to b\bar{b}]}$ factorizes into initial- and final state vertex form factors, $V_\mu^{Ze^+e^-}$ and $V_\mu^{Zb\bar{b}}$, and Z-propagator corrections, $R_Z^{\mu\nu}$.

$$\sin^2\theta_{\mathrm{eff}}^{\mathrm{b}} = \left(1-\frac{\overline{M}_W^2}{\overline{M}_Z^2}\right)\Re e\left(1+\frac{\hat{a}_b^{(1)}v_b^{(0)}-\hat{v}_b^{(1)}a_b^{(0)}}{a_b^{(0)}\left(a_b^{(0)}-v_b^{(0)}\right)}\Bigg|_{s=M_Z^2} + \frac{\hat{a}_b^{(2)}v_b^{(0)}a_b^{(0)}-\hat{v}_b^{(2)}\left(a_b^{(0)}\right)^2-\left(\hat{a}_b^{(1)}\right)^2v_b^{(0)}+\hat{a}_b^{(1)}\hat{v}_b^{(1)}a_b^{(0)}}{\left(a_b^{(0)}\right)^2\left(a_b^{(0)}-v_b^{(0)}\right)}\Bigg|_{s=M_Z^2}\right) \tag{33}$$

- In $\hat{v}_b^{(2)}$ and $\hat{a}_b^{(2)}$ the numbers in curvy brackets correspond to the loop order and the hats donate the effect of $\gamma - Z$ mixing

Vertex form factor

- The $\sin^2\theta^{\mathrm{b}}_{\mathrm{eff}}$ is contained in the residue R in $\mathcal{A}^{[e^+e^-\to b\bar{b}]}$
- The Residue R of $\mathcal{A}^{[e^+e^-\to b\bar{b}]}$ factorizes into initial- and final state vertex form factors and Z-propagator corrections
- Calculation of A_{b} rests on the calculation of the vertex form factor

$$V_\mu^{Zb\bar{b}} = \gamma_\mu[\hat{v}_b(s) - \hat{a}_b(s)\gamma_5] \tag{34}$$

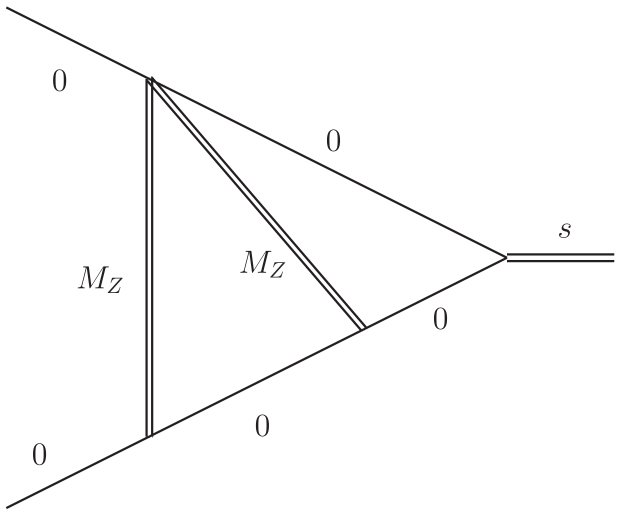
- The effective vector and axial-vector components can be projected via

$$\hat{v}_b(s) = \frac{1}{2(2-D)s}\mathrm{Tr}[\gamma^\mu \not{p}_1 V_\mu^{Zb\bar{b}} \not{p}_2] \tag{35}$$

$$\hat{a}_b(s) = \frac{1}{2(2-D)s}\mathrm{Tr}[\gamma_5\gamma^\mu \not{p}_1 V_\mu^{Zb\bar{b}} \not{p}_2] \tag{36}$$

- $D = 4 - 2\epsilon$ is the space-time dimension
- $p_{1,2}$ are the momenta of the external b-quarks and $s = (p_1 + p_2)^2$
- The hat in $\hat{v}_b(s)$ and $\hat{a}_b(s)$ donates the $Z - \gamma$ mixing

Mellin-Barnes Coefficients in the ϵ expansion, example 1



- Known analytic result for Master integrals in [Aglietti, Bonciani,2004]
- Mellin-Barnes integral coefficient in the ϵ expansion:

$$\mathcal{I}(z_1, z_2) = \frac{-(-\frac{s}{M_Z^2})^{1+z_1}\Gamma[-z_1]\Gamma[1+z_1]^2\Gamma[1+z_1-z_2]\Gamma[-z_2]\Gamma[1+z_2]^3\Gamma[-z_1+z_2]}{2\Gamma[3+z_1]\Gamma[1-z_1+z_2]\Gamma[2+z_1+z_2]} \tag{37}$$

- Mellin-Barnes integration variables $z_i = x_i + i\,t_i$, where the x_i are fixed and $t_i \in (-\infty, +\infty)$
- Here $x_1 = -\frac{2}{3}$, $x_2 = -\frac{1}{3}$