Determining symmetries of multi-Higgs potentials

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bSM model building

Why caring about discrete symmetry groups?

- The SM has many weak points: does not describe DM and baryogenesis, cannot explan the origin of fermion masses, mixing, *CP*-violation. In particular, the Higgs sector of the SM is overstretched and does not help with these issues.
- Constructions beyond the SM (bSM) based on several new fields, in particular, on extended scalar sectors, offer natural solutions (to some of them), see Ishimori et al, 1002.0211; Altarelli, Feruglio, 1003.3552; King, Luhn, 1301.1340 for classical reviews and King, 1701.04413, Ivanov, 1702.03776 for very recent ones.
- Many new fields $\rightarrow$ many interaction terms $\rightarrow$ lots of free parameters. Imposing extra global symmetries helps constrain the models.

Model-building with multiple Higgses

Two approaches:

1. postulate some symmetry setting, add extra fields to encode it and generate the desired symmetry breaking pattern,
2. fix a designed class of bSM models, then explore all symmetries which are possible with this field content.

I will show the second approach at work in two problems:

- finding all abelian symmetry groups in any class of bSM models, with illustrations from NHDM,
- finding all non-abelian discrete symmetry groups in 3HDM scalar sector.

The focus is on the method of recognizing symmetries and on establishing exhaustive lists of possibilities, not on the specific bSM models.

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Abelian (rephasing) symmetries

Rephasing symmetries in NHDM

NB: NHDM scalar potential is an illustration; the method itself is general.

Higgs potential V in NHDM is built of ϕ_j, $j = 1, \dots, N$:

$$V = Y_{ij}(\phi_i^\dagger \phi_j) + Z_{ijkl}(\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_l)\,,$$

It may be invariant under $\phi_j \mapsto e^{i\alpha_j}\phi_j$ with some α_j. The first task is to find rephasing symmetry group A of a given potential.

- If V depends only on $|\phi_j|^2$, then $A = [U(1)]^N$: any rephasing will do.
- If not, $V = V_0 + k$ rephasing-sensitive terms. For each term, write invariance condition and solve the system of k such conditions for α_j.

Seems straightforward so far...

Rephasing symmetries in NHDM

For example, $(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_3)$ changes under a general rephasing as

$$(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_3) \quad \mapsto \quad e^{i(-2\alpha_1+\alpha_2+\alpha_3)}(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_3)\,.$$

Write it as $\sum_{j=1}^N d_{1j}\alpha_j$, with $d_{1j} = (-2, 1, 1, 0, \dots, 0)$. Then if

$$d_{1j}\alpha_j = 2\pi n_1$$

with any integer n_1, this term remains invariant. Repeat for all terms to obtain

$$d_{ij}\alpha_j = 2\pi n_i \quad \text{with} \quad n_i \in \mathbb{N}.$$

The task is to solve this system for α_j and deduce the symmetry group.

NB: the rephasing group is encoded in the $k \times N$ matrix d_{ij}.

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Rephasing symmetries in NHDM

A 4HDM example:

$$V = V_0 + \lambda_1(\phi_4^\dagger\phi_1)(\phi_3^\dagger\phi_1) + \lambda_2(\phi_4^\dagger\phi_2)(\phi_1^\dagger\phi_2) + \lambda_3(\phi_4^\dagger\phi_3)(\phi_2^\dagger\phi_3) + \mathrm{h.c.}$$

gives

$$d_{ij} = \begin{pmatrix} 2 & 0 & -1 & -1 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & 2 & -1 \end{pmatrix}.$$

Rephasing symmetries in NHDM

The matrix d_{ij} always has integer entries. By certain elementary steps

- permutation of rows or columns,
- sign flips of rows or columns,
- adding a column/row to another column/row

it can be diagonalized: $d = R \cdot D \cdot C$, where $|\det R| = |\det C| = 1$ and

$$D = \begin{pmatrix} d_1 & & & & & & \\ & d_2 & & & & & \\ & & \ddots & & & & \\ & & & d_r & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 & \cdots \end{pmatrix}, \quad r = \mathrm{rank}\ d$$

with $d_i > 0$ and such that d_i divides d_{i+1}.

D is known as the Smith Normal Form (SNF) of d_{ij}.
It exists and is unique for any integer-valued matrix.

Rephasing symmetries in NHDM

The key observation: elementary steps do not change the set of solutions.

Now the equations are decoupled; each $d_i\tilde{\alpha}_i = 2\pi\tilde{n}_i$ has solutions $\tilde{\alpha}_i = 2\pi\tilde{n}_i/d_i$, which generates the group $\mathbb{Z}_{d_i}$.

The rephasing group is therefore

$$A = \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_r} \times [U(1)]^{N-r}\,.$$

The 4HDM example

$$V = V_0 + \lambda_1(\phi_4^\dagger\phi_1)(\phi_3^\dagger\phi_1) + \lambda_2(\phi_4^\dagger\phi_2)(\phi_1^\dagger\phi_2) + \lambda_3(\phi_4^\dagger\phi_3)(\phi_2^\dagger\phi_3) + \mathrm{h.c.}$$

gives

$$D = \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{pmatrix}, \quad A = \mathbb{Z}_7 \times U(1)\,.$$

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3HDM with quarks

Another example: 3HDM quark sector

$$-\mathcal{L}_Y = \Gamma^{(j_\phi)}_{j_L j_d} \bar{Q}_{L j_L} \phi_{j_\phi} d_{R j_d} + \Delta^{(j_\phi)}_{j_L j_u} \bar{Q}_{L j_L} \tilde{\phi}_{j_\phi} u_{R j_u} + h.c.$$

with the following textures:

$$\Gamma^{(1)} = \begin{pmatrix} 0 & 0 & \times \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}, \quad \Gamma^{(3)} = \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix},$$

$$\Delta^{(1)} = \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \Delta^{(2)} = \begin{pmatrix} 0 & \times & 0 \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{pmatrix}, \quad \Delta^{(3)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}.$$

There are 12 Yukawa terms; 6 with d_R's and 6 with u_R's.

3HDM with quarks

We order the 12 fields as $(\phi_{j_\phi};\, Q_{Lj_L};\, d_{Rj_d};\, u_{Rj_u})$, where $j_\phi,\, j_L,\, j_d,\, j_u = 1,2,3$. Each Yukawa term produces a row d_{ij} with entries ±1 or 0.

For example, the term with $\Gamma^{(1)}_{13}$ is $\bar{Q}_{L1}\phi_1 d_{R3}$, and its row d_{ij} is

$$\Big(\overbrace{1,\ 0,\ 0}^{\phi}\ \Big|\ \overbrace{-1,\ 0,\ 0}^{Q_L}\ \Big|\ \overbrace{0,\ 0,\ 1}^{d_R}\ \Big|\ \overbrace{0,\ 0,\ 0}^{u_R}\Big)\,,$$

and the term with $\Delta^{(2)}_{31}$ is $\bar{Q}_{L3}\tilde{\phi}_2 u_{R1}$, and its row d_{ij} is

$$(\,0,\ -1,\ 0\,|\,0,\ 0,\ -1\,|\,0,\ 0,\ 0\,|\,1,\ 0,\ 0\,)\,.$$

3HDM with quarks

The entire matrix d_{ij} is a 12×12 matrix:

$$d_{ij} = \left(\begin{array}{ccc|ccc|ccc|ccc}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
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0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right).$$

3HDM with quarks

$D = \mathrm{diag}(1, 1, 1, 1, 1, 1, 1, 1, 1, 5, 0, 0)$.

The symmetry group is $A = \mathbb{Z}_5 \times U(1)_Y \times U(1)_B$.

The $\mathbb{Z}_5$-charges of the fields are: $q_{\mathbb{Z}_5} = (0, 2, 4\,|\,2, 1, 0\,|\,3, 1, 2\,|\,2, 4, 0)$.

More examples and applications:

- Remnant discrete symmetries in GUT models: Petersen, Ratz, Schieren, 0907.4049,
- NHDM scalar sector: Ivanov, Keus, Vdovin, 1112.1660; Ivanov, Lavoura, 1302.3656; Branco, Ivanov, 1511.02764,
- 3HDM quark sector: Ivanov, Nishi, 1309.3682, Nishi, 1411.4909,
- flavor symmetry groups in $SO(10)$ GUT models with any number of Higgses in 10, $\overline{126}$, 120 irreps. Ivanov, Lavoura, 1511.02720.

Beyond case-by-case checks

Next task: find all rephasing symmetry groups possible with the given field content, and do it efficiently, avoiding case-by-case checks.

This is encoded in the structures of all possible matrices d_{ij} built of rows of special type, such as

$$(2,\,-2,\,0,\,0,\,\dots)\,,\quad (2,\,-1,\,-1,\,0,\,\dots)\,,\quad (1,\,1,\,-1,\,-1,\,0,\,\dots)\,,$$

up to permutations, for NHDM scalar potential, or

$$(1,\,-1,\,1,\,0,\,\dots)\,,\quad (-1,\,-1,\,1,\,0,\,\dots)\,,$$

up to permutations, for NHDM Yukawa sector.

Beyond case-by-case checks

The main point:

$$|\det d| = |\det D| = \prod_j d_j.$$

The procedure is then the following:

- get rid of all "automatic" $U(1)$'s. For NHDM scalar sector it implies $U(N) \to U(N)/U(1) \simeq PSU(N)$;
- using the structure of d, find all values of $|\det d| = |A|$;
- if the prime decomposition of $|A|$ involves only first powers, then A is uniquely determined without the need to explicitly find the SNF,
- if its prime decomposition involves higher powers, then one needs to explicitly find the SNF to resolve the ambiguity.

This analysis can be often done manually, without computer-algebra assistance.

Beyond case-by-case checks

For example,

- if $|A| = 5$, then the group A must be $\mathbb{Z}_5$;
- if $|A| = 30$, then the group A must be $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$;
- if $|A| = 4$, then the group A can be either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. One needs to check whether SNF is $(\dots, 1, 2, 2)$ or $(\dots, 1, 1, 4)$.

In addition, one can often place the exact upper bound on $|A|$.

- scalar sector of NHDM: $|A| \leq 2^{N-1}$ for any N;
- NHDM with quarks: $|A| \leq (N+1)^2/3$ for any N.

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What initially seemed to require a massive computer-assisted case by case check turns into an arithmetical exercise.

Non-abelian symmetries in 3HDM scalar sector

Strategy

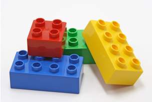
The main problem

find all discrete symmetry groups G which can be implemented in 3HDM scalar sector without producing accidental symmetries.

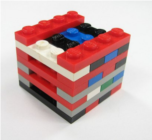
- The scalar potential in any NHDM is symmetric under the simultaneous rephasing $\alpha_j = \alpha$, which is a part of $U(1)_Y$. We are interested in additional symmetries. Therefore, we will search, within 3HDM, for G's which are subgroups not of $U(3)$ but of $PSU(3) = U(3)/U(1) = SU(3)/\mathbb{Z}_3$.
- Various families of discrete subgroups of $SU(3)$ were studied in much detail, see e.g. the recent works Grimus, Ludl, 1006.0098, 1110.6376, and used in "group scans" in search of observed flavor-physics patterns. This body of literature does not help us much with our problem we face. We need a constructive approach to find all G's which answer the question.

"Abelian LEGO" strategy

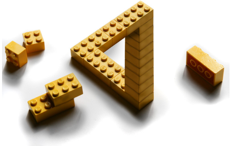
Step 1: find all possible discrete abelian groups $A_i \subset PSU(3)$; any allowed G can have only those abelian subgroups. These are "LEGO bricks" with which we will build a non-abelian model.



Step 2: build G by combining various A_i but avoid producing abelian groups not in the list!



Step 3: for each G built, check that it fits $PSU(3)$ and that it does not produce accidental symmetry.



Step 1: Abelian groups in 3HDM

For $N = 3$ we get the following finite $A_i \subset PSU(3)$:

$$A_i = \mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2\,, \quad \mathbb{Z}_3 \times \mathbb{Z}_3\,.$$

The last one is not a rephasing subgroup. Its full preimage in $SU(3)$ is the famous $\Delta(27)$:

$$\Delta(27)/Z(SU(3)) \simeq \mathbb{Z}_3 \times \mathbb{Z}_3\,.$$

For $PSU(3)$, this is the only "new" group in addition to the rephasing groups Ivanov, Keus, Vdovin, 1112.1660.

This list is complete: imposing any other finite abelian symmetry group on the potential unavoidably leads to continuous symmetry group.

Step 2: Group-theoretic part

- Any finite (non-abelian) G must contain only these A_i,
- their orders have only two prime factors: 2 and 3 $\Rightarrow$ by Cauchy's theorem, $|G| = 2^a 3^b$,
- $\Rightarrow$ by Burnside's $p^a q^b$ theorem, G is solvable (see introduction in Ivanov, Vdovin, 1210.6553): it contains a normal abelian subgroup A

$$g^{-1} A g = A \quad \forall g \in G\,.$$

- $\Rightarrow$ so far, we don't have any restriction on the size and structure of G/A.
- We proved in Ivanov, Vdovin, 1210.6553, that, inside $PSU(3)$, a stronger statement holds: G contains a normal maximal abelian subgroup (= normal self-centralizing subgroup).

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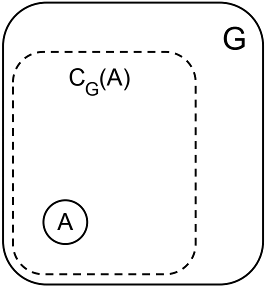
Consequences of a normal maximal abelian subgroup

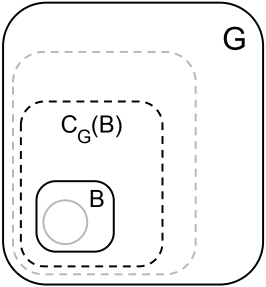


Consider A, abelian subgroup of G. Centralizer of A in G is the subgroup of all elements $g \in G$ which commute with all elements $x \in A$. We get

$$A \subseteq C_G(A) \subset G\,.$$

If $A = C_G(A)$, then A is self-centralizing.

Consequences of a normal maximal abelian subgroup



If $A \subset C_G(A)$, pick up some $b \in C_G(A)$, $b \notin A$ and consider $B = \langle A, b \rangle$, which is also an abelian subgroup of G.
We then get:

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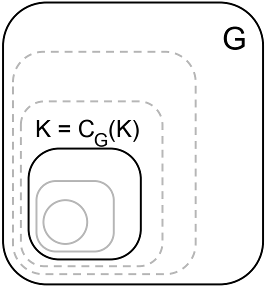

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Repeat until we hit a self-centralizing (maximal) abelian subgroup:

$$A \subset B \subset \cdots \subset K = C_G(K) \subseteq \cdots \subseteq C_G(B) \subseteq C_G(A) \subset G \,.$$

Consequences of a normal maximal abelian subgroup

What happens if a maximal abelian (=self-centralizing) subgroup A is normal in G?

- If A is normal in G, then $g^{-1}Ag = A$, so g acts on elements of A by some group-preserving permutation (automorphism of A).
- So, for every $g \in G$ we get an automorphism $\in Aut(A)$. We get a map $f : G \to Aut(A)$.
- Note that $Ker\, f = C_G(A)$. Indeed, $Ker\, f$ contains all elements g which induce the trivial permutation on A: $g^{-1}ag = a$ for all $a \in A$.
- If A is self-centralizing, $Ker\, f = A$. Therefore, map $\tilde{f} : G/A \to Aut(A)$ is injective: different elements of G/A map to different elements of $Aut(A)$.
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What happens if a maximal abelian (=self-centralizing) subgroup A is normal in G?

- If A is normal in G, then $g^{-1}Ag = A$, so g acts on elements of A by some group-preserving permutation (automorphism of A).
- So, for every $g \in G$ we get an automorphism $\in Aut(A)$. We get a map $f : G \to Aut(A)$.
- Note that $Ker\, f = C_G(A)$. Indeed, $Ker\, f$ contains all elements g which induce the trivial permutation on A: $g^{-1}ag = a$ for all $a \in A$.
- If A is self-centralizing, $Ker\, f = A$. Therefore, map $\tilde{f} : G/A \to Aut(A)$ is injective: different elements of G/A map to different elements of $Aut(A)$.
- Thus, $G/A \subseteq Aut(A)$, and G can be constructed as an extension of A by a subgroup of $Aut(A)$.

Automorphism groups

$$G = A\,.\,P, \quad \text{extension of } A \text{ by } P\,, \quad P \subseteq Aut(A)\,.$$

Overview of possibilities:

A	$Aut(A)$	"usable" subgroups P
$\mathbb{Z}_2$	$\{1\}$	—
$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$GL_2(2) \simeq S_3$	$\mathbb{Z}_2,\ \mathbb{Z}_3,\ S_3$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$GL_2(3)$	$\mathbb{Z}_2,\ \mathbb{Z}_4$

Step 3: Constructing G by extensions, $\mathbb{Z}_4$ example

Example: $A = \mathbb{Z}_4$. Then $Aut(\mathbb{Z}_4) = \mathbb{Z}_2$, so G is extension of $\mathbb{Z}_4$ by $\mathbb{Z}_2$.

There are several possibilities.
(1) extensions which lead to larger abelian groups ($\mathbb{Z}_8$, $\mathbb{Z}_4 \times \mathbb{Z}_2$) are immediately excluded;

(2) split extension $\mathbb{Z}_4 \rtimes Z_2 \simeq D_4$:

$$D_4 = \langle a, b \,|\, a^4 = 1,\ b^2 = 1,\ ab = ba^3 \rangle \,.$$

If $a = \mathrm{diag}(i, -i, 1)$, then

$$b = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ e^{-i\delta} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{with arbitrary } \delta.$$

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Step 3: Constructing G by extensions, $\mathbb{Z}_4$ example

A generic $\mathbb{Z}_4$ potential can be brought to the form $V_0 + V_{\mathbb{Z}_4}$, where

$$V_0 = -\sum_a m_a^2(\phi_a^\dagger\phi_a) + \sum_{a,b}\lambda_{ab}(\phi_a^\dagger\phi_a)(\phi_b^\dagger\phi_b) + \sum_{a\neq b}\lambda'_{ab}(\phi_a^\dagger\phi_b)(\phi_b^\dagger\phi_a)\,,$$

and

$$V_{\mathbb{Z}_4} = \lambda_1(\phi_3^\dagger\phi_1)(\phi_3^\dagger\phi_2) + \lambda_2(\phi_1^\dagger\phi_2)^2 + h.c.$$

The λ_1 term is invariant under b, while the λ_2 term transforms as

$$(\phi_1^\dagger\phi_2)^2 \mapsto e^{-4i\delta}(\phi_2^\dagger\phi_1)^2\,.$$

If we restrict parameters of V_0 ($m_{11}^2 = m_{22}^2$, $\lambda_{11} = \lambda_{22}$, $\lambda_{13} = \lambda_{23}$, $\lambda'_{13} = \lambda'_{23}$) then the potential is symmetric under one particular D_4 group in which the value of $\delta = \arg\lambda_2/2$.

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Step 3: Constructing G by extensions, $\mathbb{Z}_4$ example

(3) quaternion group $Q_4 = \langle a, b \,|\, a^4 = 1,\ b^2 = a^2,\ ab = ba^3 \rangle$.
If $a = \mathrm{diag}(i, -i, 1)$, then

$$b(Q_4) = \begin{pmatrix} 0 & e^{i\delta} & 0 \\ -e^{-i\delta} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Again, the $\mathbb{Z}_4$ part of the potential:

$$V_{\mathbb{Z}_4} = \lambda_1 (\phi_3^\dagger \phi_1)(\phi_3^\dagger \phi_2) + \lambda_2 (\phi_1^\dagger \phi_2)^2 + h.c.$$

Upon this b, the λ_1 term changes its sign. The only way to impose Q_4 is to set $\lambda_1 = 0$. But then the potential becomes invariant under a continuous transformation: $\mathrm{diag}(e^{i\alpha}, e^{i\alpha}, 1)$.

We conclude that Q_4 cannot be the finite symmetry group of potential.

Finite symmetry groups for $N = 3$

We performed this kind of analysis for all abelian groups we have.

Results:

$$\mathbb{Z}_2\,, \quad \mathbb{Z}_3\,, \quad \mathbb{Z}_4\,, \quad \mathbb{Z}_2\times\mathbb{Z}_2\,, \quad S_3\,, \quad D_4\,, \quad A_4\,, \quad S_4\,,$$

$$(\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_2 = \Delta(54)/\mathbb{Z}_3\,, \qquad (\mathbb{Z}_3\times\mathbb{Z}_3)\rtimes\mathbb{Z}_4 = \Sigma(36)\,.$$

This list is complete: trying to impose any other finite symmetry group will lead to a potential symmetric under a continuous group.

For each G, we constructed the general G-invariant potential
$\Rightarrow$ this allows us to prove the absence of accidental symmetries in each case.

Further developments

Search for GCPs

It may happen that G-invariant potential is automatically invariant under a generalized CP (GCP) transformation:

$$J : \phi_i \mapsto X_{ij}\phi_j^* \,.$$

For each G, we searched for such J satisfying conditions:

$$J^2 = XX^* \in G\,, \quad J^{-1}\rho_g J = X\rho_g X^\dagger = \rho_{g'}\,.$$

and looked whether it implies new constraints.

$\mathbb{Z}_4$, D_4, A_4, S_4, $\Sigma(36)$ indeed force explicit CP-conservation. The others do not (this possibility was absent in 2HDM).

Search for GCPs

Matrix d plays a role in the problem.

$$d(A_4) = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -2 & 2 \\ 2 & 0 & -2 \end{pmatrix}, \quad d(\Delta(54)) = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

- For A_4, $-d = d$ up to permutations $\to$ explicit CP-conservation.
- For $\Delta(27)$, $-d \neq d$ $\to$ possibility for explicit CP-violation.

CP4 3HDM

One peculiar possibility is 3HDM with CP4 (= GCP of order 4) without any other symmetry, Ivanov, Silva, 1512.09276.

- assumes very little: this is the minimal model realizing CP4. This is the first ever model based on CP4 without any accidental symmetry.
- CP4 can be extended to Yukawa sector, Aranda, Ivanov, Jimenez, 1608.08922.
- It is tractable analytically and is quite predictive.

In short, a good balance of minimality, predictiveness, and peculiarity. We are exploring its phenomenology.

Symmetry breaking patterns in NHDM

The vacuum expectation value alignment $\langle\phi_i^0\rangle = v_i e^{i\xi_i}/\sqrt{2}$ of a G-symmetric NHDM can be invariant under a residual symmetry group $G_v \subseteq G$.

Phenomenology depends on how much of G is broken! G-symmetric NHDM can lead to viable quark masses and CKM only if G is broken completely in the space of "active" doublets Leurer, Nir, Seiberg, hep-ph/9212278; Gonzalez Felipe et al, 1401.5807.

Symmetry breaking in 3HDM

Results on strongest and weakest breaking of discrete symmetries in 3HDM and on spontaneous CP-violation, Ivanov, Nishi, 1410.6139.

group	$\|G\|$	$\|G_v\|_{min}$	$\|G_v\|_{max}$	sCPv possible?
abelian	2, 3, 4, 8	1	$\|G\|$	yes
$\mathbb{Z}_3 \rtimes \mathbb{Z}_2^*$	6	1	6	yes
S_3	6	1	6	—
$\mathbb{Z}_4 \rtimes \mathbb{Z}_2^*$	8	2	8	no
$S_3 \times \mathbb{Z}_2^*$	12	2	12	yes
$D_4 \times \mathbb{Z}_2^*$	16	2	16	no
$A_4 \rtimes \mathbb{Z}_2^*$	24	4	8	no
$S_4 \times \mathbb{Z}_2^*$	48	6	16	no
CP-violating $\Delta(27)$	18	6	6	—
CP-conserving $\Delta(27)$	36	6	12	yes
$\Sigma(36)$	72	12	12	no

The moral

The moral

When building bSM models, do not ignore unconventional mathematical tools. They may help you answer questions which traditional “poor physicist’s methods” just cannot handle.