## On-shell Higgs Production in MSSM with complex parameters.

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with Stefan Liebler and Georg Weiglein based on [arXiV:1611.09308]

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Motivations

- O The MSSM with complex parameters
- 8 Gluon fusion Higgs production
- O Phenomenology with SusHiMi and FeynHiggs
- $_{\odot}$  Interference effects in Higgs production and decay

6 Conclusions

4th July 2012: Higgs discovery completes the Standard Model?



## Looking beyond the Standard Model

4th July 2012: Higgs discovery completes the Standard Model?

- Essential part of the LHC program in Run II is to look for additional Higgses
- The 125 GeV Higgs can be accommodated into several BSM models with extended Higgs sectors
- To effectively search for new physics we need
  - precise knowledge of cross-sections for additional Higgs states
  - test for deviations from SM expectations for the observed Higgs



 $\mathcal{CP}$ -violating scenarios remain largely unexplored in the searches so far

2 Higgs doublets  $\Phi_1$  and  $\Phi_2$  with  $\Phi_i = 1/\sqrt{2}(\phi_i + i\chi_i)$ 

Physical states

- ▶ CP-even:  $\phi_1^0, \phi_2^0 \to h, H$
- $\mathcal{CP}$ -odd:  $\chi_1^0, \chi_2^0 \to A, G$
- $\blacktriangleright\,$  charged:  $\phi_1^\pm, \phi_2^\pm \to H^\pm, G^\pm$

Input Parameters

► 
$$\tan \beta = \frac{v_2}{v_1}$$
  
►  $m_A^2 = \frac{2|m_{12}|^2}{\sin(\beta)}$ 

Electroweak Symmetry Breaking  $\rightarrow 5$  physical Higgs states  $h, H, A, H^{\pm}$ 

$$\begin{pmatrix} h \\ H \\ A \\ G \end{pmatrix} = \begin{pmatrix} -s_{\alpha} & c_{\alpha} & 0 & 0 \\ c_{\alpha} & s_{\alpha} & 0 & 0 \\ 0 & 0 & s_{\beta_n} & c_{\beta_n} \\ 0 & 0 & -c_{\beta_n} & s_{\beta_n} \end{pmatrix} \begin{pmatrix} \phi_1^0 \\ \phi_2^0 \\ \chi_1^0 \\ \chi_2^0 \end{pmatrix} \quad \textcircled{\textbf{for all } for all } \textbf{for a$$

 $\operatorname{MSSM}$  Higgs sector is  $\mathcal{CP}\text{-}\mathrm{conserving}$  at lowest order

**Motivation**: Baryon asymmetry of the universe requires more  $\mathcal{CP}$ -violation than in CKM matrix

- ▶ 105 new parameters + 19 from the SM
  - appear as masses, mixing angles and  $\mathcal{CP} ext{-violating phases}$
  - Minimal flavour violation  $\Rightarrow$  41 independent parameters
- ▶ 12 of 41 parameters are CP-violating phases
  - Trilinear couplings  $A_f, f = u, d, c, s, t, b, e, \mu, \tau \rightarrow A_f = |A_f| e^{i\phi_A f}$
  - Higgsino mass parameter  $\mu \rightarrow \mu = |\mu| e^{i\phi\mu}$
  - Gluino mass parameter  $M_3 \to M_3 = |M_3| e^{i\phi_{M_3}}$
  - Gaugino mass parameters  $M_1$ ,  $M_2$  (Only  $\phi_{M_1}$  physical by convention)
- Experimental bounds on these phases mainly come from the EDMs of the electron and the muon eg:[Barger, Falk, Han, Jiang, Li, Plehn '01], [Ellis, Lee, Pilaftsis '09], [Li, Profumo, Ramsey-Musolf '10], [Arbey, Ellis, Godbole, Mahmoudi '14]

- $\blacktriangleright$  CP-violating phases enter the Higgs sector through higher order corrections
- ▶ CP-violating self-energies appear at 1-loop order
- ▶ Dominant phases in the Higgs sector:  $\phi_{A_{t,b}}, \phi_{M_3}, \phi_{\mu}$

### $\operatorname{MSSM}$ with $\operatorname{\mathbb{R}eal}$ parameters

- $\blacktriangleright CP$  conserved
- ▶ Mixing b/w CP-even states
- ▶ Input mass:  $M_A$  or  $M_{H^{\pm}}$

MSSM with Complex parameters

- $\blacktriangleright$  *CP* violated
- ▶  $3 \times 3$  mixing b/w h, H, A
- ▶ A not a CP eigenstate ⇒Input mass:  $M_{H^{\pm}}$

h, H, A mix into new mass eigenstates  $h_1, h_2, h_3$ with  $m_{h_1} \le m_{h_2} \le m_{h_3}$ 





- ▶ Gluon fusion is the dominant production channel for Higgs bosons at the LHC
- ▶ Coupling of gluons to the Higgs mediated by loops of coloured (s)fermions
- ▶ Primary contribution from (s)top and (s)bottom (s)quarks.
- In the CP-conserving MSSM, XS for gluon fusion Higgs production known at the N<sup>3</sup>LO for top-quark contribution, and analytically/various expansions at NLO for other contributions

- ▶ CP-violating phases mainly enter the XS through **Z** factors, squark loops or  $\Delta_b$  corrections to the bottom Yukawa coupling
- The Z-matrix determines the mixings between the Higgses h, H and A [Hahn, Heinemeyer, Hollik, Rzehak, Weiglein: feynhiggs.de].[E. Fuchs, G. Weiglein: 1610.06193]



▶ The Higgs-squark vertex contains CP-violating phases:



An explicit calculation of the LO XS with Z-factors for a general case of arbitrary parameters can be found in [S. Liebler, SP, G. Weiglein '16, arXiV:1611.09308]

## Resummation of SUSY-QCD Corrections

- > Leading  $\tan \beta$  enhanced corrections from gluino-sbottom loops absorbed in effective bottom Yukawa coupling
- ▶  $\Delta_b$  corrections appear in the effective Lagrangian as follows

$$\mathcal{L}_{\text{eff}} = -\lambda_b \bar{b}_R \left[ h_d^0 + \frac{\Delta_b}{t_\beta} h_u^{0*} \right] b_L + h.c.$$

with  $\Delta_b = \frac{2}{3\pi} \alpha_s \mu^* M_3^* t_\beta I\left(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, m_{\tilde{g}}^2\right)$ 

 Effective bottom Yukawa couplings with full resummed SUSY-QCD Corrections at LO XS:

$$\mathbf{g}_{b_L}^h = (g_{b_R}^h)^* = \frac{1}{1+\Delta_b} \left[ \frac{\sin\alpha}{\cos\beta} + \frac{\cos\alpha}{\sin\beta} \Delta_b \right]$$

$$\mathbf{g}_{b_L}^h = (g_{b_R}^h)^* = \frac{1}{1+\Delta_b} \left[ \frac{\cos\alpha}{\cos\beta} + \frac{\sin\alpha}{\sin\beta} \Delta_b \right]$$

$$\mathbf{g}_{b_L}^A = (g_{b_R}^A)^* = \frac{1}{1+\Delta_b} \left[ \tan\beta \left( 1 - \frac{\Delta_b}{\tan^2\beta} \right) \right]$$

▶ For NLO XS, a simplified  $\Delta_b$  correction is used, with effective Yukawa coupling

$$g^{\phi}_{b_L} = g^{\phi}_{b_R} = \frac{1}{|1 + \Delta_b|} f(\alpha, \beta)$$

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LO production XS of the mass eigenstates  $h_a$ :

$$\sigma_{\rm LO}(pp \to h_a) = \sigma_0^{h_a} \tau_{h_a} \mathcal{L}^{gg}(\tau_{h_a}) \quad \text{with} \quad \mathcal{L}^{gg}(\tau) = \int_{\tau}^1 \frac{dx}{x} g(x) g(\tau/x), \ \tau_{h_a} = m_{h_a}^2/s$$

$$\begin{split} \boldsymbol{\sigma}_{0}^{h_{a}} &= \frac{G_{F} \boldsymbol{\alpha}_{s}^{2}(\boldsymbol{\mu}_{R})}{288 \sqrt{\pi}} \left[ \left| \mathcal{A}^{h_{a}, \mathbf{e}} \right|^{2} + \left| \mathcal{A}^{h_{a}, \mathbf{o}} \right|^{2} \right] \\ & \text{with} \quad \mathcal{A}^{h_{a}, \mathbf{e}} = \hat{\mathbf{Z}}_{ah} \mathcal{A}_{+}^{h} + \hat{\mathbf{Z}}_{aH} \mathcal{A}_{+}^{H} + \hat{\mathbf{Z}}_{aA} \mathcal{A}_{-}^{A} \\ & \text{and} \quad \mathcal{A}^{h_{a}, \mathbf{o}} = \hat{\mathbf{Z}}_{ah} \mathcal{A}_{-}^{h} + \hat{\mathbf{Z}}_{aH} \mathcal{A}_{-}^{H} + \hat{\mathbf{Z}}_{aA} \mathcal{A}_{+}^{A}, \end{split}$$

 $\mathcal{A}^{h_a,\mathrm{e}}, \mathcal{A}^{h_a,\mathrm{e}}$  can be identified with a contribution that stems from  $\mathcal{L} \supset G^{\mu\nu}G_{\mu\nu}\phi$  and  $\mathcal{L} \supset \tilde{G}^{\mu\nu}G_{\mu\nu}\phi$  respectively and  $\mathcal{A}^{\phi}_{\pm} \propto (g^{\phi}_{f_L} \pm g^{\phi}_{f_R})$ 

Higher order contributions are of the form



▶ The total gluon-fusion cross section is the sum of the two parts:

$$\sigma_{\mathbf{N}^k \mathbf{LO}}(pp \to h_a + X) = \sigma^{\mathbf{e}}_{\mathbf{N}^k \mathbf{LO}}(pp \to h_a + X) + \sigma^{\mathbf{o}}_{\mathbf{N}^k \mathbf{LO}}(pp \to h_a + X) \,.$$

Result beyond NLO QCD:

$$\begin{split} \sigma^{\rm e}_{\rm N^k_{\rm LO}} &= \sigma^{\rm e}_{\rm NLO}(1+\delta^{\rm lf}_{\rm EW}) + \left(\sigma^{t,\rm e}_{\rm N^k_{\rm LO,~EFT}} - \sigma^{t,\rm e}_{\rm NLO,~EFT}\right) \\ \sigma^{\rm o}_{\rm N^k_{\rm LO}} &= \sigma^{\rm o}_{\rm NLO} + \left(\sigma^{t,\rm o}_{\rm N^k_{\rm LO,~EFT}} - \sigma^{t,\rm o}_{\rm NLO,~EFT}\right) \,, \end{split}$$

with  $k \in \{1, 2, 3\}$ .

- ▶ Only the CP-even component of  $h_1$  XS  $\sigma_{N^kLO, EFT}^{t,e}$  is implemented up to k = 3
- $\blacktriangleright \sigma_{\rm NLO}^{{\rm e}/{\rm o}}$  are the NLO cross sections including real contributions and the interpolated NLO virtual corrections
- ▶ The virtual NLO amplitude  $\mathcal{A}^{\phi}_{\mathrm{NLO}}(\phi_z)$  approximated using

$$\mathcal{A}^{\phi}_{\rm NLO}(\phi_z) = \frac{1 + \cos \phi_z}{2} \mathcal{A}^{\phi}_{\rm NLO}(0) + \frac{1 - \cos \phi_z}{2} \mathcal{A}^{\phi}_{\rm NLO}(\pi)$$

## SusHi & FeynHiggs

 ${\tt SusHi}$  calculates neutral Higgs boson production XS through gluon fusion and bottom-quark annihilation (5FS) in the SM, the 2HDM, MSSM and the NMSSM.

[Harlander Liebler Mantler '12; Liebler '15: sushi.hepforge.org]

FeynHiggs calculates the masses, couplings and wavefunction renormalization factors of the Higgs sector in the MSSM. [Hahn, Heinemeyer, Hollik, Rzehak, Weiglein: feynhiggs.de]



[S. Liebler, SP, G. Weiglein '16, arXiV:1611.09308]



The effect of  $\phi_{A_t}$  studied in two scenarios:

### lightstop – inspired:

- $\blacktriangleright$   $M_{\rm SUSY} = 500 \, {\rm GeV}$
- $\blacktriangleright X_t^{OS} = 2 M_{SUSY}$
- $\blacktriangleright |A_t| = |A_b| = |A_\tau|$
- $\blacktriangleright \mu = 400 \text{ GeV}$
- $\blacktriangleright \tan \beta = 16$
- ▶ M<sub>3</sub> = 1500 GeV
- $\blacktriangleright$   $M_{H\pm} = 500 \text{ GeV}$
- $M_2 = 400 \text{ GeV}$
- ▶ M<sub>1̃</sub> = 1000 GeV

# $\mathbf{M}_{\mathbf{h}}^{\text{mod}+} - \mathbf{inspired}:$ $\blacktriangleright M_{\text{SUSY}} = 1000 \text{ GeV}$

 $\blacktriangleright X_t^{\rm OS} = 1.5 \ M_{\rm SUSY}$ 

$$\blacktriangleright |A_t| = |A_b| = |A_\tau|$$

- $\blacktriangleright \mu = 1000 \text{ GeV}$
- $\blacktriangleright \tan \beta = 10$
- ▶  $M_3 = 1500 \text{ GeV}$
- ▶  $M_{H\pm} = 900 \text{ GeV}$
- ▶  $M_2 = 500 \text{ GeV}$
- $M_{\tilde{l}_3} = 1500 \text{ GeV}$

[S. Liebler, SP, G. Weiglein '16, arXiV:1611.09308]

## lightstop-inspired

### Mass and XS variation with $\phi_{A_t}$ for $h_1$



XS variation with  $\phi_{A_t}$  for  $h_2$  and  $h_3$ 



At  $\phi_{A_t} = 0$  the squark effects reduce the XS: 89% for  $h_2$  and 22% for  $h_3$ 

### Mass and XS variation with $\phi_{A_t}$ for $h_2$ and $h_3$



### $PDF + \alpha_s$ uncertainties

- $\blacktriangleright$  Uncertainties in cross-section predictions can be induced by PDF fits and associated  $\alpha_s$
- ▶ MMHT2017 PDF sets used for LO, NLO and NNLO for  $gg\Phi$  and  $bb\Phi$
- ▶ PDF +  $\alpha_s$  uncertainties function of  $m_{h_a}$
- Can be taken over from description for MSSM Higgses in LHCHXSWG report 4 [arXiV:1610.07922]

### Scale uncertainties

- Central scale choice:  $(\mu_R^0, \mu_F^0) = (m_{h_a}/2, m_{h_a}/2)$  for  $gg\Phi$  and  $(\mu_R^0, \mu_F^0) = (m_{h_a}, m_{h_a}/4)$  for  $bb\Phi$
- ▶ Scale unc. obtained by taking maximal deviation from central scale choice  $\{(2\mu_R^0, 2\mu_F^0), (2\mu_R^0, \mu_F^0), (\mu_R^0, 2\mu_F^0), (\mu_R^0, \mu_F^0/2), (\mu_R^0/2, \mu_F^0), (\mu_R^0/2, \mu_F^0/2)\}$

$$\blacktriangleright \Delta \sigma^{\text{scale}} = \sqrt{\left(\Delta \sigma_{\text{N}^{k}\text{LO}}^{\Delta_{b1}}\right)^{2} + \left(\Delta \sigma_{\text{LO}}^{\Delta_{b2}} - \Delta \sigma_{\text{LO}}^{\Delta_{b1}}\right)^{2}}$$

### Interpolation uncertainties

- Interpolation of two-loop virtual squark-gluino contributions induces an unc. due to non-exact results
- $\Delta \sigma^{\text{int}} = \sin^2(\phi_z) |\sigma(0) \sigma(\pi)|/2$

## Interpolation Uncertainties

### Interpolation uncertainties in XS variation with $\phi_{M_3}$ for $h_2$ and $h_3$



## Interference effects in $\{b\bar{b}, gg\} \to h_1, h_2, h_3 \to \tau^+ \tau^-$

[E. Fuchs, G. Weigline (in preparation)]

In scenarios where the Higgses are heavily admixed and therefore almost mass degenerate, we need to account for interference effects. Consider the processes:



Diagrammatic representation of the amplitude  $\mathcal{A}_{h_a}$ 

$$\begin{aligned} |\mathcal{A}|_{\text{int}}^2 &= |\mathcal{A}|_{\text{coh}}^2 - |\mathcal{A}|_{\text{incoh}}^2 = \sum_{a < b} 2 \operatorname{Re} \left[ \mathcal{A}_{h_a} \mathcal{A}_{h_b}^* \right] \\ |\mathcal{A}|_{\text{coh}}^2 &= \left| \sum_{a=1}^3 \mathcal{A}_{h_a} \right|^2, |\mathcal{A}|_{\text{incoh}}^2 = \sum_{a=1}^3 \left| \mathcal{A}_{h_a} \right|^2 \end{aligned}$$

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▶ Relative interference term for a production channel  $P \rightarrow b\bar{b}, gg$ :

$$\eta^P = rac{\sigma_{
m int}^P}{\sigma_{
m incoh}^P}$$

• Express the total XS into separate Higgs contributions by introducing a correction term for the interference effect:

$$\sigma^{P} = \sigma^{P}_{h_{1}}(1 + \eta^{P}_{1}) + \sigma^{P}_{h_{2}}(1 + \eta^{P}_{2}) + \sigma^{P}_{h_{3}}(1 + \eta^{P}_{3})$$

• The interference factors  $\eta_a^P$  are defined as:

$$\eta_a^P = \frac{\sigma_{\rm int_{ab}}^P}{\sigma_{h_a}^P + \sigma_{h_b}^P} + \frac{\sigma_{\rm int_{ac}}^P}{\sigma_{h_c}^P + \sigma_{h_c}^P}$$

This is stable even if one of the contributions  $\sigma_{h_a}$  is suppressed

▶ Keeping the branching ratios unchanged, the final XS for the process is:

$$\sigma(pp \to P \to h_{1,2,3} \to \tau^+ \tau^-) \simeq \sum_{a=1}^3 \sigma(pp \to P \to h_a) \cdot (1 + \eta_a^P) \cdot \mathrm{BR}(h_a \to \tau^+ \tau^-)$$

## Effect of interference on exclusion bounds





Comparison of predicted XS times BR with and without interference for a fixed value of  $\tan \beta$  to CMS exclusion bounds in the complex  $\mathbf{M}_{\mathbf{h}}^{\mathrm{mod}+}$  scenario

## Effect of interference on exclusion bounds

#### [E. Fuchs, G. Weigline (in preparation)]



Bounds for the complex  $M_h^{mod+}$  scenario by modifying the input data for HiggsBounds with  $\eta$ : without the interference term and including the interference term for  $\mu = 1000 \text{ GeV}$ 

## Summary

### **Conclusions** :

- ▶ Complex parameters produce significant effects on neutral Higgs production XS
  - Through squark loops
  - Through Higgs mixings described by Z factors
- $\blacktriangleright$  Mixing between the heavy Higgses through the Z factors  $\rightarrow$  can lead to interference effects

### **Outlook**:

▶ Integrating interference factors in SusHiMi to study interference effects including final states in Higgs production and decay → work in collaboration with E. Fuchs, S.Liebler and G. Weiglein



### APPENDIX

The gluino  $\tilde{g}$  does not mix with other fields and enters the Lagrangian in the form

$$\mathcal{L} \supset -\frac{1}{2}\overline{\tilde{g}}m_{\tilde{g}}\tilde{g}\,,$$

where  $m_{\tilde{g}}$  is the absolute value of the complex soft-breaking parameter  $M_3 = m_{\tilde{g}} e^{i\phi_{M_3}}$ 

In the Feynman diagrams for the Higgs boson self-energies and the Higgs boson production via gluon fusion, the gluino only contributes beyond the one-loop level.

However it affects the bottom-quark Yukawa coupling already at the one-loop level, where it enters the leading corrections to the relation between the bottom-quark mass and the bottom-quark Yukawa coupling which can be resummed to all orders.

## Squark Sector

In the MSSM without flavour mixing in the squark sector, squarks  $\tilde{q}_{L,R}$  of one generation mix into mass eigenstates  $\tilde{q}_{1,2}$ .

$$\mathcal{L} \supset -(\tilde{q}_{L}^{\dagger}, \tilde{q}_{R}^{\dagger}) M_{\tilde{q}}^{2} \begin{pmatrix} \tilde{q}_{L} \\ \tilde{q}_{R} \end{pmatrix} \quad \text{with} \\ M_{\tilde{q}}^{2} = \begin{pmatrix} M_{\tilde{q}_{L}}^{2} + m_{q}^{2} + M_{Z}^{2} \cos 2\beta (I_{q}^{3} - Q_{q}s_{W}^{2}) & m_{q}X_{q}^{*} \\ m_{q}X_{q} & M_{\tilde{q}_{R}}^{2} + m_{q}^{2} + M_{Z}^{2} \cos 2\beta Q_{q}s_{W}^{2} \end{pmatrix}.$$
(1)

with  $X_q := A_q - \mu^* \cdot \{\cot \beta, \tan \beta\}$ , where  $\cot \beta$  and  $\tan \beta$  apply to up- and down-type quarks, respectively.

The soft-breaking masses  $M_{\tilde{q}_L}^2$  and  $M_{\tilde{q}_R}^2$ , the third component of the weak isospin  $I_q^3$ , the electric charge  $Q_q$  and the mass of the quark  $m_q$  are real parameters.

In the CP-violating MSSM the parameters  $A_q = |A_q|e^{i\phi_{A_q}}$  and  $\mu = |\mu|e^{i\phi_{\mu}}$ , and hence  $X_q$ , can be complex.

The mass matrix is diagonalised through the unitary matrix  $U_{\tilde{q}}$  having real diagonal elements and complex off-diagonal elements

$$\begin{pmatrix} \tilde{q}_1\\ \tilde{q}_2 \end{pmatrix} = U_{\tilde{q}} \begin{pmatrix} \tilde{q}_L\\ \tilde{q}_R \end{pmatrix}$$

The squark masses (with  $m_{\tilde{q}_1} \leq m_{\tilde{q}_2}$ ) are eigenvalues of the mass matrix.

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The neutral fields of the two Higgs doublets can be decomposed in CP-even  $(\phi_1^0, \phi_2^0)$  and CP-odd  $(\chi_1^0, \chi_2^0)$  components

$$\mathcal{H}_{1} = \begin{pmatrix} h_{d}^{0} \\ h_{d}^{-} \end{pmatrix} = \begin{pmatrix} v_{d} + \frac{1}{\sqrt{2}}(\phi_{1}^{0} + i\chi_{1}^{0}) \\ \phi_{1}^{-} \end{pmatrix} \\
\mathcal{H}_{2} = \begin{pmatrix} h_{u}^{+} \\ h_{u}^{0} \end{pmatrix} = e^{i\xi} \begin{pmatrix} \phi_{2}^{+} \\ v_{u} + \frac{1}{\sqrt{2}}(\phi_{2}^{0} + i\chi_{2}^{0}) \end{pmatrix},$$
(2)

Higgs potential  $V_H$  in terms of the neutral Higgs states is given by

$$\begin{split} V_{H}^{0} = &(|\mu|^{2} + m_{\mathcal{H}_{2}}^{2})|h_{u}^{0}|^{2} + (|\mu|^{2} + m_{\mathcal{H}_{1}}^{2})|h_{d}^{0}|^{2} \\ &- [m_{12}^{2}h_{u}^{0}h_{d}^{0} + h.c.] + \frac{g_{1}^{2} + g_{2}^{2}}{8}[|h_{u}^{0}|^{2} - |h_{d}^{0}|^{2}]^{2} \end{split}$$

## ${\cal Z}$ factors

- ► Diagonal propagator:  $\Delta_{ii}(p^2) = \frac{i}{p^2 m_i^2 \hat{\Sigma}_{ii}^{\text{eff}}(p^2)}$
- On-shell properties of external Higgses given by wave-function normalization ratios  $\hat{Z}_{ij}$  taken at the complex pole:

$$\hat{Z}_{ai} = \frac{1}{1 + \hat{\Sigma}_{ii}^{\text{eff}'}(\mathcal{M}_{H_a}^2)} \qquad \hat{Z}_{aj} = \frac{\Delta_{ij}(\mathcal{M}_{H_a}^2)}{\Delta_{ii}(\mathcal{M}_{H_a}^2)}$$

 $\blacktriangleright$  With the S-matrix normalization factor: non-unitary  $\hat{\mathbf{Z}}$  matrix

$$\hat{\mathbf{I}}_{H_{1}}^{\hat{\Gamma}_{H_{1}}} ) = \hat{\mathbf{Z}} \cdot \begin{pmatrix} \hat{\Gamma}_{h} \\ \hat{\Gamma}_{H} \\ \hat{\Gamma}_{A} \end{pmatrix}$$

$$\hat{\mathbf{Z}} = \begin{pmatrix} \sqrt{\hat{Z}_{1}} \hat{Z}_{1h} & \sqrt{\hat{Z}_{1}} \hat{Z}_{1H} & \sqrt{\hat{Z}_{1}} \hat{Z}_{1A} \\ \sqrt{\hat{Z}_{2}} \hat{Z}_{2h} & \sqrt{\hat{Z}_{2}} \hat{Z}_{2H} & \sqrt{\hat{Z}_{2}} \hat{Z}_{2A} \\ \sqrt{\hat{Z}_{3}} \hat{Z}_{3h} & \sqrt{\hat{Z}_{3}} \hat{Z}_{3H} & \sqrt{\hat{Z}_{3}} \hat{Z}_{3A} \end{pmatrix} .$$