Moments of the Hadronic Vacuum Polarization at the Physical Point – Results & Challenges

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Outline

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Hadronic vacuum polarization (HVP)

- EM current of quarks

$$j_\mu(x) = \sum_f Q_f \, \overline{\psi}^{(f)}(x)\gamma_\mu \psi^{(f)}(x)$$

Q_f: electric charge of flavor f.

$$Q_u = \frac{2}{3}, \quad Q_d = -\frac{1}{3}, \quad Q_s = -\frac{1}{3}, \quad Q_c = \frac{2}{3}$$

- Hadronic vacuum polarization $\Pi(Q^2)$ is defined through

$$\Pi_{\mu\nu}(Q) = (Q^2\delta_{\mu\nu} - Q_\mu Q_\nu)\,\Pi(Q^2),$$

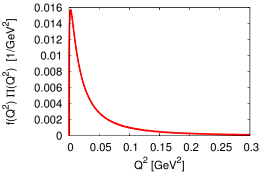
where

$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}x \, e^{iQx} \, \langle j_\mu(x) j_\nu(0)\rangle =$$

on the lattice $\longrightarrow$ $$\frac{1}{TV}\sum_{x,y}\left(e^{iQ(x-y)} - 1\right)\langle j_\mu(x) j_\nu(y)\rangle$$

Moments of HVP

- $a_\mu^{\text{LO-HVP}} = 4\alpha^2 \int_0^\infty \mathrm{d}Q^2\, f(Q^2)\,\Pi(Q^2)$ $\longleftarrow$ low Q^2 region dominates



- Π_1 and Π_2 provides $a_\mu^{\text{LO-HVP}}$ within less then 2 %

$$\Pi(Q^2) = \sum_n Q^{2n}\Pi_n$$

- Π_n can be computed as moments of current-current correlator

[Feng *et al.* 2013; Chakraborty *et al.* 2014]

$$\Pi_n = \frac{(-1)^{(n+1)}}{(2(n+1))!}\,\frac{1}{T}\sum_{t,\bar{t}}(t-\bar{t})^{2(n+1)}\,\langle j_\mu(t)j_\mu(\bar{t})\rangle$$

Current-current correlator

- Introduce $U(1)$ phase $c_{x,\mu}$ on links:

$$\left(M^{(f)}(c)\right)^{ab}_{x,y} = \frac{1}{2}\sum_{\mu}\left(\delta_{y,x+\mu}U^{ab}_{x,\mu}\, e^{iQ_f c_{x,\mu}} - \delta_{y,x-\mu}\left(U^{\dagger}_{x-\mu,\mu}\right)^{ab} e^{-iQ_f c_{x-\mu,\mu}}\right) + m_f \delta_{x,y}\delta^{ab}$$

  Staggered phase is absorbed into $U_{x,\mu}$.

- Partition function:

$$Z(c) = \int dU\, e^{-S_g(U)} \prod_f \left(\det M^{(f)}(c)\right)^{1/4}$$

- Define current-current correlator as 2^{nd} derivative:

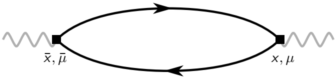
$$\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle = i\frac{\partial}{\partial c_{x,\mu}}\, i\frac{\partial}{\partial c_{\bar x,\bar\mu}} \log Z(c)\Big|_{c=0} =$$

$$= \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{connected}} + \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{contact}} + \langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{disconnected}}$$

Contributions to correlator

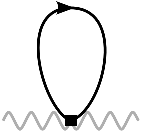
- $\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{connected}} =$

$$-\sum_f \frac{Q_f^2}{8}\,\mathrm{Re}\,\mathrm{tr}_c\left(M^{(f)}{}^{-1}_{x+\mu,\bar x} U_{\bar x,\bar\mu} M^{(f)}{}^{-1}_{\bar x+\bar\mu,x} U_{x,\mu} + M^{(f)}{}^{-1}_{x+\mu,\bar x+\bar\mu} U^\dagger_{\bar x,\bar\mu} M^{(f)}{}^{-1}_{\bar x,x} U_{x,\mu}\right)$$



- $\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{contact}} = \delta_{x,\bar x}\delta_{\mu,\bar\mu} \sum_f \frac{Q_f^2}{4}\,\mathrm{Re}\,\mathrm{tr}_c\left(U_{x,\mu} M^{(f)}{}^{-1}_{x+\mu,x}\right)$

  vanishes due to $q = 0$ subtraction



- $\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{disconnected}} =$

$$-\left[\sum_f \frac{Q_f}{4}\,\mathrm{Im}\,\mathrm{tr}_c\left(U_{x,\mu} M^{(f)}{}^{-1}_{x+\mu,x}\right)\right] \times \left[\sum_{\bar f} \frac{Q_{\bar f}}{4}\,\mathrm{Im}\,\mathrm{tr}_c\left(U_{\bar x,\bar\mu} M^{(\bar f)}{}^{-1}_{\bar x+\bar\mu,\bar x}\right)\right]$$

Current conservation

- Current is conserved in continuum:

$$\frac{\partial}{\partial x_\mu} \langle j_\mu(x) j_{\bar{\mu}}(\bar{x}) \rangle = \frac{\partial}{\partial \bar{x}_{\bar{\mu}}} \langle j_\mu(x) j_{\bar{\mu}}(\bar{x}) \rangle = 0$$

- On lattice, on each configuration:

$$\nabla^{(b)}_\mu \Big(\langle j_\mu(x) j_{\bar{\mu}}(\bar{x}) \rangle_{\text{conn.}} + \langle j_\mu(x) j_{\bar{\mu}}(\bar{x}) \rangle_{\text{contact}} \Big) = \\ = \bar{\nabla}^{(b)}_{\bar{\mu}} \Big(\langle j_\mu(x) j_{\bar{\mu}}(\bar{x}) \rangle_{\text{conn.}} + \langle j_\mu(x) j_{\bar{\mu}}(\bar{x}) \rangle_{\text{contact}} \Big) = 0$$

$$\nabla^{(b)}_\mu \, \langle j_\mu(x) j_{\bar{\mu}}(\bar{x}) \rangle_{\text{disc.}} = \bar{\nabla}^{(b)}_{\bar{\mu}} \, \langle j_\mu(x) j_{\bar{\mu}}(\bar{x}) \rangle_{\text{disc.}} = 0$$

with backward derivatives

$$\nabla^{(b)}_\mu \, f_{\mu,\bar{\mu}}(x,\bar{x}) = \sum_\mu \Big(f_{\mu,\bar{\mu}}(x,\bar{x}) - f_{\mu,\bar{\mu}}(x-\mu,\bar{x}) \Big)$$

$$\bar{\nabla}^{(b)}_{\bar{\mu}} \, f_{\mu,\bar{\mu}}(x,\bar{x}) = \sum_{\bar{\mu}} \Big(f_{\mu,\bar{\mu}}(x,\bar{x}) - f_{\mu,\bar{\mu}}(x,\bar{x}-\bar{\mu}) \Big)$$

- Consistency check of code possible

Outline

Action

Gluon: Tree-level improved Symanzyk gauge action

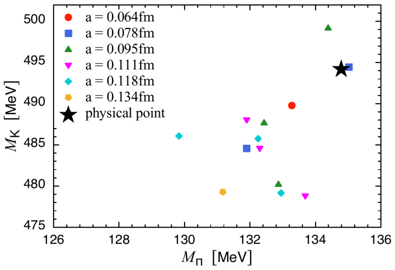
Quark: $N_f = 2+1+1$ staggered fermions
with 4 steps of $\varrho = 0.125$ stout smearing

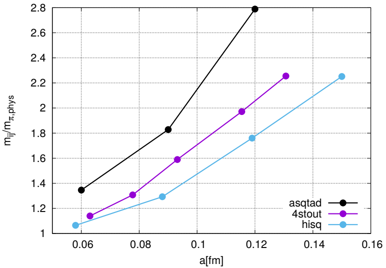
- $m_u = m_d = m_l$ and m_s fixed via m_π and m_K
- charm quark mass is fixed via $m_c/m_s = 11.85$
- RHMC
  - $\det\left(M_l^2 M_s\right)^{1/4} = \det\left(M_l^2 M_s M_{\bar{q}}^{-3}\right)^{1/4} \cdot \det\left(M_{\bar{q}}\right)^{3/4}$
    with $m_{\bar{q}} = (m_u + m_d + m_s)/3 = (2m_l + m_s)/3$
  - RHMC for M_c
  - Force gradient integrator

$$\mathcal{T}_{\mathrm{FG}} = e^{\frac{1}{6}\hat{V}\tau}\, e^{\frac{1}{2}\hat{T}\tau}\, e^{\frac{2}{3}\hat{V}\tau - \frac{1}{72}\tau^3\{V,\widehat{\{V,T\}}\}}\, e^{\frac{1}{2}\hat{T}\tau}\, e^{\frac{1}{6}\hat{V}\tau}$$

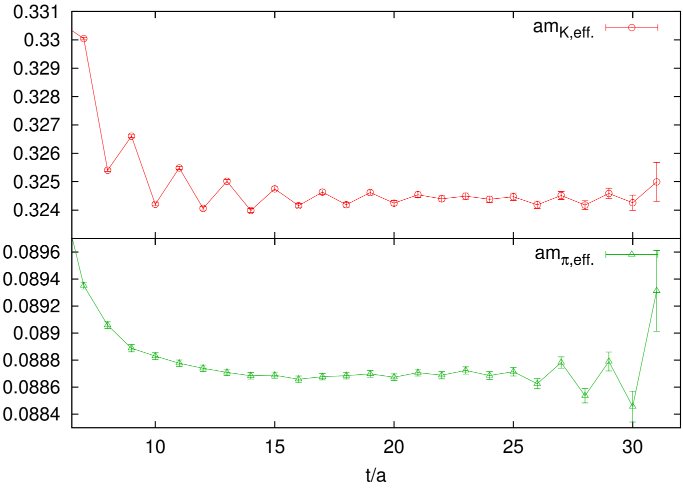
Gauge ensembles

β	a [fm]	$T \times L$	#conf-conn	#conf-disc
3.7000	0.134	64 × 48	1000	1000
3.7500	0.118	96 × 56	1500	1500
3.7753	0.111	84 × 56	1500	1500
3.8400	0.095	96 × 64	2500	1500
3.9200	0.078	128 × 80	3500	1000
4.0126	0.064	144 × 96	450	-

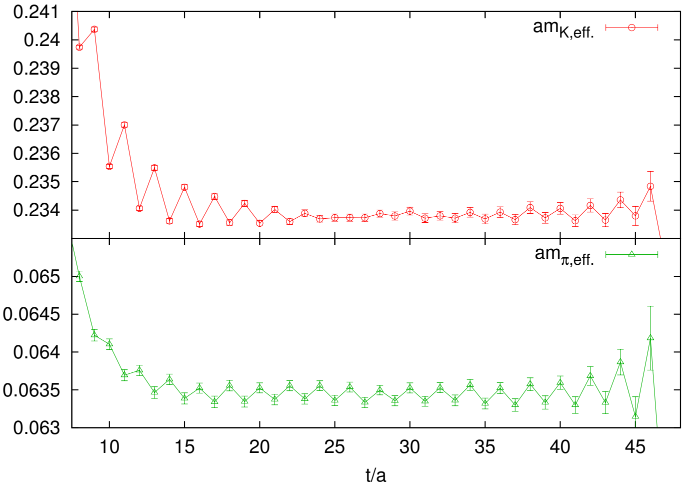




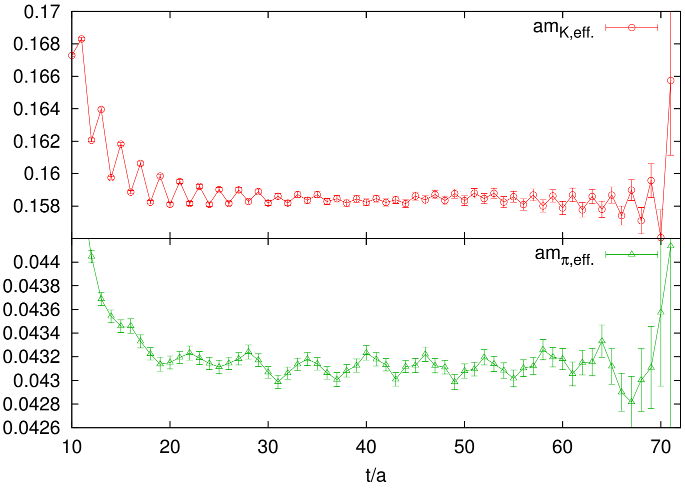
Effective masses, $\beta = 3.7000$



Effective masses, $\beta = 3.8400$



Effective masses, $\beta = 4.0126$



Outline

Disconnected contribution

$$\langle j_\mu(x) j_{\bar\mu}(\bar x)\rangle_{\text{disc.}} = - \left[\sum_f \frac{Q_f}{4} \operatorname{Im}\operatorname{tr}_c \left(U_{x,\mu} {M^{(f)}}^{-1}_{x+\mu,x}\right)\right] \times \left[\sum_{\bar f} \frac{Q_{\bar f}}{4} \operatorname{Im}\operatorname{tr}_c \left(U_{\bar x,\bar\mu} {M^{(\bar f)}}^{-1}_{\bar x+\bar\mu,\bar x}\right)\right]$$

- $\Pi(Q^2)$ depends only on $Q^2 \quad \longrightarrow \quad$ take $Q = (\underline{\mathbf{0}}, q)$ and $\mu = 1, 2, 3$

$$\langle j_\mu(t) j_{\bar\mu}(\bar t)\rangle_{\text{disc.}} = -\frac{1}{V} \times \underbrace{\left[\sum_{\underline{\mathbf{x}}} \sum_f \frac{Q_f}{4} \operatorname{Im}\operatorname{tr}_c \left(U_{\underline{\mathbf{x}},t,\mu} {M^{(f)}}^{-1}_{\underline{\mathbf{x}}+\mu,t\,;\,\underline{\mathbf{x}},t}\right)\right]}_{j_\mu(t)} \times \underbrace{\left[\sum_{\underline{\bar{\mathbf{x}}}} \sum_{\bar f} \frac{Q_{\bar f}}{4} \operatorname{Im}\operatorname{tr}_c \left(U_{\underline{\bar{\mathbf{x}}},\bar t,\bar\mu} {M^{(\bar f)}}^{-1}_{\underline{\bar{\mathbf{x}}}+\bar\mu,\bar t\,;\,\underline{\bar{\mathbf{x}}},\bar t}\right)\right]}_{j_{\bar\mu}(\bar t)}$$

- Way to proceed: calculate $j_\mu(t)$ on each configuration, then correlate with itself.

Reduction to even sites

$$\tilde{j}_\mu(t) = \sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr}_c \left(U_{\underline{\mathbf{x}},t,\mu} {M^{-1}}_{\underline{\mathbf{x}}+\mu,t\,;\,\underline{\mathbf{x}},t} \right) =$$

$$= \sum_{\underline{\mathbf{x}}\text{ even}} \mathrm{Im}\,\mathrm{tr}_c \left(U_{\underline{\mathbf{x}},t,\mu} {M^{-1}}_{\underline{\mathbf{x}}+\mu,t\,;\,\underline{\mathbf{x}},t} \right) + \sum_{\underline{\mathbf{x}}\text{ odd}} \mathrm{Im}\,\mathrm{tr}_c \left(U_{\underline{\mathbf{x}},t,\mu} {M^{-1}}_{\underline{\mathbf{x}}+\mu,t\,;\,\underline{\mathbf{x}},t} \right)$$

Since $\quad {M^{-1}}_{x+\mu,x} = \epsilon_{x+\mu}\,\epsilon_x \left(M^{-1}\right)^\dagger_{x+\mu,x} = -\left(M^{-1}\right)^\dagger_{x+\mu,x}$,

Odd part can be rewritten:

$$\sum_{\underline{\mathbf{x}}\text{ odd}} \mathrm{Im}\,\mathrm{tr}_c \left(U_{\underline{\mathbf{x}},t,\mu} {M^{-1}}_{\underline{\mathbf{x}}+\mu,t\,;\,\underline{\mathbf{x}},t} \right) = -\sum_{\underline{\mathbf{x}}\text{ odd}} \mathrm{Im}\,\mathrm{tr}_c \left(\left(M^{-1}\right)^\dagger_{\underline{\mathbf{x}}+\mu,t\,;\,\underline{\mathbf{x}},t} U_{\underline{\mathbf{x}},t,\mu} \right) =$$

$$= \sum_{\underline{\mathbf{x}}\text{ odd}} \mathrm{Im}\,\mathrm{tr}_c \left(U^\dagger_{\underline{\mathbf{x}},t,\mu} {M^{-1}}_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}}+\mu,t} \right) = \sum_{\underline{\mathbf{x}}\text{ even}} \mathrm{Im}\,\mathrm{tr}_c \left(U^\dagger_{\underline{\mathbf{x}}-\mu,t,\mu} {M^{-1}}_{\underline{\mathbf{x}}-\mu,t\,;\,\underline{\mathbf{x}},t} \right)$$

Combined:

$$\tilde{j}_\mu(t) = \sum_{\underline{\mathbf{x}}\text{ even}} \mathrm{Im}\,\mathrm{tr}_c \left(\left(U^{+-}_\mu\, M^{-1} \right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t} \right)$$

U^{+-}_μ = (covariant shift in direction μ) + (covariant shift in direction $-\mu$)

Estimation with random vectors

$$\tilde{j}_\mu(t) = \sum_{\underline{\mathbf{x}}\text{ even}} \mathrm{Im}\,\mathrm{tr_c}\left(\left(U_\mu^{+-}\,M^{-1}\right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t}\right)$$

- N random vectors $\xi^{(r)}_{\underline{\mathbf{x}},t,a}$ on even sites, $\left\langle \xi^{(r)}_{\underline{\mathbf{x}},t,a}\left(\xi^{(r)}_{\bar{\underline{\mathbf{x}}},\bar{t},\bar{a}}\right)^*\right\rangle = \delta_{\underline{\mathbf{x}},\bar{\underline{\mathbf{x}}}}\,\delta_{t,\bar{t}}\,\delta_{a,\bar{a}}$

$$\tilde{j}_\mu(t) = \left\langle \frac{1}{N}\sum_{r=1}^{N} \mathrm{Im}\left(\sum_{\underline{\mathbf{x}}\text{ even}}\sum_{\bar{\underline{\mathbf{x}}}\text{ even}}\sum_{\bar{t}}\sum_{a,\bar{a}}\left(\xi^{(r)}_{\underline{\mathbf{x}},t,a}\right)^*\left(U_\mu^{+-}\,M^{-1}\right)_{\underline{\mathbf{x}},t,a\,;\,\bar{\underline{\mathbf{x}}},\bar{t},\bar{a}}\,\xi^{(r)}_{\bar{\underline{\mathbf{x}}},\bar{t},\bar{a}}\right)\right\rangle =$$

$$= \left\langle \frac{1}{N}\sum_{r=1}^{N}\mathrm{Im}\left\langle \xi^{(r)}\middle|\,U_\mu^{+-}\,M^{-1}\,\xi^{(r)}\right\rangle_t\right\rangle = \left\langle \frac{1}{N}\sum_{r=1}^{N} j_\mu^{(r)}(t)\right\rangle$$

Our choice: Z_2 random sources.

- We use isospin symmetric masses: $m_u = m_d = m_l$

$$j^{(u)} + j^{(d)} + j^{(s)} = \frac{2}{3}\tilde{j}^{(l)} - \frac{1}{3}\tilde{j}^{(l)} - \frac{1}{3}\tilde{j}^{(s)} = \frac{1}{3}\cdot\left(\tilde{j}^{(l)} - \tilde{j}^{(s)}\right) = \frac{1}{3}\tilde{j}^{(l-s)}$$

Noise reduction

1. Use same random vectors for l and s [Gülpers *et al.* 2014]

$$\tilde{j}^{(l)} - \tilde{j}^{(s)} = \left\langle \frac{1}{N} \sum_{r=1}^{N} \mathrm{Im} \left\langle \xi^{(r)} \middle| U_\mu^{+-} \left(M_l^{-1} - M_s^{-1} \right) \xi^{(r)} \right\rangle_t \right\rangle$$

$$M_l^{-1} - M_s^{-1} = \frac{m_s - m_l}{M_l M_s}$$

$\longrightarrow$ UV-part of noise is suppressed

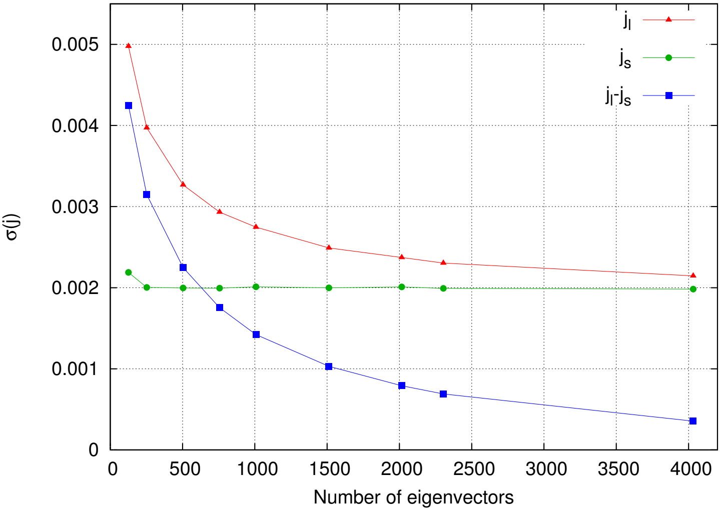
2. Compute lowest N_{eig} eigenpairs of M: $\lambda_i,\ v_i$

$$M^{-1} = \sum_{i=1}^{N_{\mathrm{eig}}} \frac{1}{\lambda_i} \left| v_i \right\rangle \left\langle v_i \right| + M^{-1} P, \qquad P = \left(1 - \sum_{i=1}^{N_{\mathrm{eig}}} \left| v_i \right\rangle \left\langle v_i \right| \right)$$

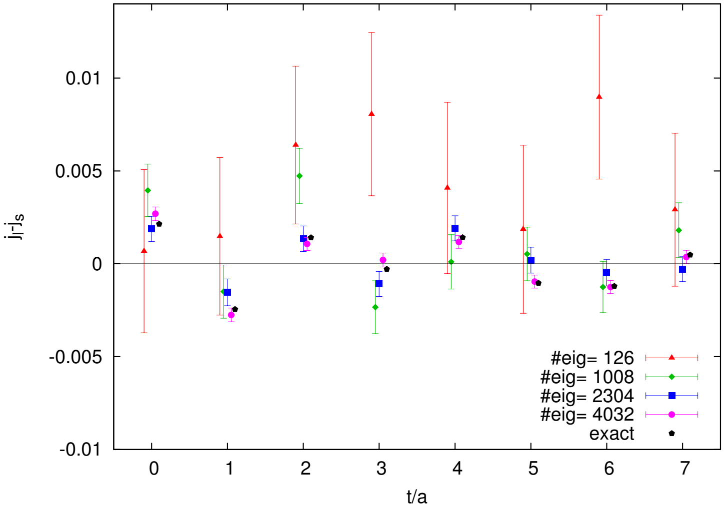
$$\tilde{j}_\mu(t) = \sum_{i=1}^{N_{\mathrm{eig}}} \mathrm{Im} \left[\frac{1}{\lambda_i} \left\langle v_i \middle| U_\mu^{+-} v_i \right\rangle_t \right] + \left\langle \frac{1}{N} \sum_{r=1}^{N} \mathrm{Im} \left\langle P \xi^{(r)} \middle| U_\mu^{+-} M^{-1} P \xi^{(r)} \right\rangle_t \right\rangle$$

$\longrightarrow$ IR-part of noise is suppressed

Noise of j_μ on one configuration, $\beta = 3.7000$



Noise of j_μ on one configuration, $\beta = 3.7000$



Truncated solver method

[Collins *et al.* 2007]

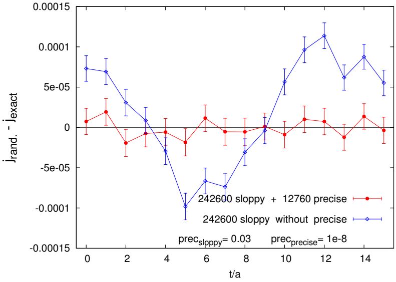
- Need to compute $M^{-1}\xi$ on many random vectors.
- Reduce work: $N_p \ll N_s, \quad N_p + N_s = N$

  For $r = 1, 2, \ldots, N_s,$ compute $M^{-1}\xi^{(r)}$ with low precision $\longrightarrow j_s^{(r)}$

  For $r = N_s+1, \ldots, N,$ both low and high precision $\longrightarrow j_s^{(r)}, j_p^{(r)}$

$$j = \Big\langle (\text{sloppy}) \Big\rangle + \Big\langle (\text{precise}) - (\text{sloppy}) \Big\rangle =$$

$$= \left\langle \frac{1}{N_s} \sum_{r=1}^{N_s} j_s^{(r)} \right\rangle + \left\langle \frac{1}{N_p} \sum_{r=N_s+1}^{N} \left(j_p^{(r)} - j_s^{(r)} \right) \right\rangle$$



Estimator for the correlator

$$\langle j_\mu(t) j_{\bar\mu}(\bar t)\rangle_{\text{disc.}} = -\frac{1}{V}\times$$

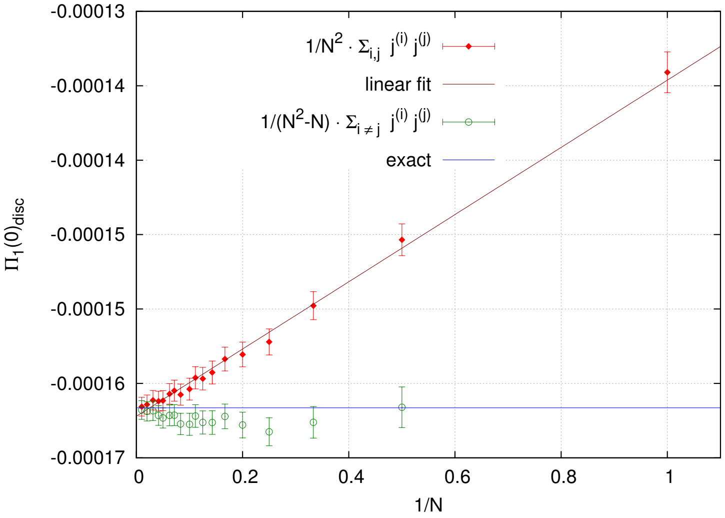
$$\underbrace{\left[\sum_{\underline{\mathbf{x}}}\sum_f \frac{Q_f}{4}\,\mathrm{Im}\,\mathrm{tr}_c\left(U_{\underline{\mathbf{x}},t,\mu} {M^{(f)}}^{-1}_{\underline{\mathbf{x}}+\mu,t\,;\,\underline{\mathbf{x}},t}\right)\right]}_{j_\mu(t)} \times \underbrace{\left[\sum_{\bar{\underline{\mathbf{x}}}}\sum_{\bar f} \frac{Q_{\bar f}}{4}\,\mathrm{Im}\,\mathrm{tr}_c\left(U_{\bar{\underline{\mathbf{x}}},\bar t,\bar\mu} {M^{(\bar f)}}^{-1}_{\bar{\underline{\mathbf{x}}}+\bar\mu,\bar t\,;\,\bar{\underline{\mathbf{x}}},\bar t}\right)\right]}_{j_{\bar\mu}(\bar t)}$$

- $\frac{1}{N}\sum_{r=1}^{N} j_\mu^{(r)}(t)$ is unbiased estimator for $\tilde{j}_\mu(t)$.
- But $\left(\frac{1}{N}\sum_{r=1}^{N} j_\mu^{(r)}(t)\right)\left(\frac{1}{N}\sum_{\bar r=1}^{N} j_{\bar\mu}^{(\bar r)}(\bar t)\right)$ is biased estimator for $\tilde{j}_\mu(t)\tilde{j}_{\bar\mu}(\bar t)$:

$$\tilde{j}_\mu(t)\tilde{j}_{\bar\mu}(\bar t) - \left\langle \left(\frac{1}{N}\sum_{r=1}^{N} j_\mu^{(r)}(t)\right)\left(\frac{1}{N}\sum_{\bar r=1}^{N} j_{\bar\mu}^{(\bar r)}(\bar t)\right)\right\rangle = \mathcal{O}\left(\frac{1}{N}\right)$$

- Unbiased estimator: $\frac{1}{N^2-N}\sum_{r\neq\bar r} j_\mu^{(r)}(t) j_{\bar\mu}^{(\bar r)}(\bar t)$

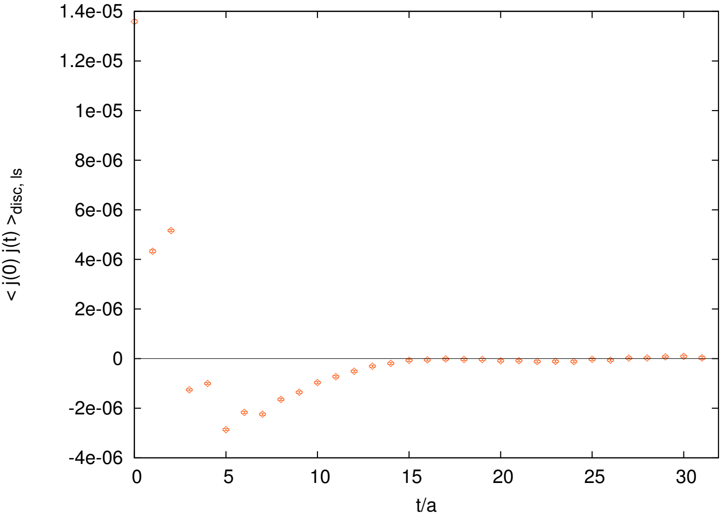
Unbiasing


1/N^2 · Σ_{i,j} j^(i) j^(j)
linear fit
1/(N^2-N) · Σ_{i≠j} j^(i) j^(j)
exact
-0.00013
-0.00014
-0.00014
-0.00015
-0.00015
-0.00016
-0.00017
0
0.2
0.4
0.6
0.8
1
1/N
Π_1(0)_disc

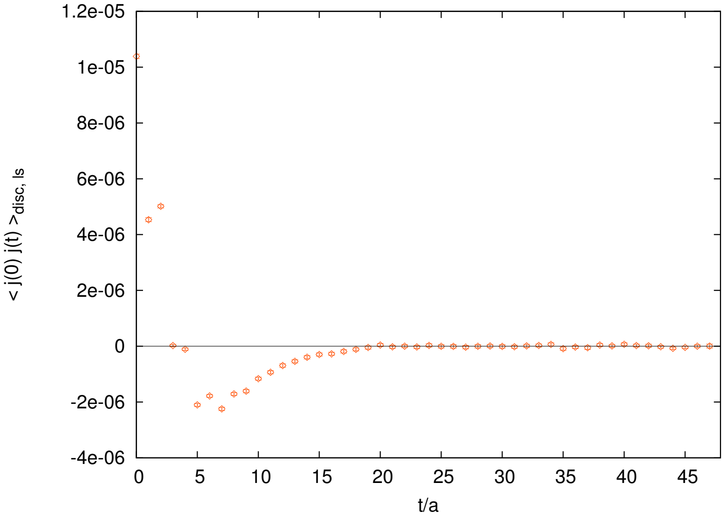
Disconnected computation parameters

β	a [fm]	$T \times L$	#eig	#rand
3.7000	0.134	64×48	1920	9000
3.7500	0.118	96×56	960	6000
3.7753	0.111	84×56	960	6144
3.8400	0.095	96×64	960	3600
3.9200	0.078	128×80	960	6144

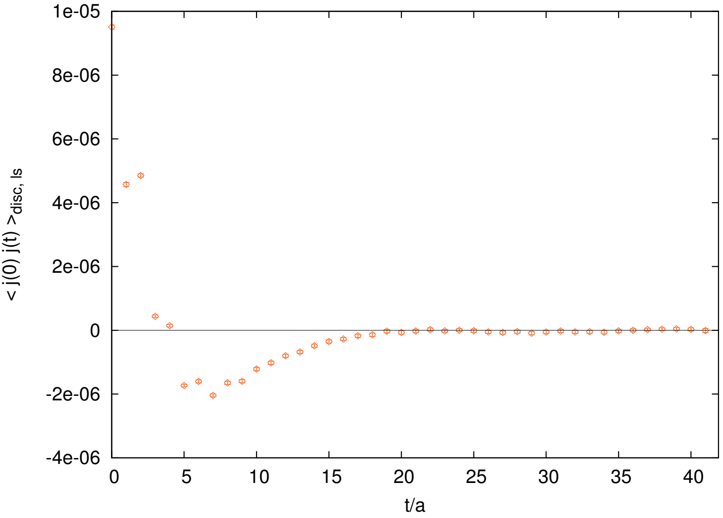
Disconnected correlator, $\beta = 3.7000$



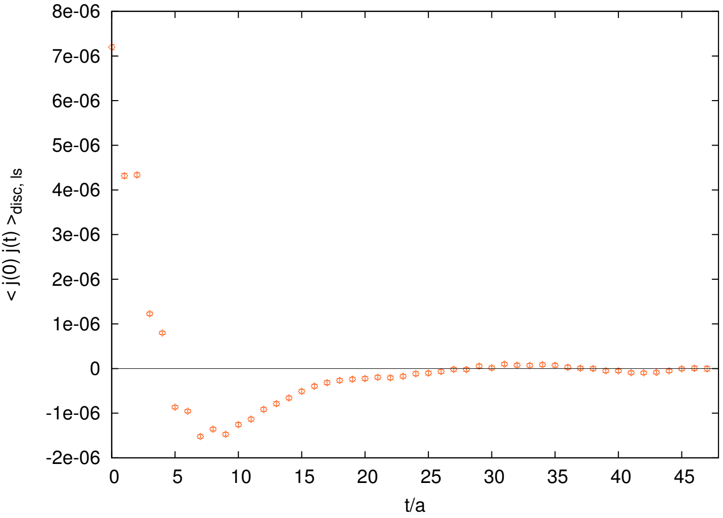
Disconnected correlator, $\beta = 3.7500$



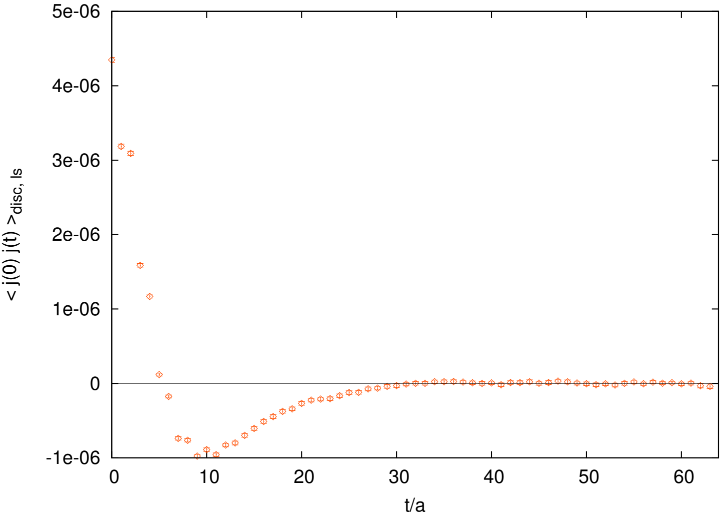
Disconnected correlator, $\beta = 3.7753$



Disconnected correlator, $\beta = 3.8400$



Disconnected correlator, $\beta = 3.9200$



Inclusion of charm in the current

$$j^{(u)} + j^{(d)} + j^{(s)} + j^{(c)} = \frac{1}{3} \cdot \left(\tilde{j}^{(l)} - \tilde{j}^{(s)} \right) + \frac{2}{3} \tilde{j}^{(c)}$$

- No cancellation for $\tilde{j}^{(c)}$
- IR-noise is irrelevant

$\longrightarrow$ UV-noise is dominant

- Noise reduction via hopping parameter expansion (HPE)

Hopping parameter expansion

[Thron *et al.* 1998; Bali *et al.* 2010]

- $M = m + D,\qquad M^\dagger = m - D,\qquad M^\dagger M = m^2 - D^2$

$$M^{-1} = M^\dagger \left(M^\dagger M\right)^{-1} = \frac{m-D}{m^2-D^2}$$

- Then the current:

$$\tilde{j}_\mu(t) = \sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr}_c \left(\left(U_\mu \frac{m-D}{m^2-D^2} \right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t} \right) =$$

$$= \sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr}_c \left(\left(U_\mu (-D) \frac{1}{m^2-D^2} \right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t} \right)$$

- Rewrite the inverse using n^{th} degree polynomial $\quad a_0 + a_1 x + \cdots + a_n x^n$

$$\frac{1}{m^2-D^2} = \sum_{k=0}^{n} a_k \left(D^2\right)^k + \frac{1}{m^2-D^2} \sum_{k=0}^{n+1} \underbrace{\left(a_{k-1} - m^2\, a_k\right)}_{b_k} \cdot \left(D^2\right)^k$$

with $\quad a_{-1} = 1 \quad$ and $\quad a_{n+1} = 0.$

Hopping parameter expansion

- Current consists of two parts:

$$\tilde{j}_\mu(t) = \sum_{k=0}^{n} a_k \overbrace{\sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr}_c \left(U_\mu(-D)\,(D^2)^k \right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t}}^{K_\mu^{(k)}(t)} + \\ + \sum_{k=0}^{n+1} b_k \underbrace{\sum_{\underline{\mathbf{x}}} \mathrm{Im}\,\mathrm{tr}_c \left(U_\mu(-D)\,\frac{(D^2)^k}{m^2 - D^2} \right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t}}_{R_\mu^{(k)}(t)}$$

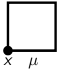
- Recipe:
  - Calculate $K_\mu^{(k)}(t)$ exactly.
  - Calculate $R_\mu^{(k)}(t)$ using random vectors:

$$R_\mu^{(k)}(t) = \frac{1}{N} \sum_{r=1}^{N} \mathrm{Im} \left\langle \xi^{(r)} \middle| U_\mu\,(-D)\,\frac{(D^2)^k}{m^2 - D^2}\,\xi^{(r)} \right\rangle_t$$

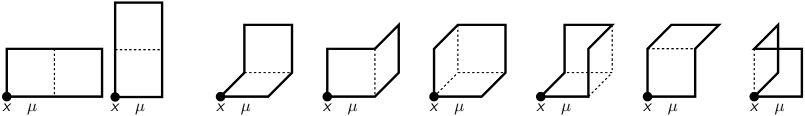
    Use same random vector set for all $k = 0, 1, \ldots, n+1$.
  - Choose coefficients a_k such that the noise of $\sum_k b_k R_\mu^{(k)}(t)$ is minimal.

Hopping parameter expansion

$$K_\mu^{(k)}(t) = \sum_{\underline{\mathbf{x}}} \operatorname{Im} \operatorname{tr}_c \left(U_\mu (-D) \left(D^2\right)^k \right)_{\underline{\mathbf{x}},t\,;\,\underline{\mathbf{x}},t}$$

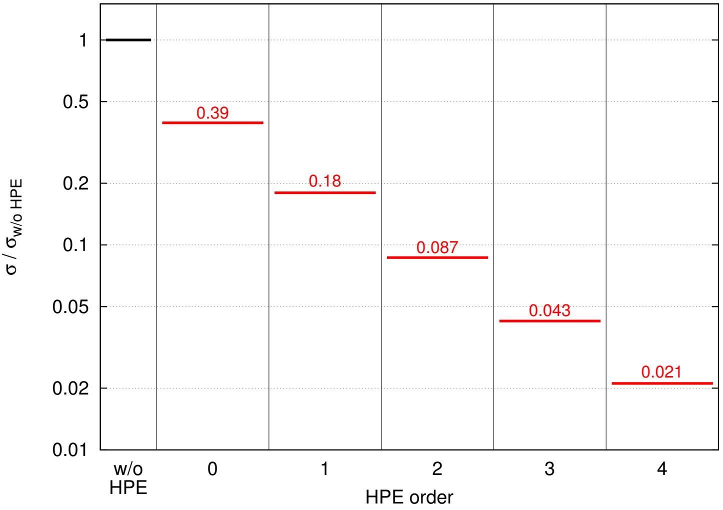
- Computing $K_\mu^{(k)}(t)$ $\longrightarrow$ calculate loops
  - $k = 0$ $K_\mu^{(0)}(t) = 0$
  - $k = 1$ 1 loop of length 4 $\longrightarrow$ 

  - $k = 2$ 8 additional loops of length 6

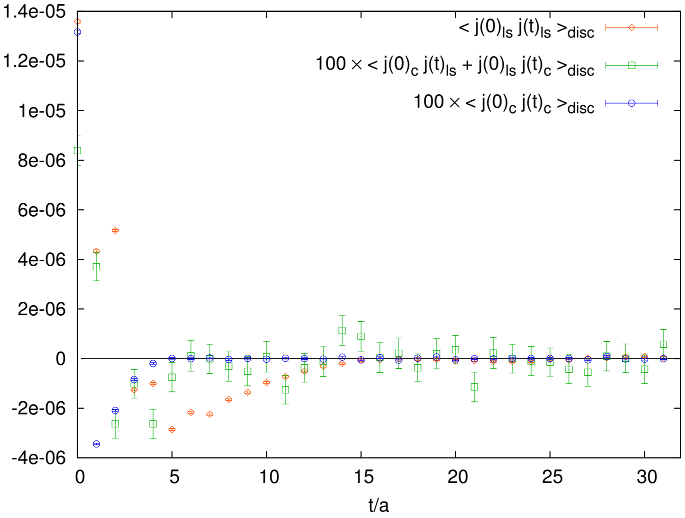


  - $k = 3$ 167 additional loops of length 8
  - $k = 4$ 4402 additional loops of length 10
  - ...

Hopping parameter expansion



Effect of charm in correlator, $\beta = 3.7000$



Outline

Reduction to even sites

- $\Pi(Q^2)$ depends only on $Q^2 \quad \longrightarrow \quad$ take $Q = (\underline{\mathbf{0}}, q)$ and $\mu = 1, 2, 3$

$$\langle j_\mu(t) j_{\bar\mu}(\bar t)\rangle_{\text{conn.}} =$$
$$= -\sum_{\underline{\mathbf{x}},\underline{\bar{\mathbf{x}}}} \sum_f \frac{Q_f^2}{8V} \mathrm{Re}\,\mathrm{tr_c} \left({M^{(f)}}^{-1}_{\underline{\mathbf{x}}+\mu,t;\underline{\bar{\mathbf{x}}},\bar t} U_{\bar{\mathbf{x}},\bar t,\bar\mu} {M^{(f)}}^{-1}_{\underline{\bar{\mathbf{x}}}+\bar\mu,\bar t;\underline{\mathbf{x}},t} U_{\underline{\mathbf{x}},t,\mu} + \right.$$
$$\left. + {M^{(f)}}^{-1}_{\underline{\mathbf{x}}+\mu,t;\underline{\bar{\mathbf{x}}}+\bar\mu,\bar t} U^\dagger_{\underline{\bar{\mathbf{x}}},\bar t,\bar\mu} {M^{(f)}}^{-1}_{\underline{\bar{\mathbf{x}}},\bar t;\underline{\mathbf{x}},t} U_{\underline{\mathbf{x}},t,\mu} \right)$$

- Sufficient to take even sites at source

$$\langle j_\mu(t) j_{\bar\mu}(\bar t)\rangle_{\text{conn.}} =$$
$$= -\sum_{\underline{\bar{\mathbf{x}}}\text{ even}} \sum_{\underline{\mathbf{x}}} \sum_f \frac{Q_f^2}{4V} \mathrm{Re}\,\mathrm{tr_c} \left({M^{(f)}}^{-1}_{\underline{\mathbf{x}}+\mu,t;\underline{\bar{\mathbf{x}}},\bar t} U^{+-}_{\bar{\mathbf{x}},\bar t,\bar\mu} {M^{(f)}}^{-1}_{\underline{\bar{\mathbf{x}}}+\bar\mu,\bar t;\underline{\mathbf{x}},t} U^{+-}_{\underline{\mathbf{x}},t,\mu} \right)$$

with

$$U^{+-}_\mu = (\text{covariant shift in direction } \mu) + (\text{covariant shift in direction } -\mu)$$

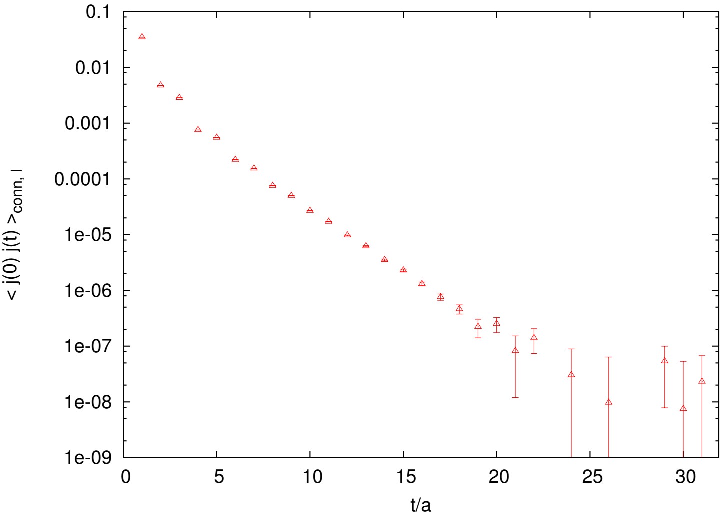
Connected computation parameters

- Use randomly placed point sources ξ
- Precondition inversion using eigenvectors

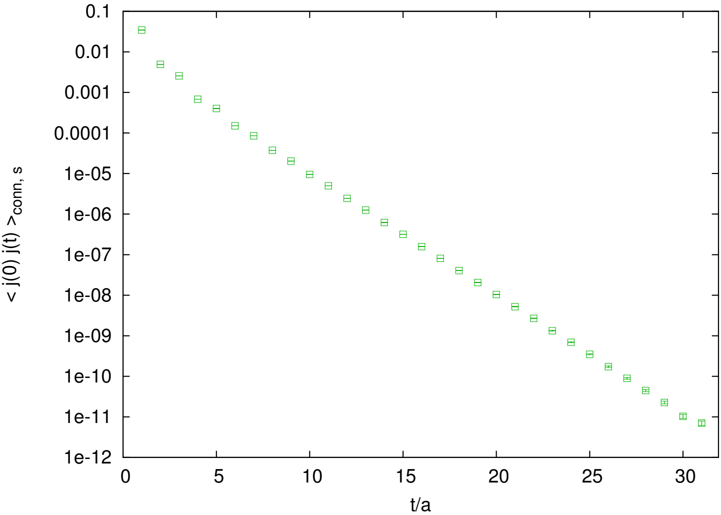
$$M^{-1}\left|\xi\right\rangle = \sum_{i=1}^{N_{\text{eig}}} \frac{1}{\lambda_i} \left|v_i\right\rangle \left\langle v_i \mid \xi\right\rangle + M^{-1}\left|P\xi\right\rangle$$

β	a [fm]	$T \times L$	#src l	#src s	#src c
3.7000	0.134	64×48	768	128	64
3.7500	0.118	96×56	768	64	64
3.7753	0.111	84×56	768	64	64
3.8400	0.095	96×64	768	64	64
3.9200	0.078	128×80	768	64	64
4.0126	0.064	144×96	768	64	64

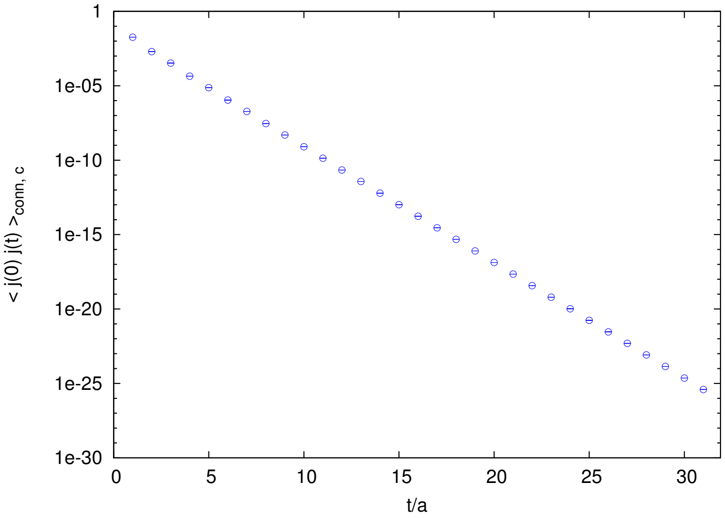
Light connected correlator, $\beta = 3.7000$



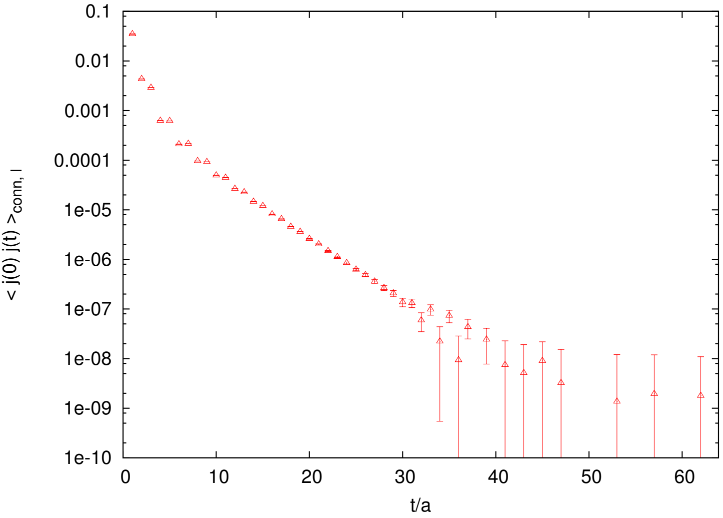
Strange connected correlator, $\beta = 3.7000$



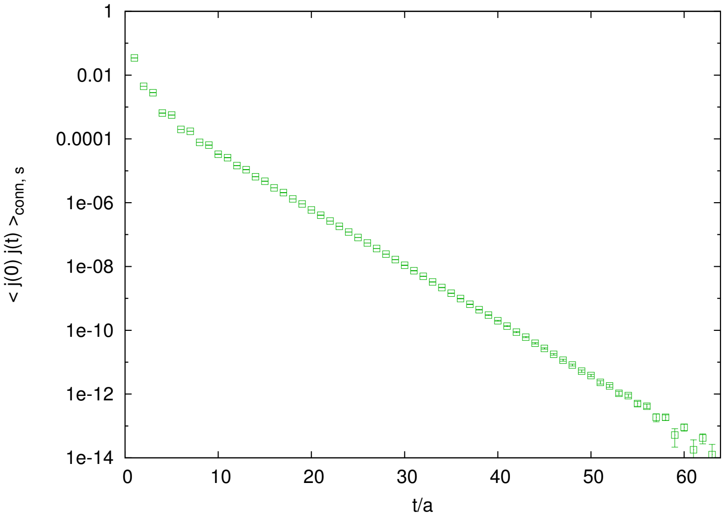
Charm connected correlator, $\beta = 3.7000$



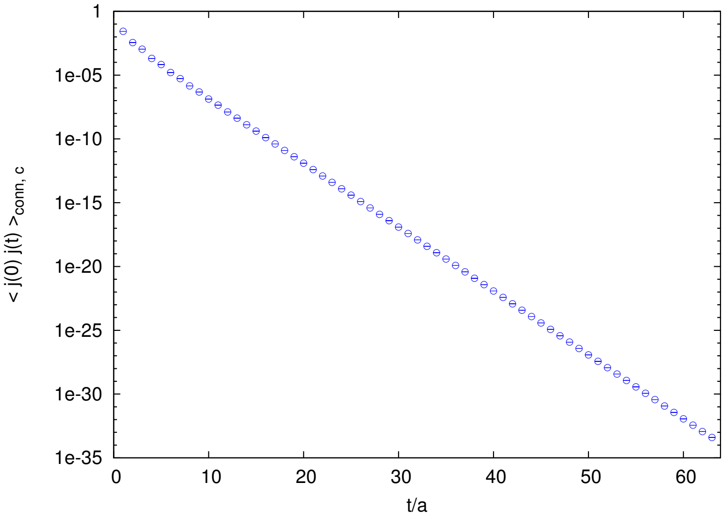
Light connected correlator, $\beta = 3.9200$



Strange connected correlator, $\beta = 3.9200$



Charm connected correlator, $\beta = 3.9200$



Outline

Upper and lower bounds

[Lehner 2014]

$$\langle j(0) j(t) \rangle = e^2 \left[\frac{5}{9} C^l(t) + \frac{1}{9} C^s(t) + \frac{4}{9} C^c(t) + \frac{1}{9} C^{\text{disc}}(t) \right]$$

$$\Pi_n = \frac{5}{9} \Pi_n^l + \frac{1}{9} \Pi_n^s + \frac{4}{9} \Pi_n^c + \frac{1}{9} \Pi_n^{\text{disc}}(t)$$

$$\Pi_n^f = \frac{(-1)^{n+1}}{(2(n+1))!} \sum_t t^{2(n+1)} \, C^f(t)$$

- Isospin triplet/singlet correlators

$$C^{I=1} = \frac{1}{2} C^l \qquad C^{I=0} = \frac{1}{18} \left[C^l + 2C^s + 8C^c + 2C^{\text{disc}} \right]$$

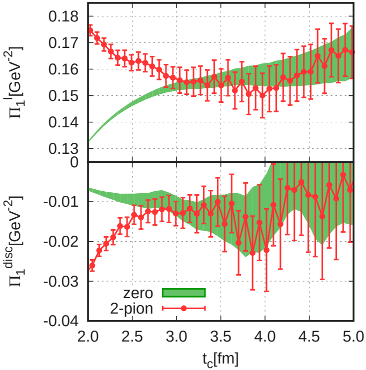
- Bounds for C^l at $t > t_c$:

$$0 \leq C^l(t) \leq C^l(t_c) \exp\left(-E_{2\pi}(t - t_c)\right)$$

- At large t: $\quad 0 \leq C^{I=0}(t) \ll C^{I=1}(t) \leq (2\pi \text{ state})$
- Bounds for C^{disc} at $t > t_c$:

$$0 \geq 2C^s(t) + 8C^c(t) + 2C^{\text{disc}}(t) \geq -C^l(t_c) \exp\left(-E_{2\pi}(t - t_c)\right)$$

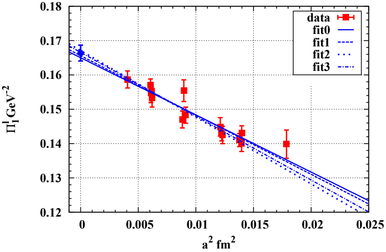
Upper and lower bounds

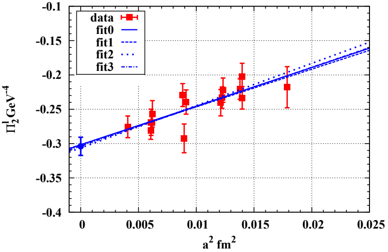


- C^l: $t_c = 3.1\,\mathrm{fm}$
- C^{disc}: $t_c = 2.7\,\mathrm{fm}$
- For $t > t_c$:

  replace $C^{l,\mathrm{disc}}$ with average of upper and lower bounds

Π_1^l and Π_2^l

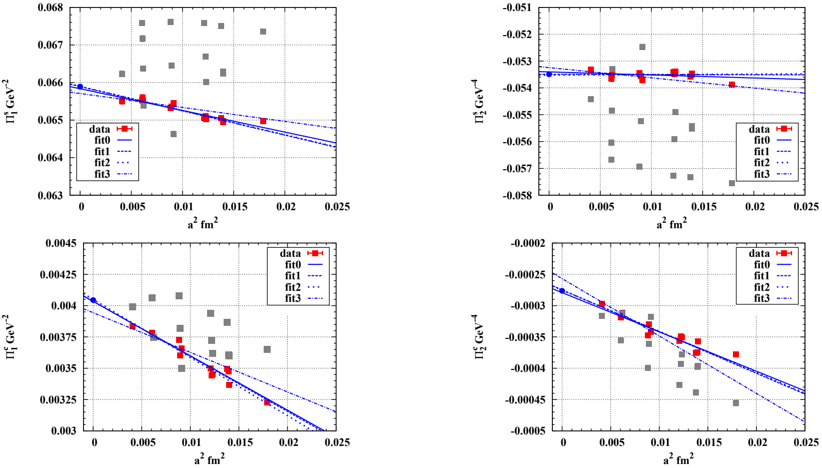




- Combined fit to all ensembles:
  - continuum limit
  - interpolation to physical point

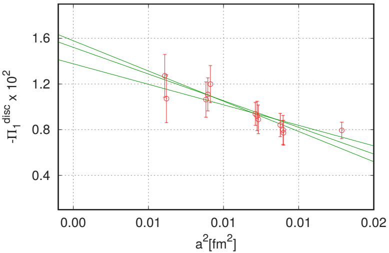
$$\Pi_n^{\mathrm{lat}} = \Pi_n \left(1 + \mathrm{Pade}(A_i a^i) + B(M_\pi^2 - M_{\pi,\mathrm{phys.}}^2) + C(M_K^2 - M_{K,\mathrm{phys.}}^2)\right)$$

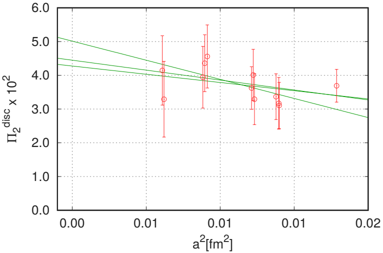
Π_n^s and Π_n^c



grey points: without M_K correction

Π_1^{disc} and Π_2^{disc}





Summary table of moments

	Π_1 [GeV^{-2}]	Π_2 [GeV^{-4}]
light	0.1653(17)(16)	−0.295(10)(7)
strange $\times 10^2$	6.57(1)(3)	−5.33(1)(4)
charm $\times 10^4$	40.3(2)(6)	−2.66(3)(11)
disconnected $\times 10^2$	−1.5(2)(1)	4.4(1.0)(0.4)
$I = 0$	0.0167(2)(2)	−0.018(1)(1)
$I = 1$	0.0827(8)(8)	−0.147(5)(4)
total	0.0993(10)(9)	−0.165(6)(4)
I=1 FV corr.	0.0006(23)	−0.016(10)
total + FV	0.0999(10)(9)(23)(13)	−0.181(6)(4)(10)(2)

- cf. Phenomenology [Benayoun *et al.* 2016]

$$\Pi_1 = 0.990(7)\,\text{GeV}^{-2} \qquad \Pi_2 = -0.206(2)\,\text{GeV}^{-4}$$

Outline

Conclusions & Outlook

	Π_1 [GeV^{-2}]	Π_2 [GeV^{-4}]
total + FV	0.0999(10)(9)(23)(13)	−0.181(6)(4)(10)(2)

- Preliminary estimate for $a_\mu^{\text{LO–HVP}}$:

$$a_\mu^{\text{LO–HVP}} = 4\alpha^2 \int_0^\infty \mathrm{d}Q^2\, f(Q^2)\, \frac{Q^2 \cdot \Pi_1}{1 - Q^2 \cdot \frac{\Pi_2}{\Pi_1}}$$

$$\sim [691 \pm \mathcal{O}(10)|_{\text{stat.}} \pm \mathcal{O}(10)|_{\text{syst.}} \pm \mathcal{O}(10)|_{\text{FV}}] \times 10^{10}$$

- Outlook
  - Investigate FV from lattice data
  - Compute $a_\mu^{\text{LO–HVP}}$ with all statistical/systematic uncertainties
  - Include isospin breaking and EM effects