On Symmetry-Protected Topological States: From Free Fermions to the Haldane Phase

Martin R. Zirnbauer (Cologne)

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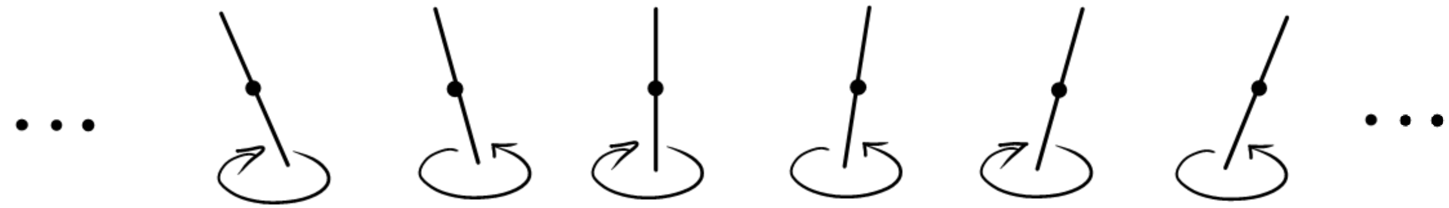
Introduction

Antiferromagnetic quantum spin chains



$$H = J \sum_n S_n \cdot S_{n+1} \quad (J > 0)$$

Spin 1/2 (Heisenberg chain; Bethe Ansatz):

gapless ↖ Lieb–Schultz–Mattis 1961

Spin 1: Haldane (1983) predicts excitation gap from O(3) nonlinear sigma model with topological angle $\Theta = 2\pi |S|$

F. D. M. Haldane

Nobel Prize Physics 2016

— neutron scattering experiments on $CsNiCl_3$

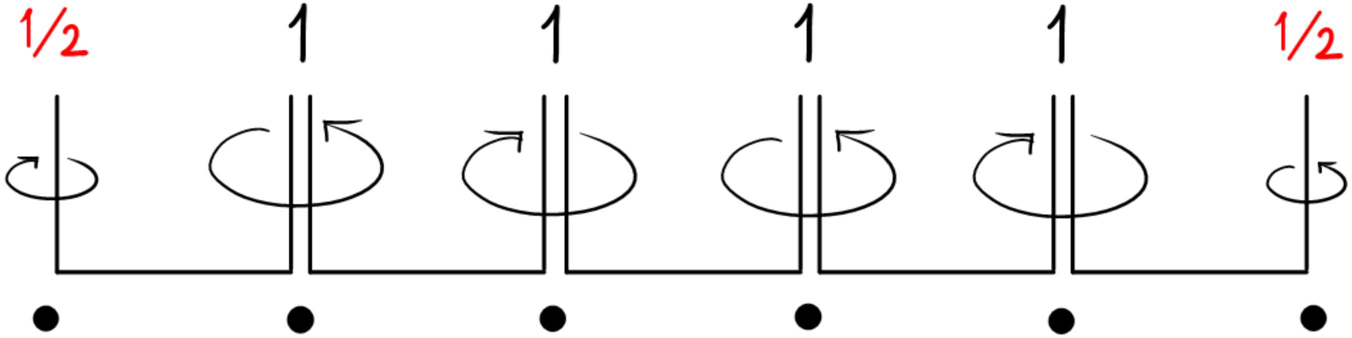
Haldane phase (spin 1)

exponential decay of correlations (↔ mass gap), BUT

hidden topological order: edge excitations of spin 1/2 (!)

→ "fractionalization"

Example: Affleck–Kennedy–Lieb–Tasaki (AKLT)



matrix-product state

New paradigm (beyond Landau–Ginzburg–Wilson)

for (short-range) topological order

Haldane phase as an SPT phase

symmetry-protected topological

Q: protection by what symmetry?

A1 (Pollmann–Berg–Turner–Oshikawa, 2010)

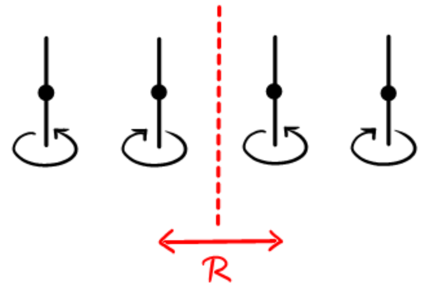
T (time reversal)

OR $\mathbb{Z}_2 \times \mathbb{Z}_2$ (dihedral group)

OR R (space refl)

~~A1~~ (Anfuso–Rosch, 2007): local charge fluctuations!

A2 (Moudgalya–Pollmann, 2015): bond inversion



disorder?

Def. Two Hamiltonians H_0 and H_1 are said to be in the same topological phase (or topologically equivalent, $H_0 \sim H_1$) if there exists a homotopy $[0,1] \ni t \mapsto H(t)$, $H(0) = H_0$, $H(1) = H_1$ such that $H(t)$ has a unique ground state with a finite energy gap for excitations, for all t.

Def. A Hamiltonian H is said to be of symmetry class AIII if

$$H = e^{i\theta Q} H e^{-i\theta Q} = C H C^{-1}.$$

(charge operator Q, particle-hole conjugation C)

$\rightsquigarrow$ symmetry-protected topological phase of class AIII

A3 (here): free fermion SPT phase in 1D with symmetries Q, C $\overset{\mathrm{AIII}}{\sim}$ Haldane phase

OUTLINE

0. Introduction

I. Dyson's Threefold Way

II. The Tenfold Way

III. Free fermions $\rightsquigarrow$ Haldane phase

Symmetries in quantum mechanics

Q: What's a symmetry in quantum mechanics?

A: An operator $T : \mathscr{R}\psi_1 \mapsto \mathscr{R}\psi_2$ on Hilbert rays that preserves all transition probabilities: $\left|\langle T\mathscr{R}\psi_2 , T\mathscr{R}\psi_1\rangle\right|^2 = \left|\langle \mathscr{R}\psi_2 , \mathscr{R}\psi_1\rangle\right|^2.$

Wigner's Theorem: cf. D. Freed, arXiv: 1112.2133

A symmetry T in quantum mechanics can always be represented on Hilbert space by an operator $\hat{T}$ which is either unitary or anti-unitary.

$$\langle \hat{T}\psi_2 | \hat{T}\psi_1 \rangle = \overline{\langle \psi_2 | \psi_1 \rangle}$$

Eugene P. Wigner

Remark 1: The symmetries form a group, G.

Remark 2: Symmetries commute with the Hamiltonian ($\hat{T}H = H\hat{T}$).
Thus "chiral symmetry" ($\gamma_5 D \gamma_5 = -D$) is not a symmetry.

Threefold Way (Dyson, 1962)

$$\underbrace{(\,G_0 \cup G_1 = G\,)}_{\text{symmetry group}} \times \underbrace{V}_{\text{Hilbert space}} \longrightarrow V$$

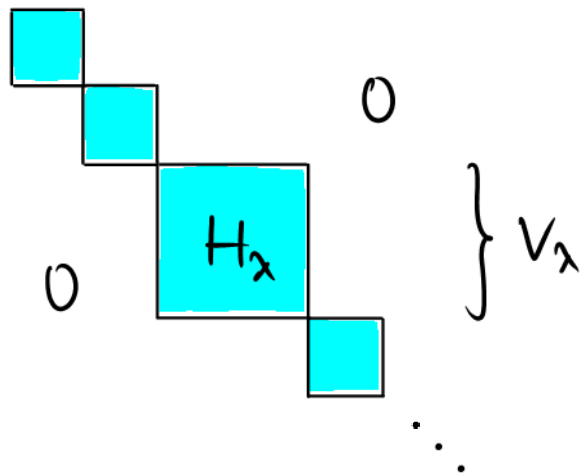
G_0 unitaries, G_1 anti-unitaries (Wigner's Thm), $G/G_0 = \mathbb{Z}_2$

- Assume G_0 reductive. Then $V = \bigoplus_\lambda V_\lambda$ ← irreps of type λ

  V_λ isotypic component

Block form of Hamiltonians:

$$H = g H g^{-1} \quad = \quad (\text{for all } g \in G_0)$$



For fixed $T \in G_1$ and variable $g \in G_0$:

$$g \mapsto T g T^{-1} \equiv a_T(g) \in G_0$$

automorphism of normal subgroup G_0

- Assume: $\exists\, T \in G_1$ s.th. $a_T^2 = id$ $\quad (\leftrightarrow T^2 \in Z(G_0))$

T "time reversal"

Then $T^2 = \bigoplus_\lambda T_\lambda^2$, $\quad T_\lambda^2 = e^{i\theta_\lambda}\, Id_{V_\lambda}$ (Schur);

and if $T V_\lambda = V_\lambda$ then $e^{i\theta_\lambda} T_\lambda = T_\lambda^2 \cdot T_\lambda = T_\lambda \cdot T_\lambda^2 = T_\lambda \cdot e^{i\theta_\lambda}$

so $T_\lambda^2 = \pm 1$.

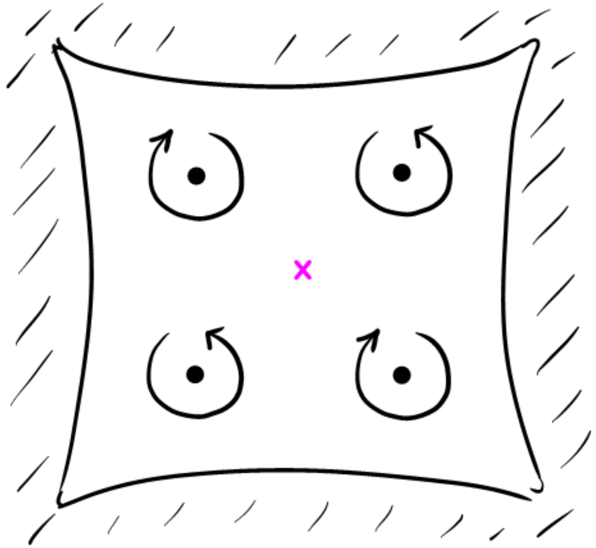
$\rightsquigarrow$ Trichotomy

Special case: G_0 Abelian

	H_λ	RMT	class
$T V_\lambda = V_{\lambda'}$, $\lambda \neq \lambda'$	Hermitian	GUE	A
$T V_\lambda = V_\lambda$, $T^2 = +1$	real symm.	GOE	AI
$T V_\lambda = V_\lambda$, $T^2 = -1$	quaternion	GSE	AII

Example.

quantum chaotic billiard
with magnetic flux insertions



$$G_0 = \{ \mathrm{id}, R_\pi \}, \quad G_1 = \{ \underbrace{R_{\pi/2}\Theta}_{=:\,T}, R_{-\pi/2}\Theta \}, \quad T^2 = R_\pi$$

$$V = V_{\text{even}} \oplus V_{\text{odd}}$$

AI   AII ← Kramers degeneracy

General case (G_0 non-Abelian).

multiplicity

$$V_\lambda = R_\lambda \otimes M_\lambda, \qquad M_\lambda = \mathrm{Hom}_{G_0}(R_\lambda, V_\lambda) \cong \mathbb{C}^{m_\lambda}$$

standard irrep

(Hamiltonian is matrix in M_λ)

Transfer T from V_λ to M_λ

J. Math. Phys. 3 (1962) 1199

The Threefold Way.
Algebraic Structure of Symmetry Groups and Ensembles in Quantum Mechanics

FREEMAN J. DYSON
Institute for Advanced Study, Princeton, New Jersey
(Received June 22, 1962)

Using mathematical tools developed by Hermann Weyl, the Wigner classification of group-representations and co-representations is clarified and extended. The three types of representation, and the three types of co-representation, are shown to be directly related to the three types of division algebra with real coefficients, namely, the real numbers, complex numbers, and quaternions. The author's theory of matrix ensembles, in which again three possible types were found, is shown to be in exact correspondence with the Wigner classification of co-representations. In particular, it is proved that the most general kind of matrix ensemble, defined with a symmetry group which may be completely arbitrary, reduces to a direct product of independent irreducible ensembles each of which belongs to one of the three known types.

II. The Tenfold Way

Tenfold Way: Setting & Result

$$\left(G_0 \cup G_1 \cup G_C \cup G_{CT} = G \right) \times \underset{\text{Fock space}}{\mathcal{F}} \longrightarrow \mathcal{F} = \overset{\text{fermions}}{\Lambda(V)}$$

G_0 arbitrary (Dyson!), $G/G_0 = \mathbb{Z}_2^T \times \mathbb{Z}_2^C$

particle-hole conjugation $C : \Lambda^{\text{half}+n}(V) \longrightarrow \Lambda^{\text{half}-n}(V)$

C anti-unitary, $C H C^{-1} = H$.

Heinzner, Huckleberry & Z. (Commun. Math. Phys., 2004) classify quadratic Hamiltonians ("free fermions") by studying the induced action of (G, H_{free}) on $W = V \oplus V^*$ (fields).

<u>Thm.</u> G_0-reduced block data $\overset{10 \longleftrightarrow 10}{\longleftrightarrow}$ classical irred. symmetric spaces

<u>Cor.</u> $H(k)$ (Bloch) is of one of 10 types ($\rightsquigarrow$ TCI).

What's a symmetric space?

Riemann tensor: $R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj} \Gamma^i_{km} - \Gamma^m_{kj} \Gamma^i_{lm}$

Def.: A (locally) symmetric space is a Riemannian manifold $X = U/K$ with covariantly constant curvature: $\nabla R = 0$.

Ex. 1: the round two-sphere $X = \mathrm{S}^2$, $\quad ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

Ex. 2: the set $X = \mathrm{Gr}_n(\mathbb{C}^N) = \mathrm{U}(N)/\mathrm{U}(n) \times \mathrm{U}(N-n)$ of all subspaces $\mathbb{C}^n \simeq V \subset \mathbb{C}^N$

Classification:

Globally symmetric spaces classified by E. Cartan (1926)

10 large families: *A*, *A*I, *A*II, *A*III, *BD*, *BD*I, *C*, *C*I, *C*II, *D*III

U, U/O, U/Sp, U/U×U, O, O/O×O, Sp, Sp/U, Sp/Sp×Sp, O/U

Tenfold Way: basic realizations ($T^2 = -1$, $C^2 = +1$)

class	G_0	T	C	physical realization
D	$\{e\}$	no	no	superconducting Majorana chain
DIII	$\{e\}$	yes	no	superfluid ^{3}He-B
AII	$U(1)_Q$	yes	no	HgTe (strong spin-orbit scattering)
CII	$U(1)_Q$	yes	yes	massless Dirac fermions, adjoint repn
A	$U(1)_Q$	no	no	quantum Hall systems
AIII	$U(1)_Q$	no	yes	polyacetylene (Su-Schrieffer-Heeger)
C	$SU(2)_{spin}$	no	no	spin-singlet superconductor (vortex phase)
CI	$SU(2)_{spin}$	yes	no	... (Meissner phase)
AI	$SU(2)_{spin} \times U(1)_Q$	yes	no	atomic nucleus; chaotic billiard
BDI	$SU(2)_{spin} \times U(1)_Q$	yes	yes	massless Dirac fermions, $G = SU(2)$

Tenfold Way

Communications in
Mathematical
Physics

Symmetry Classes of Disordered Fermions

P. Heinzner[1], A. Huckleberry[1], M.R. Zirnbauer[2]

[1] Fakultät für Mathematik, Ruhr-Universität Bochum, Germany.
E-mail: heinzner@cplx.ruhr-uni-bochum.de; ahuck@cplx.ruhr-uni-bochum.de
[2] Institut für Theoretische Physik, Universität zu Köln, Germany.
E-mail: zirn@thp.uni-koeln.de

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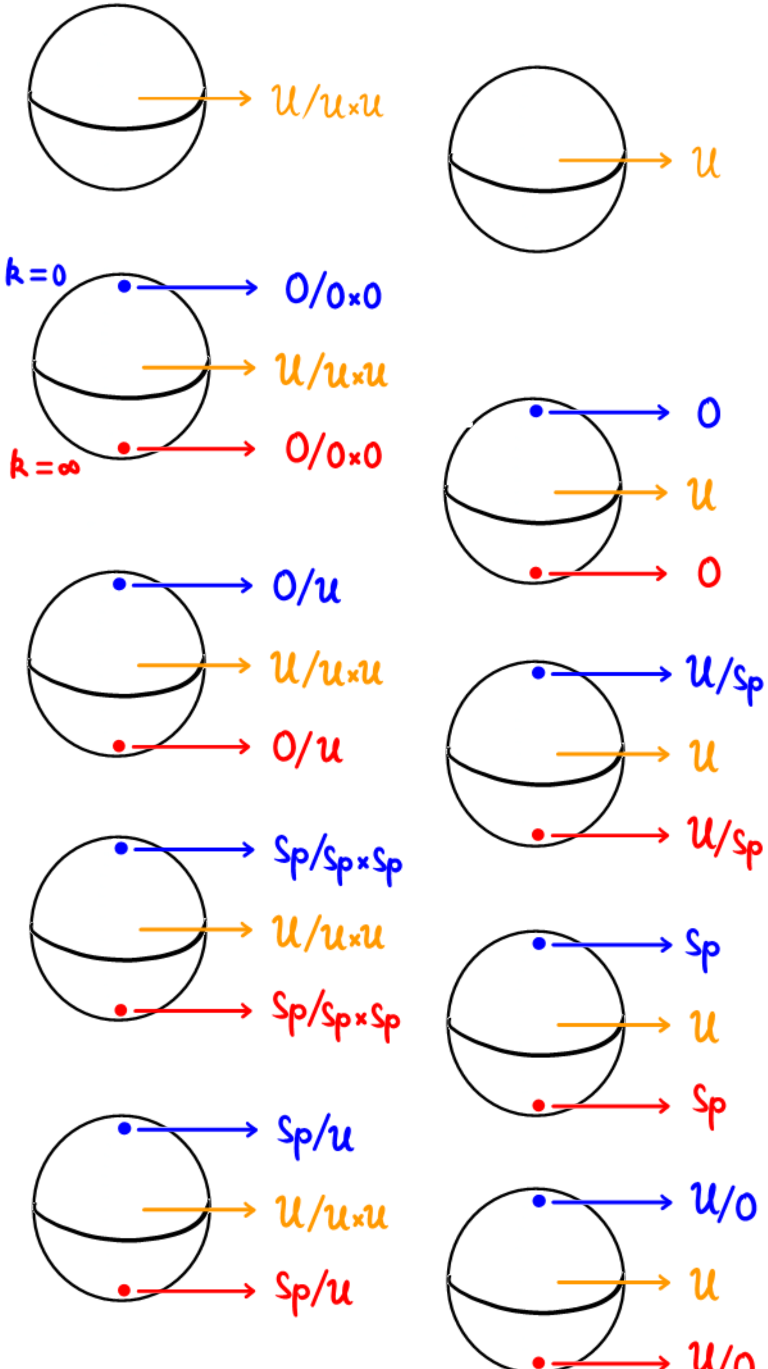
Abstract: Building upon Dyson's fundamental 1962 article known in random-matrix theory as *the threefold way*, we classify disordered fermion systems with quadratic Hamiltonians by their unitary and antiunitary symmetries. Important physical examples are afforded by noninteracting quasiparticles in disordered metals and superconductors, and by relativistic fermions in random gauge field backgrounds.

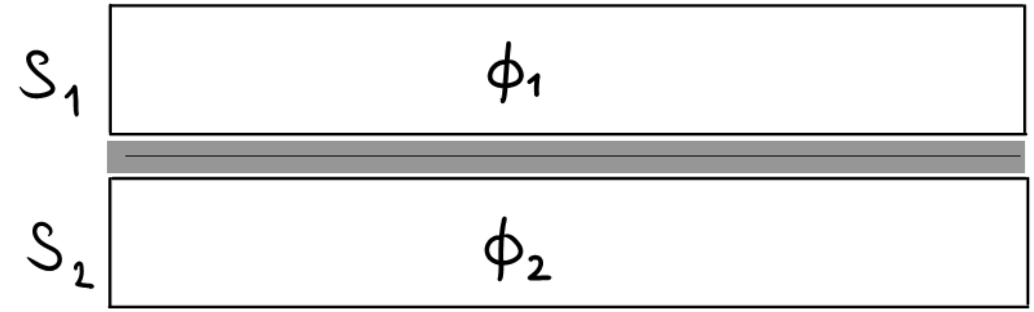
The primary data of the classification are a Nambu space of fermionic field operators which carry a representation of some symmetry group. Our approach is to eliminate all

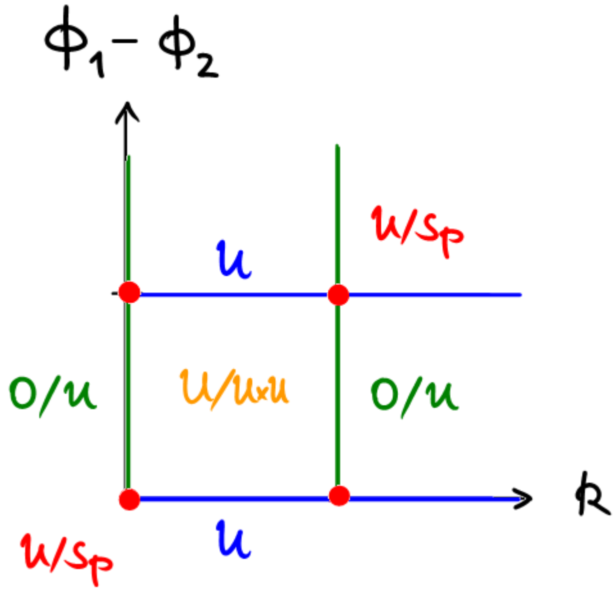
In this paper, it is proved that the symmetry classes of disordered fermions are in one-to-one correspondence with the 10 large families of symmetric spaces.



Symmetry				d					
AZ	Θ	Ξ	Π	1	2	3	4	5	6
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Example: anomalous topological pump (Zhang & Kane 2014)





$S_{1,2}$ T-invariant superconductors with spin-orbit scattering

Further examples.

- topological crystalline insulators
- statistical topological insulators

Comment on Particle-Hole (or Charge) Conjugation, C

Consistency with $U(1)_Q$ requires $\boxed{C \circ e^{i\theta Q} = e^{i\theta Q} \circ C}$

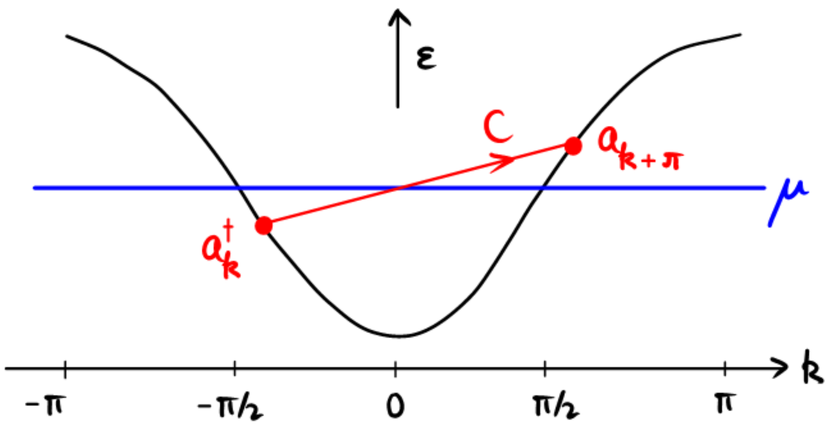
Option A (particle physics): C unitary, $\theta \overset{C}{\mapsto} -\theta$

$A_\mu \mapsto -A_\mu$ (inverts E <u>and</u> B; exact symmetry of E.M. theory)

Option B (condensed matter physics): C anti-unitary, $\theta \overset{C}{\mapsto} +\theta$

Example: cosine band

$$H - \mu N = \sum_k \varepsilon(k)\, a_k^\dagger a_k\,, \qquad \varepsilon(k) = -\cos k$$



$$a_k^\dagger \overset{C}{\longmapsto} a_{k+\pi}$$

$$\varepsilon(k+\pi) = -\varepsilon(k)$$

$$\int \varepsilon(k)\, dk = 0\,.$$

III. Free fermions $\rightsquigarrow$ Haldane phase

Example 2: Hubbard model at half filling

Hamiltonian $H = -\sum_{n\in\mathbb{Z}} \sum_{\sigma=\pm 1/2} \left(t\, a^\dagger_\sigma(n)\, a_\sigma(n+1) + h.c. \right) + U \sum_n Q^2(n)$

Charge (normal ord.) at site n : $Q(n) = \frac{1}{2} \sum_{\sigma=\pm 1/2} \left(a^\dagger_\sigma(n)\, a_\sigma(n) - a_\sigma(n)\, a^\dagger_\sigma(n) \right)$

Particle-hole conjugation $C\, a^\dagger_\sigma(n)\, C^{-1} = (-1)^n a_\sigma(n)$

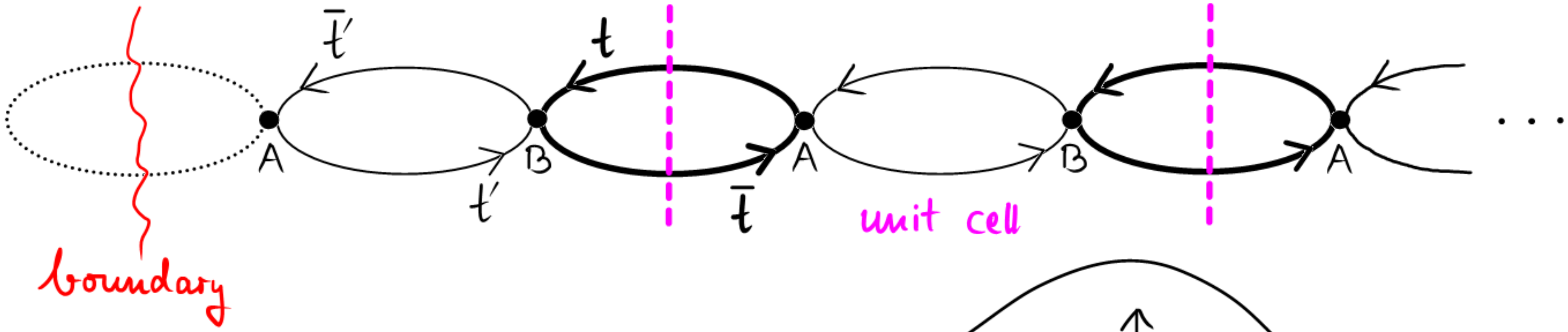
is a symmetry :

$$\left.\begin{aligned} C\, Q(n)\, C^{-1} &= -\, Q(n) \\ C\; t\, a^\dagger_\sigma(n)\, a_\sigma(n+1)\, C^{-1} &= -\, \bar{t}\, a_\sigma(n)\, a^\dagger_\sigma(n+1) \end{aligned}\right\}\; C H C^{-1} = H$$

and leaves the ground state (at half filling) invariant.

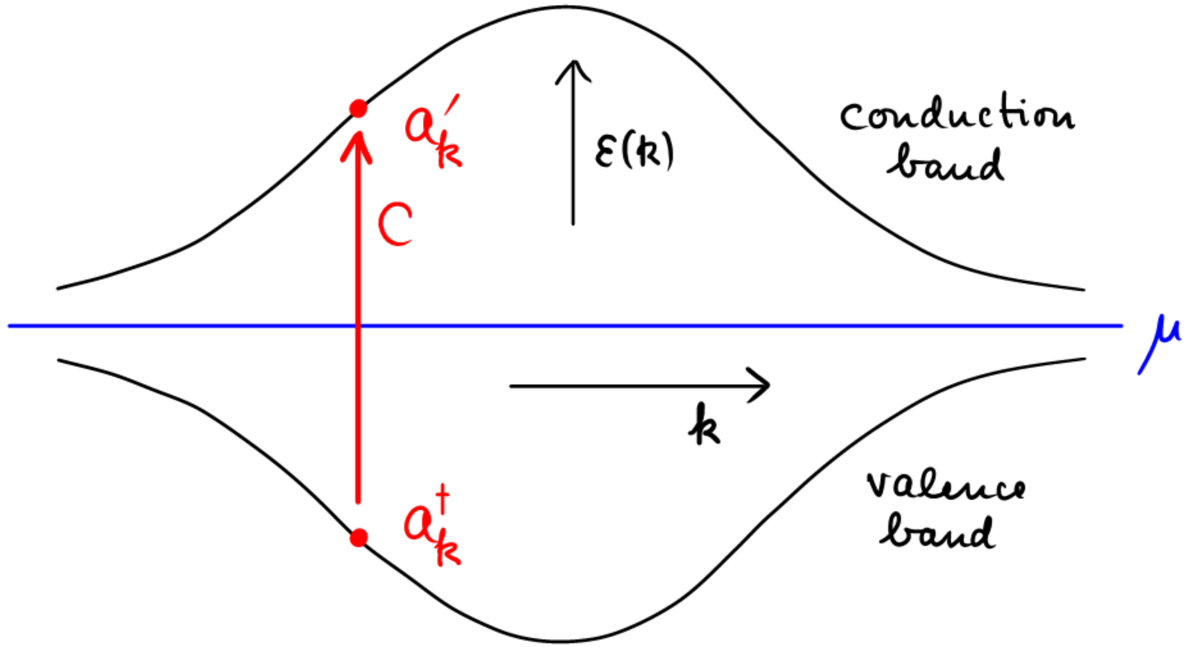
Note also : $C\, S^j(n)\, C^{-1} = -\, S^j(n)$ (spin operators)

Su–Schrieffer–Heeger model ("polyacetylene", class AIII)



$|t| > |t'|$ (Peierls)

$\rightsquigarrow$ double the unit cell:



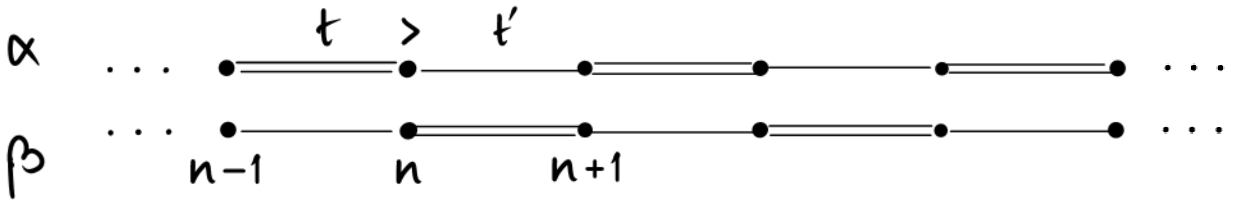
Valence band carries non-trivial topological invariant (e.g. from bulk–boundary correspondence).

Zero mode localized at boundary (cut strong bond):

$$\psi(A_{n+1}) = -\frac{t'}{t}\,\psi(A_n)\,, \quad \psi(B_n) = 0\,.$$

From SSH to Haldane-AKLT

starting point: two chains of SSH



spin $S = 1/2$ (each)

Symmetry group $G = U(1)_Q \times \mathbb{Z}_2^C$ (class AIII)

Recall $a^\dagger(n) \xrightarrow{C} (-1)^n a(n)$, $Q(n) \xrightarrow{C} -Q(n)$, $S(n) \xrightarrow{C} -S(n)$.

Hamiltonian (path in class AIII):

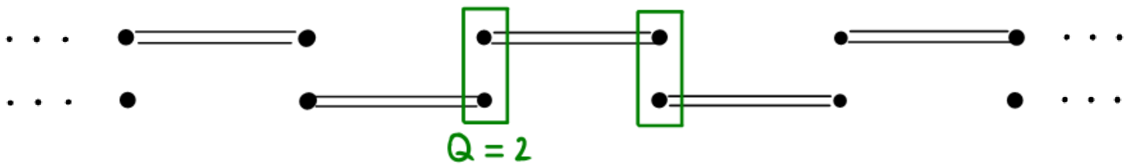
$$H(t, t', U, J) = H_{\text{free}} + \underbrace{U \sum_n Q^2(n)}_{\text{"Hubbard"}} - \underbrace{J \sum_n S_\alpha(n) \cdot S_\beta(n)}_{\text{"Hund's rule"}}$$

From SSH to Haldane-AKLT (II)

Step 1. Turn off the hopping t' $\rightsquigarrow$ flat bands

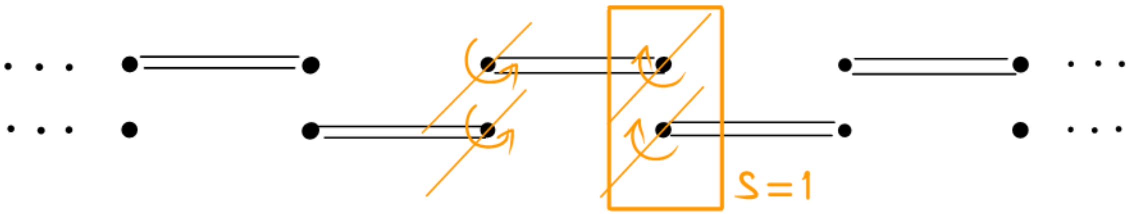
(energy gap stays open; topological invariant remains unchanged)

Step 2. Turn on the Hubbard coupling ($U \gg |t|$) $\rightsquigarrow$ 2 charges/site



(singlet bonds form; energy gap stays open)

Step 3. Turn on the Hund's rule coupling ($J \sim U$) $\rightsquigarrow$ S=1 on each site



(antiferromagnetic exchange coupling; energy gap stays open)

The resulting ground state is the AKLT state $\hookrightarrow$ Haldane phase

Thank you!

(The End)