Dark-Matter Bound States

Juri Smirnov
Florence division INFN

Many thanks to: A. Mitridate, M. Redi and A.Strumia

The Effects of Unstable Dark-Matter Bound States

Weak Dark Matter: The Gauge Portal

Juri Smirnov, INFN Florence division

The Gauge Portal
(single parameter models)

Quantum numbers $SU(2)_L$	$U(1)_Y$	Spin	DM can decay into	DD bound?	Stable?
2	1/2	S	EL	×	×
2	1/2	F	EH	×	×
3	0	S	HH^*	√	×
3	0	F	LH	√	×
3	1	S	HH, LL	×	×
3	1	F	LH	×	×
4	1/2	S	HHH^*	×	×
4	1/2	F	(LHH^*)	×	×
4	3/2	S	HHH	×	×
4	3/2	F	(LHH)	×	×
5	0	S	(HHH^*H^*)	√	×
5	0	F	–	√	√
5	1	S	$(HH^*H^*H^*)$	×	×
5	1	F	–	×	√
5	2	S	$(H^*H^*H^*H^*)$	×	×
5	2	F	–	×	√
6	1/2, 3/2, 5/2	S	–	×	√
7	0	S	–	√	√
8	1/2, 3/2 ...	S	–	×	√

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5	2	S	$(H^*H^*H^*H^*)$	×	×
5	2	F	–	×	√
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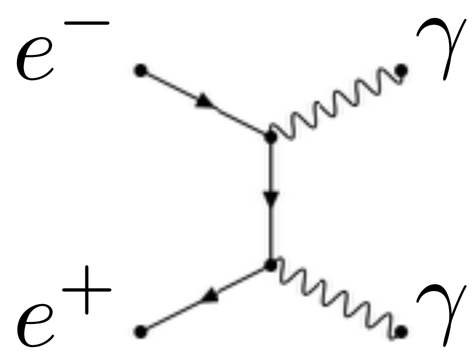
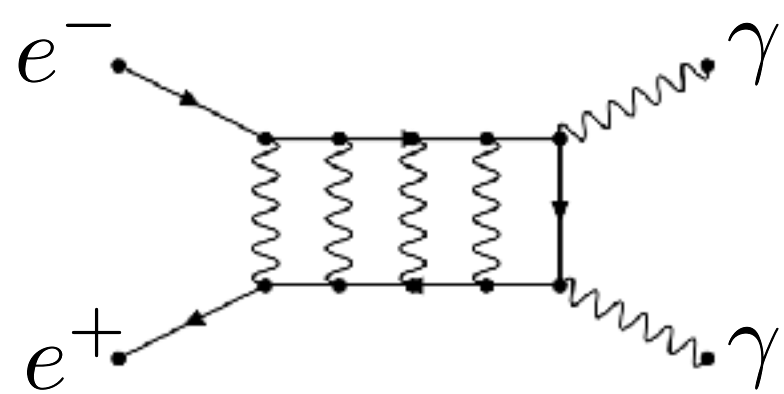
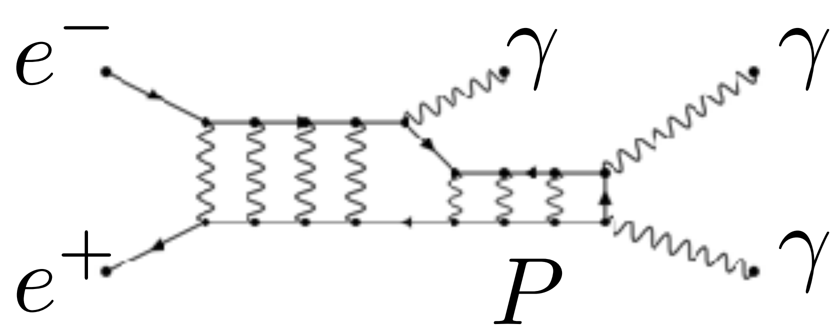
Process	Diagram	Cross-Section area	
I) $e^+ e^- \to \gamma\gamma$			
II) $e^+ e^- \to P^* \to \gamma\gamma$			
III) $e^+ e^- \to P^* \to P\,\gamma$ $P \to \gamma\gamma$			

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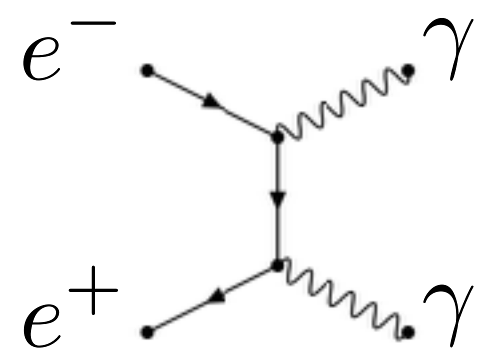
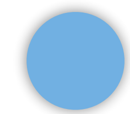
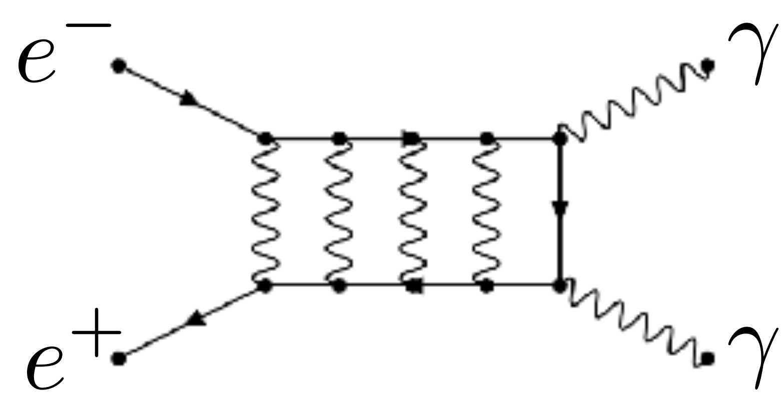
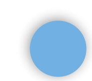
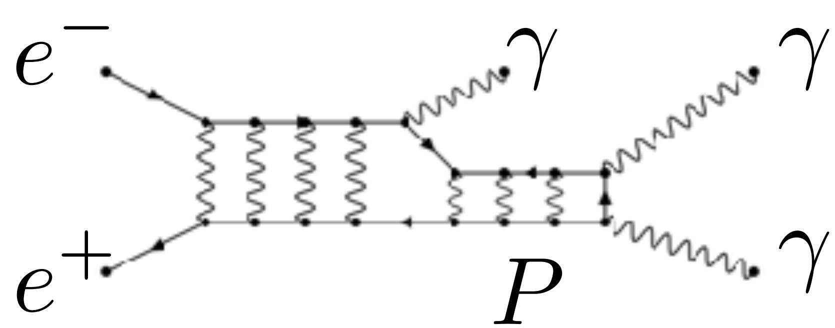
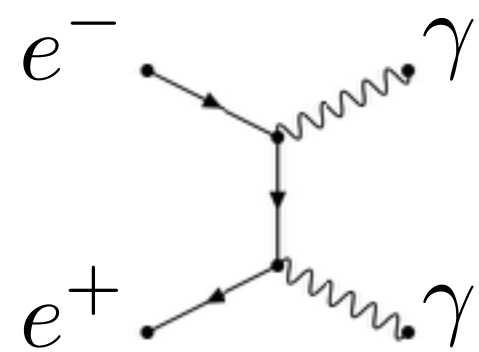
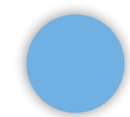
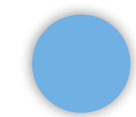
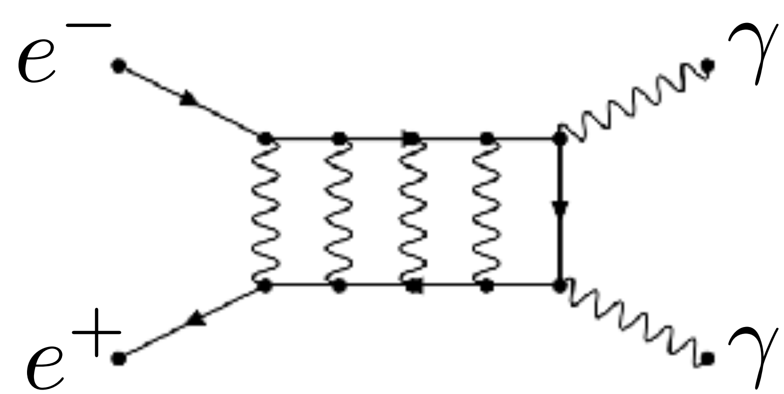
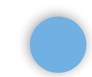
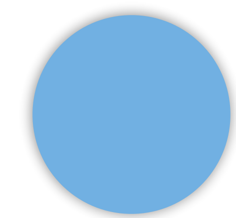
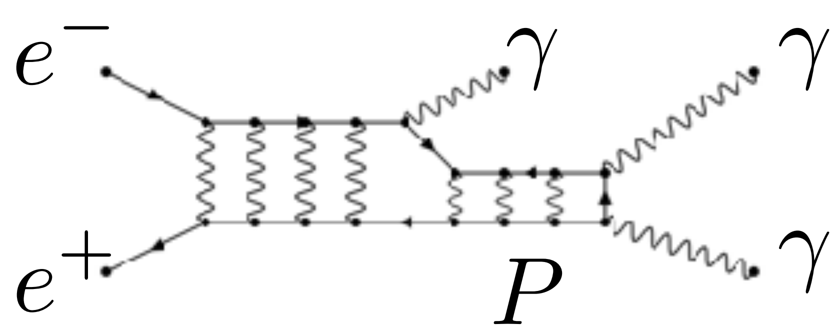
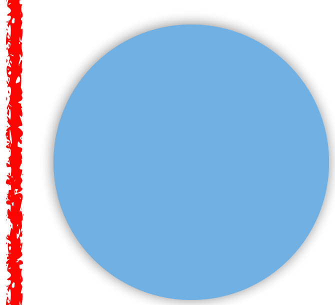
I) $e^+ e^- \to \gamma\gamma$		large velocity 	
II) $e^+ e^- \to P^* \to \gamma\gamma$			
III) $e^+ e^- \to P^* \to P\,\gamma$ $P \to \gamma\gamma$			

Process	Diagram	Cross-Section area	
		large velocity	small velocity
I) $e^+ e^- \to \gamma\gamma$			
II) $e^+ e^- \to P^* \to \gamma\gamma$			
III) $e^+ e^- \to P^* \to P\,\gamma$ $P \to \gamma\gamma$			

Non perturbative Effects

$$M_V < \alpha_{\rm eff}\, M_\chi$$

$$R_{\rm Bohr} < R_{\rm Yukawa}$$

$$V(r) = -\alpha_{\rm eff}\,\frac{e^{-M_V r}}{r} \approx -\alpha_{\rm eff}\left(\frac{1}{r} - M_V\right)$$

$$E_{n\ell} \simeq \frac{\alpha_{\rm eff}^2 M_\chi}{4n^2} - \alpha_{\rm eff} M_V + \mathcal{O}(M_V^2).$$

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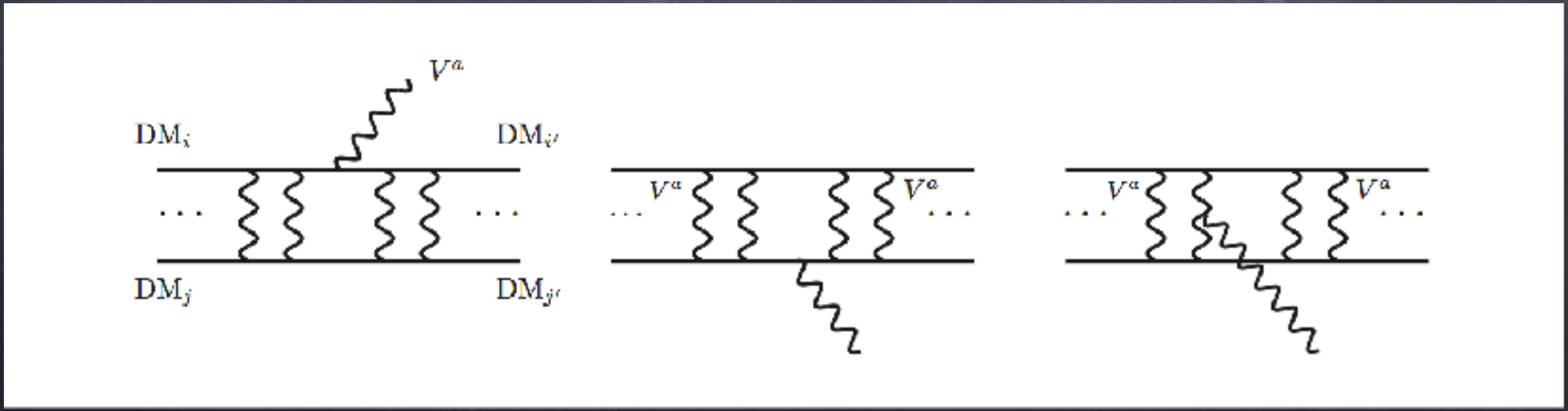
Bound State Selection Rules

- The Group theory structure
- The wave function symmetry
- Angular momentum conservation ΔL = 1
- Energy conservation



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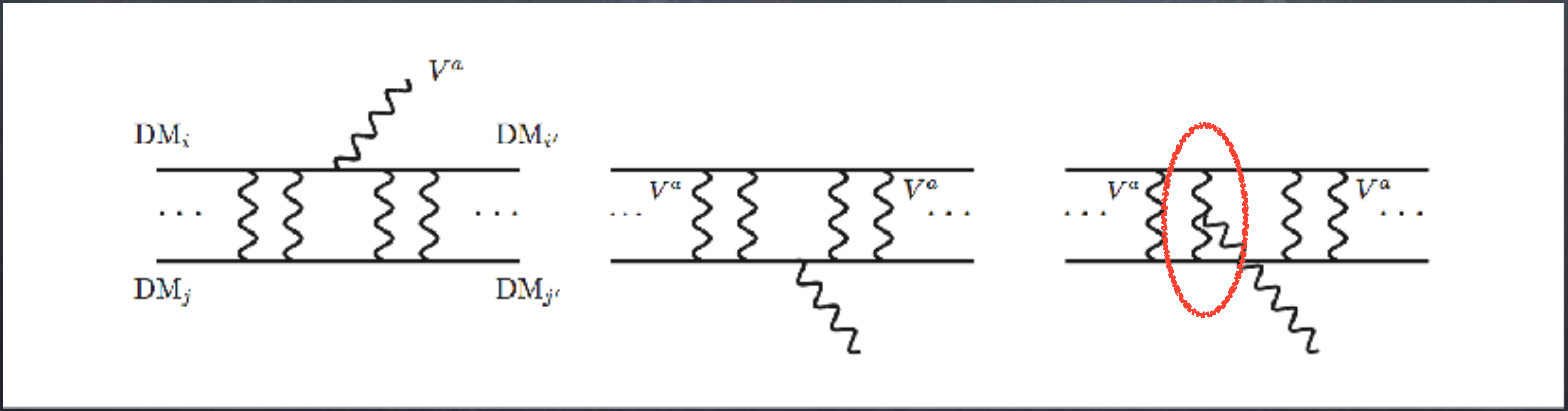

Cosmological Impact

$$\frac{dY_{DM}}{dz} = -\frac{\langle\sigma v\rangle_{\rm ann}}{z^2}\left(Y_{DM}^2 - Y_{eq.}^2\right) - \sum_i \frac{\langle\sigma v\rangle_{\rm bsf}}{z^2}\left(Y_{DM}^2 - Y_{B_i}\frac{Y_{eq.}^2}{Y_{eq.}^{B_i}}\right)$$

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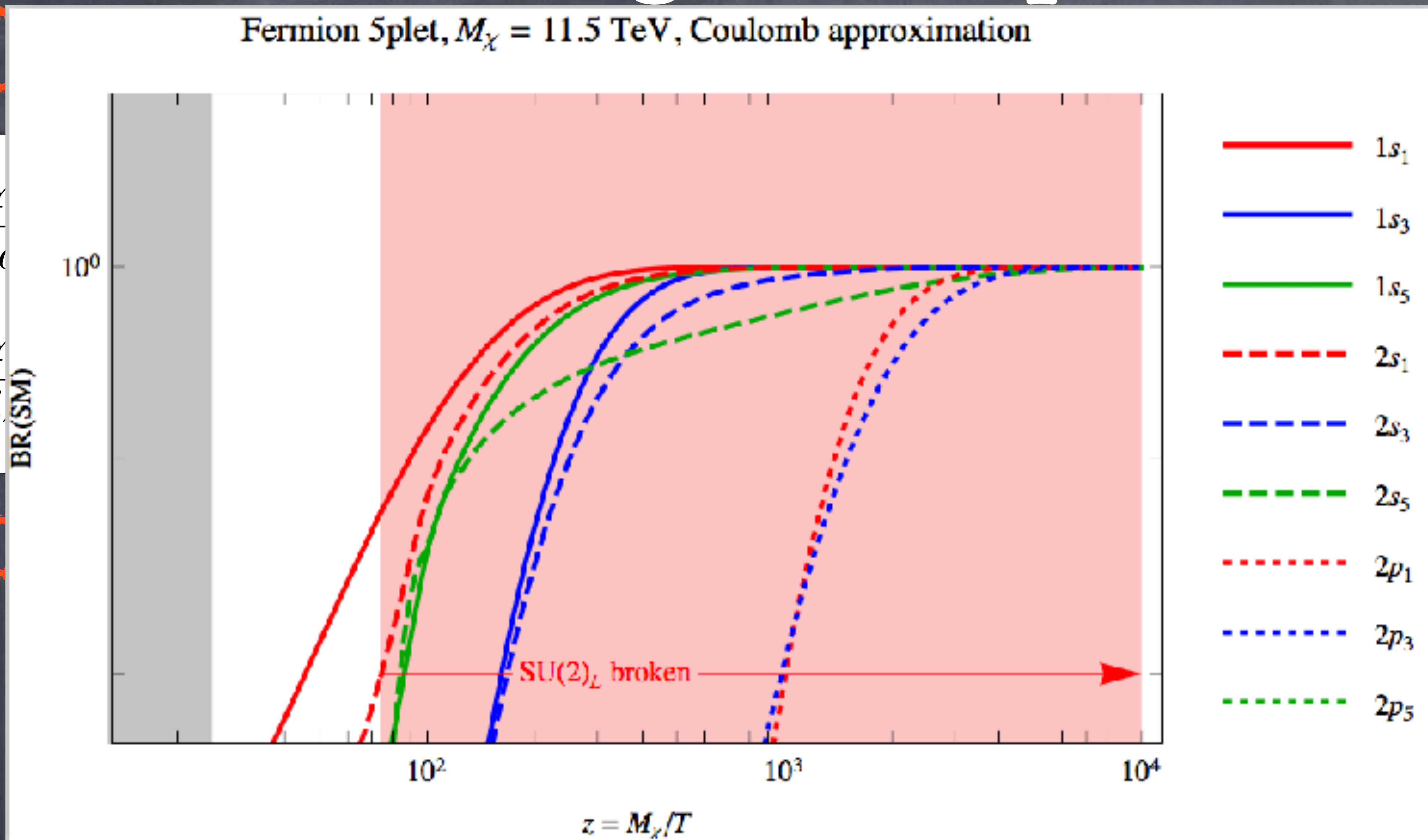

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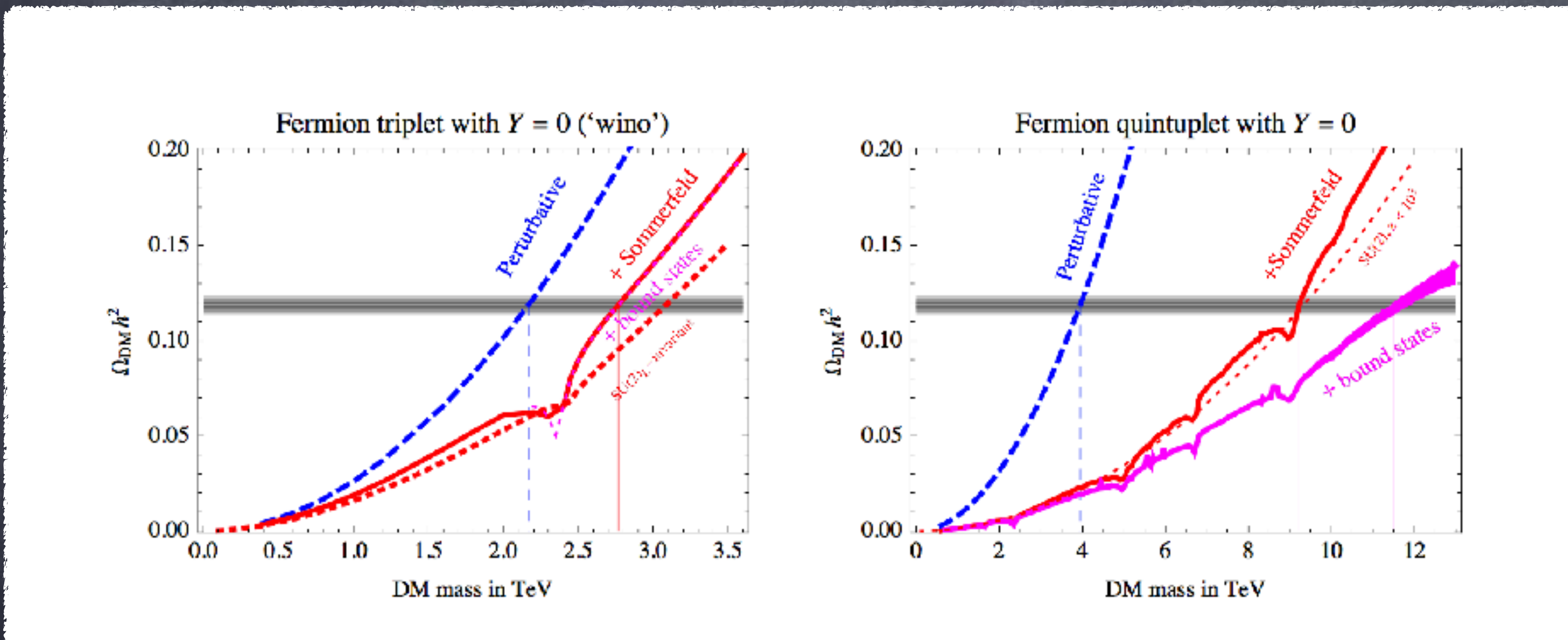
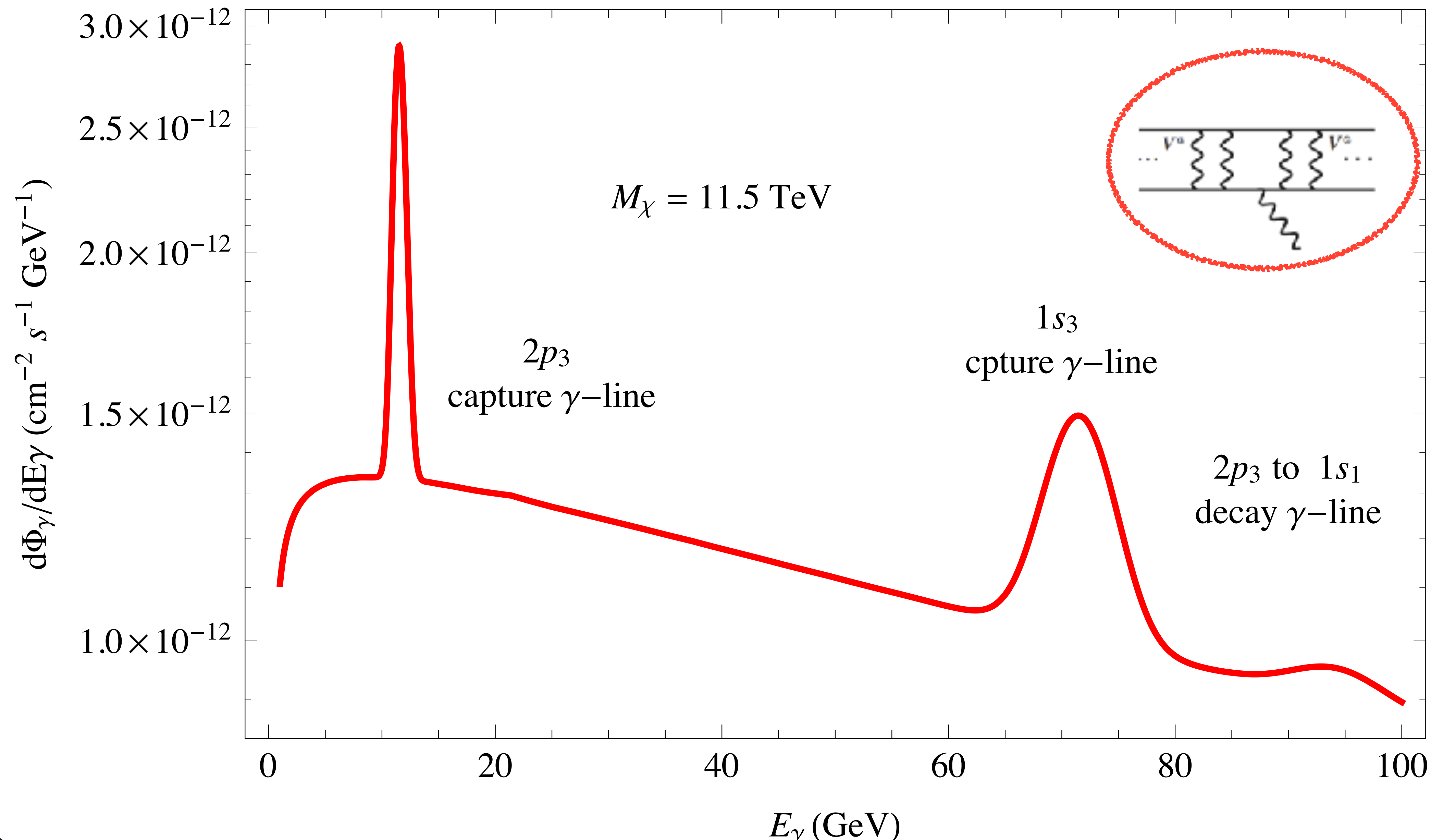
Applications: The gauge portal



The Triplet
(Wino)

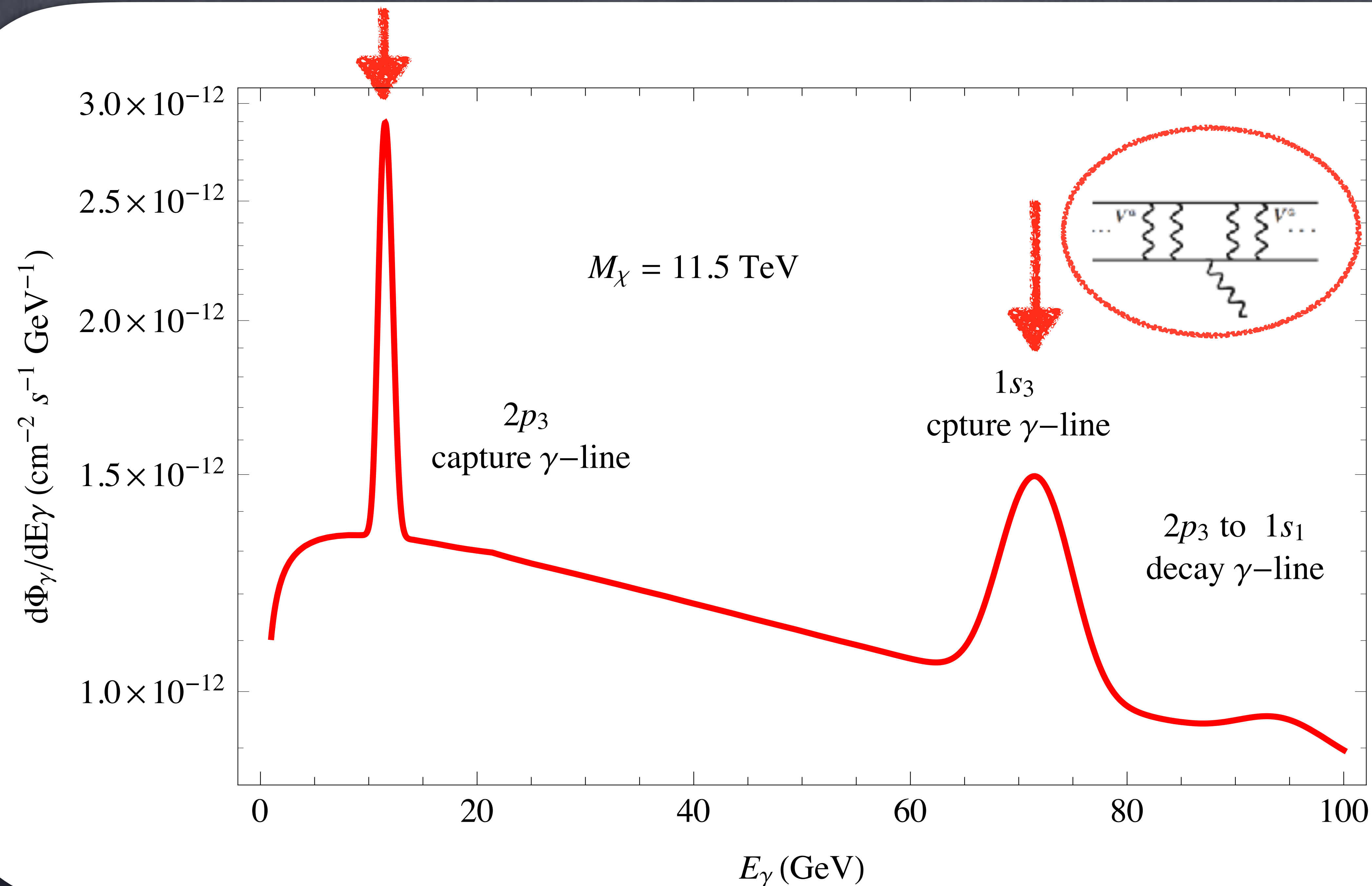
The Quintuplet
(Minimal Dark Matter)

The Quintonium Spectrum



The minimal dark matter spectrum

The Quintonium Spectrum



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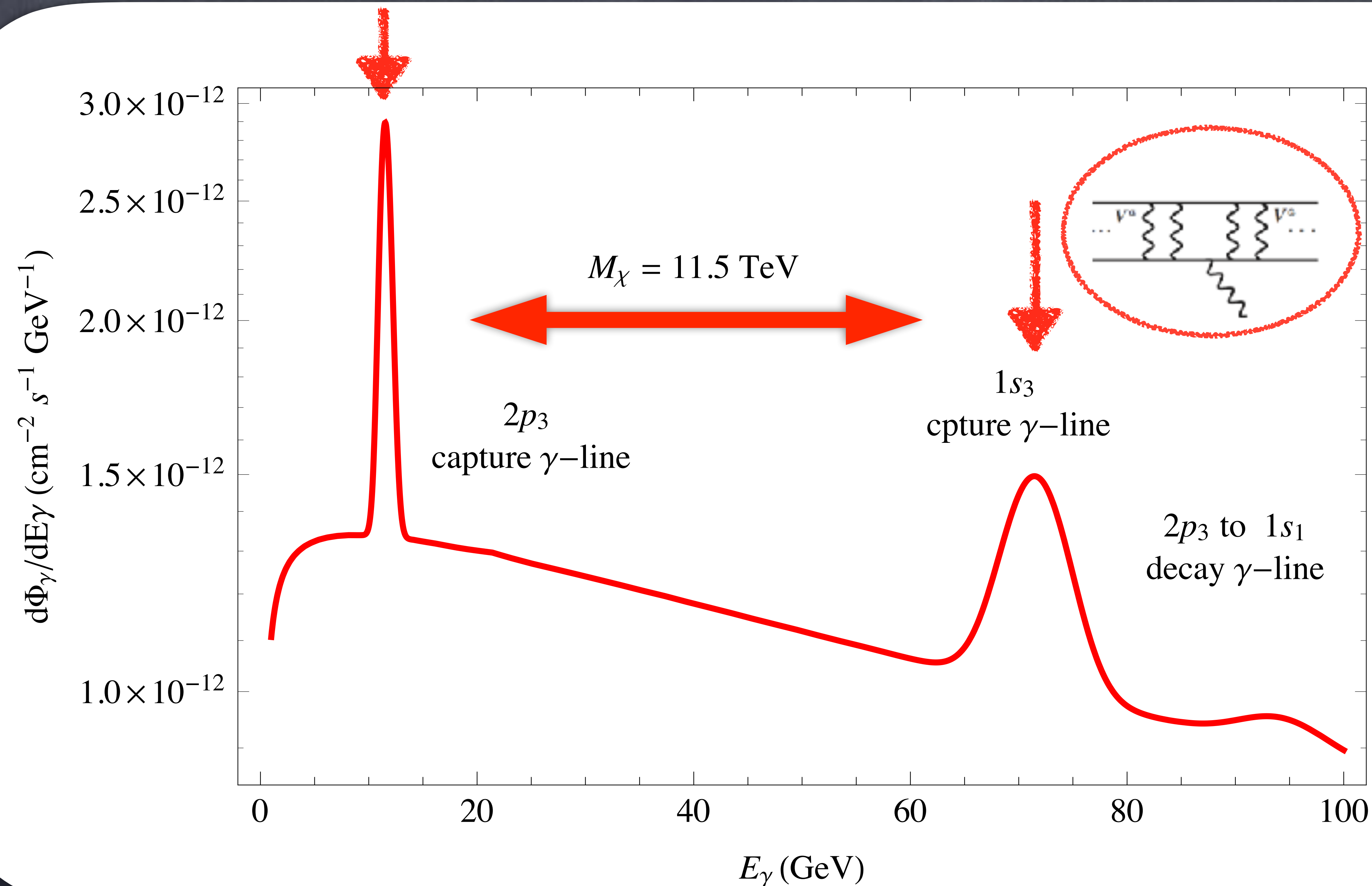

The minimal dark matter spectrum

Stable Bound States of Dark Matter

Dark Matter stability

$$SU(N)_{\rm DC} \times SU(3)_c \times SU(2)_L \times U(1)_{\rm em}$$

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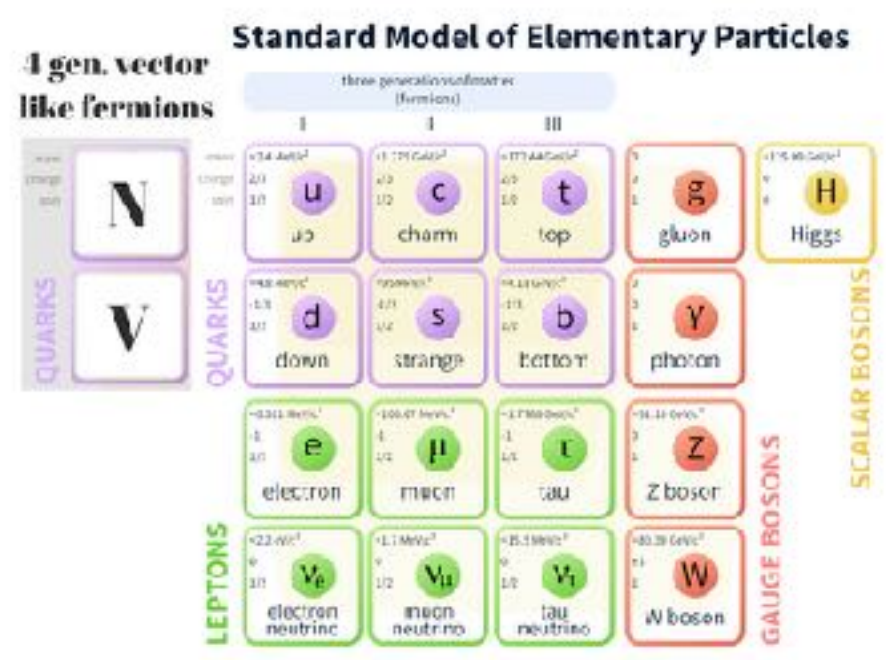

1503.08749

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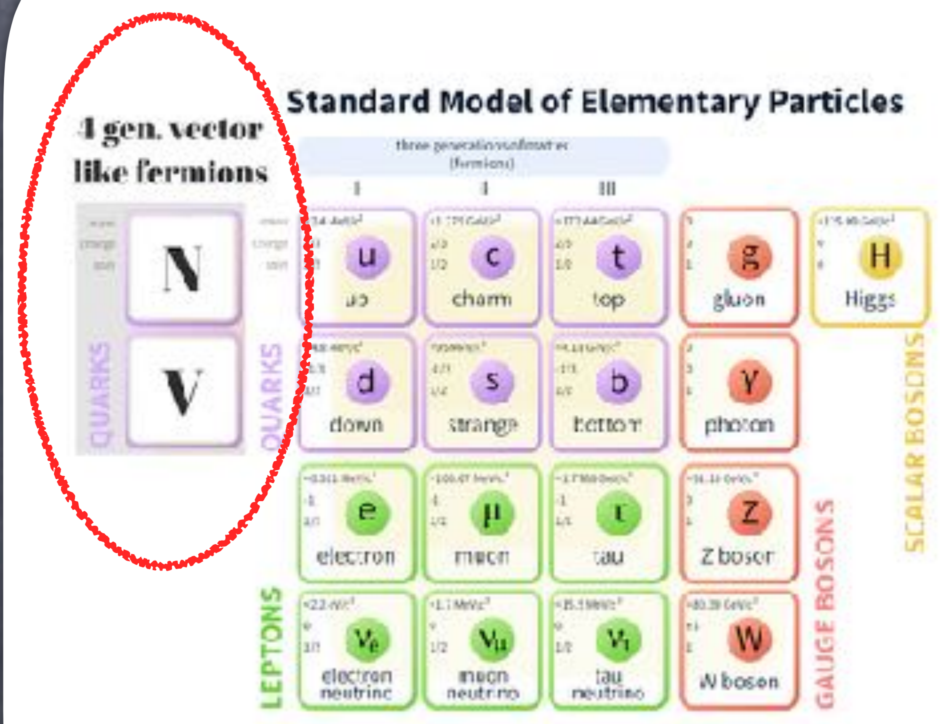

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1503.08749

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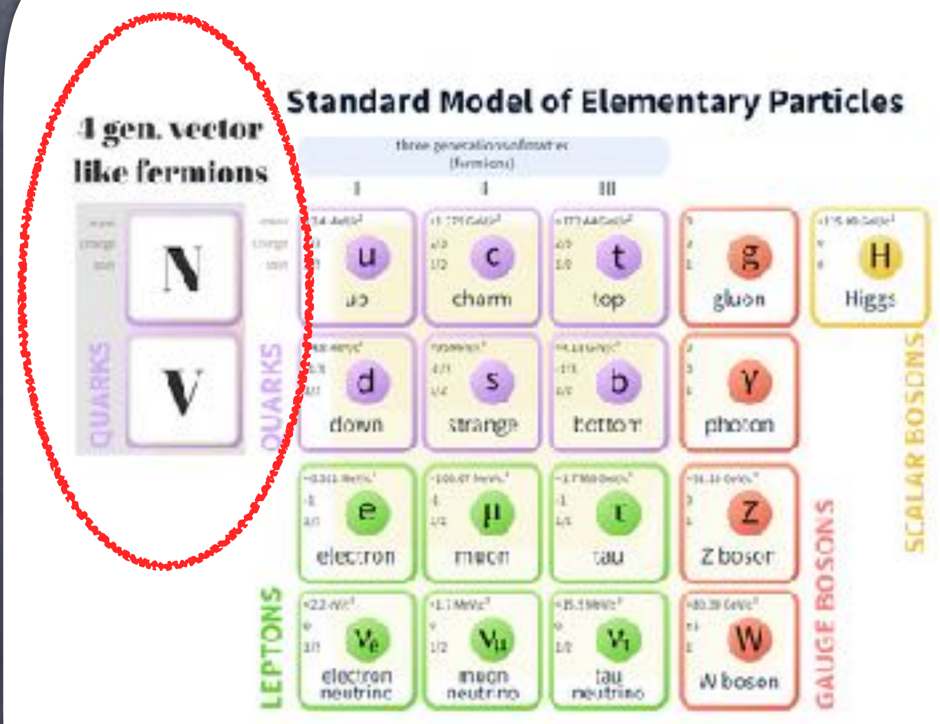

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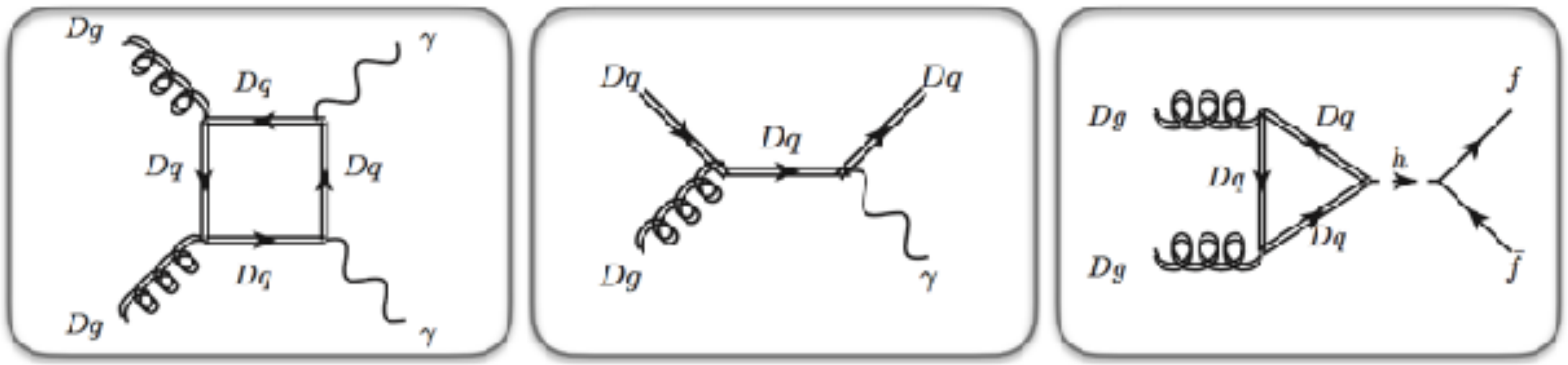
Thermal contact with the SM sector

1503.08749



Consequences of Dark Color II

Simple example Model-V:

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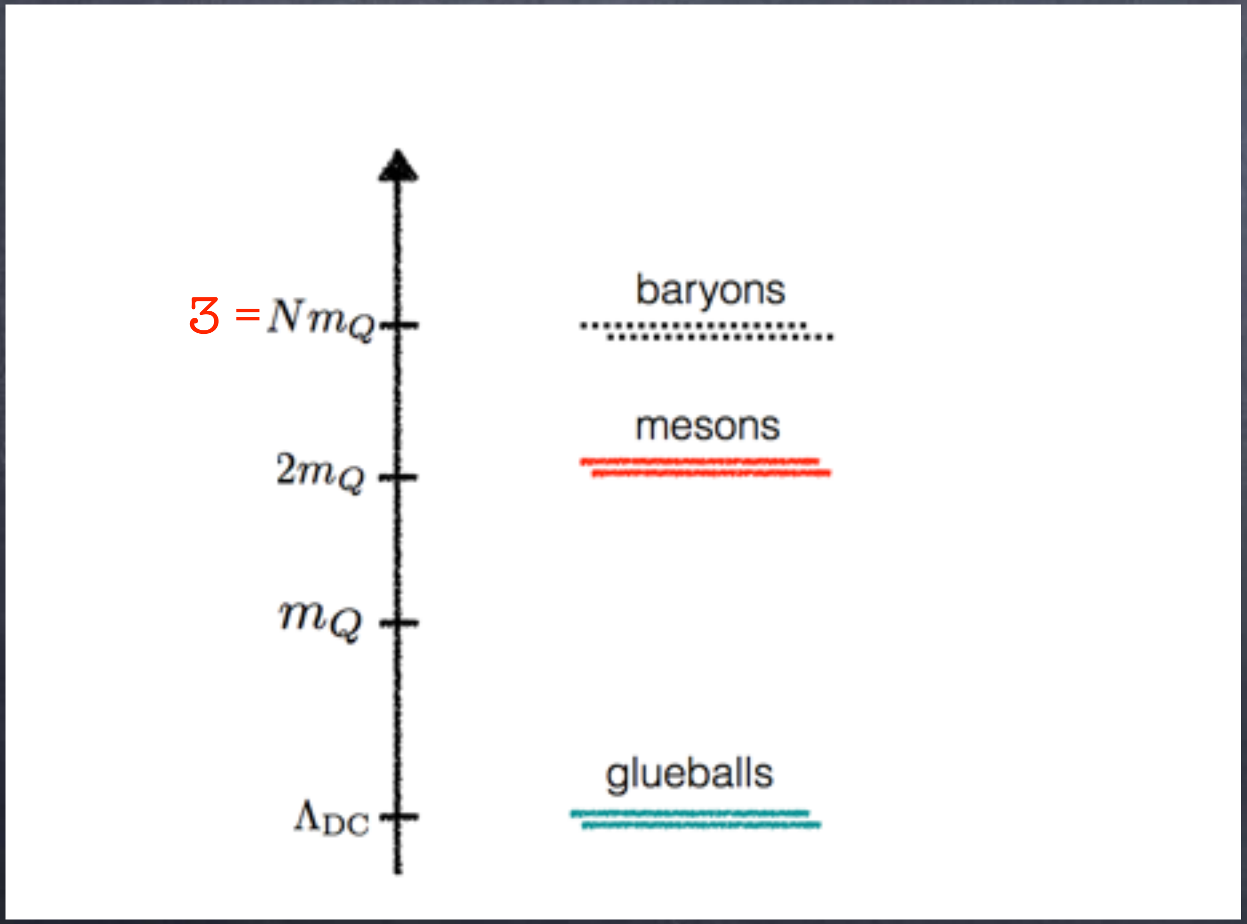

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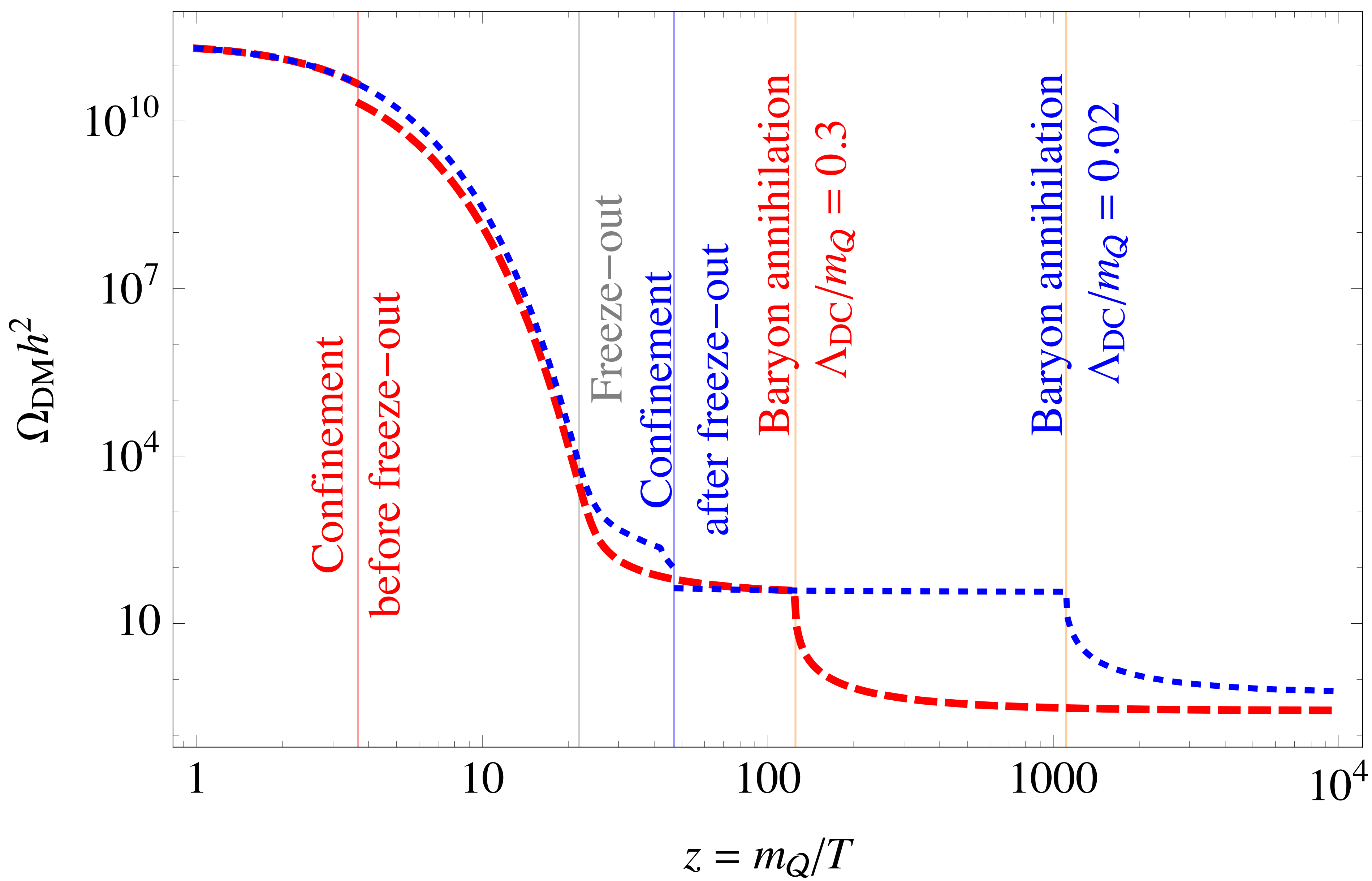
Freeze-out and Confinement



Geometrical Confinement

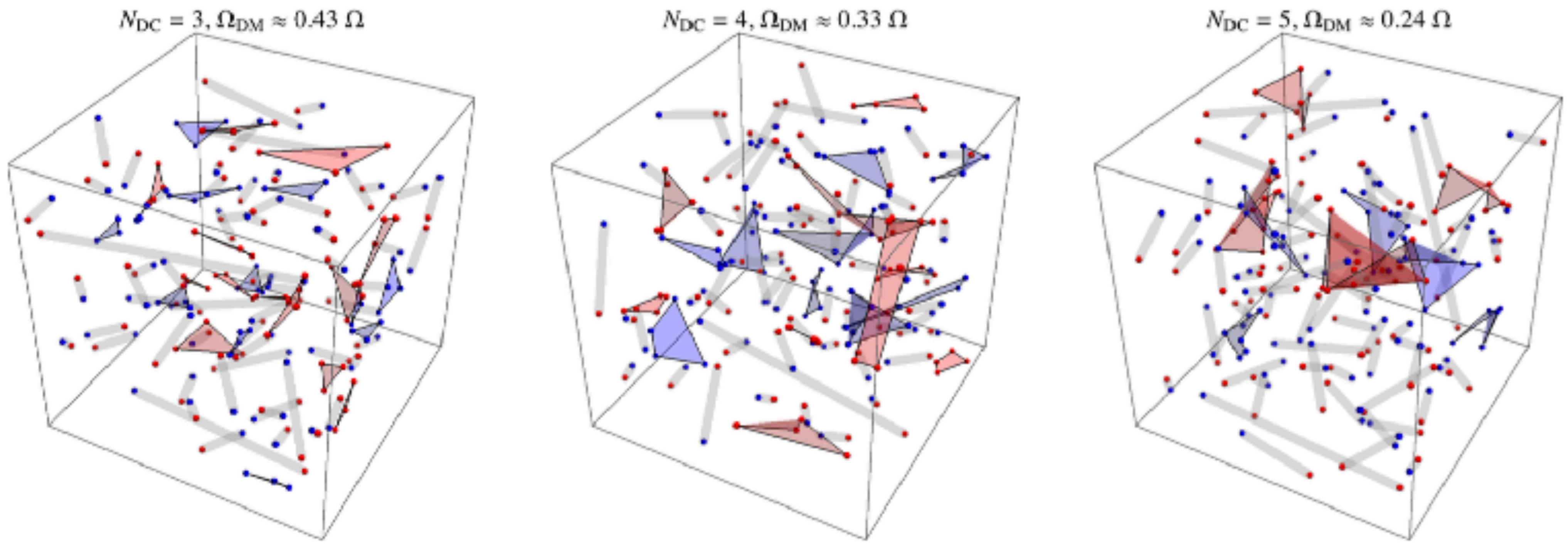
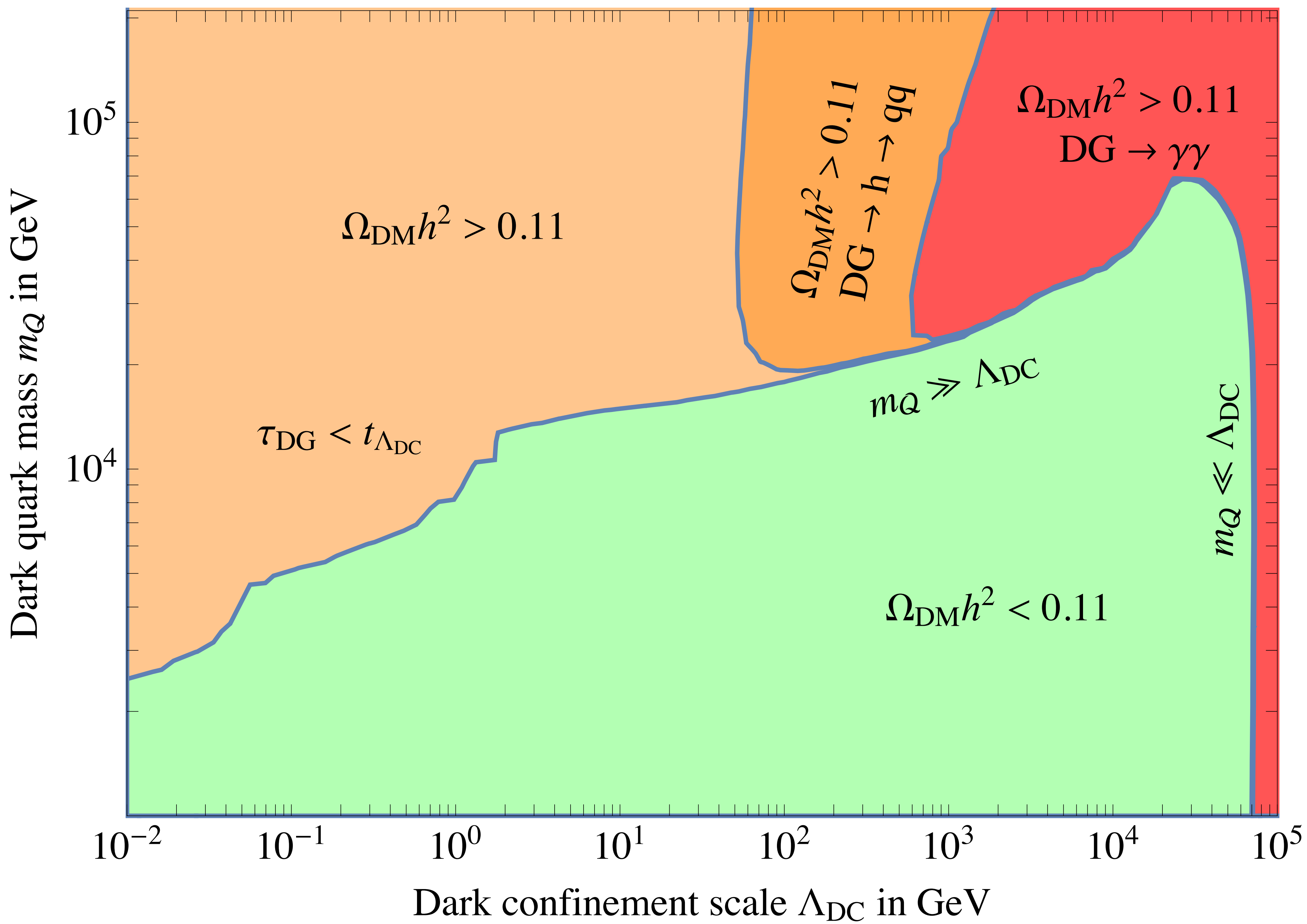
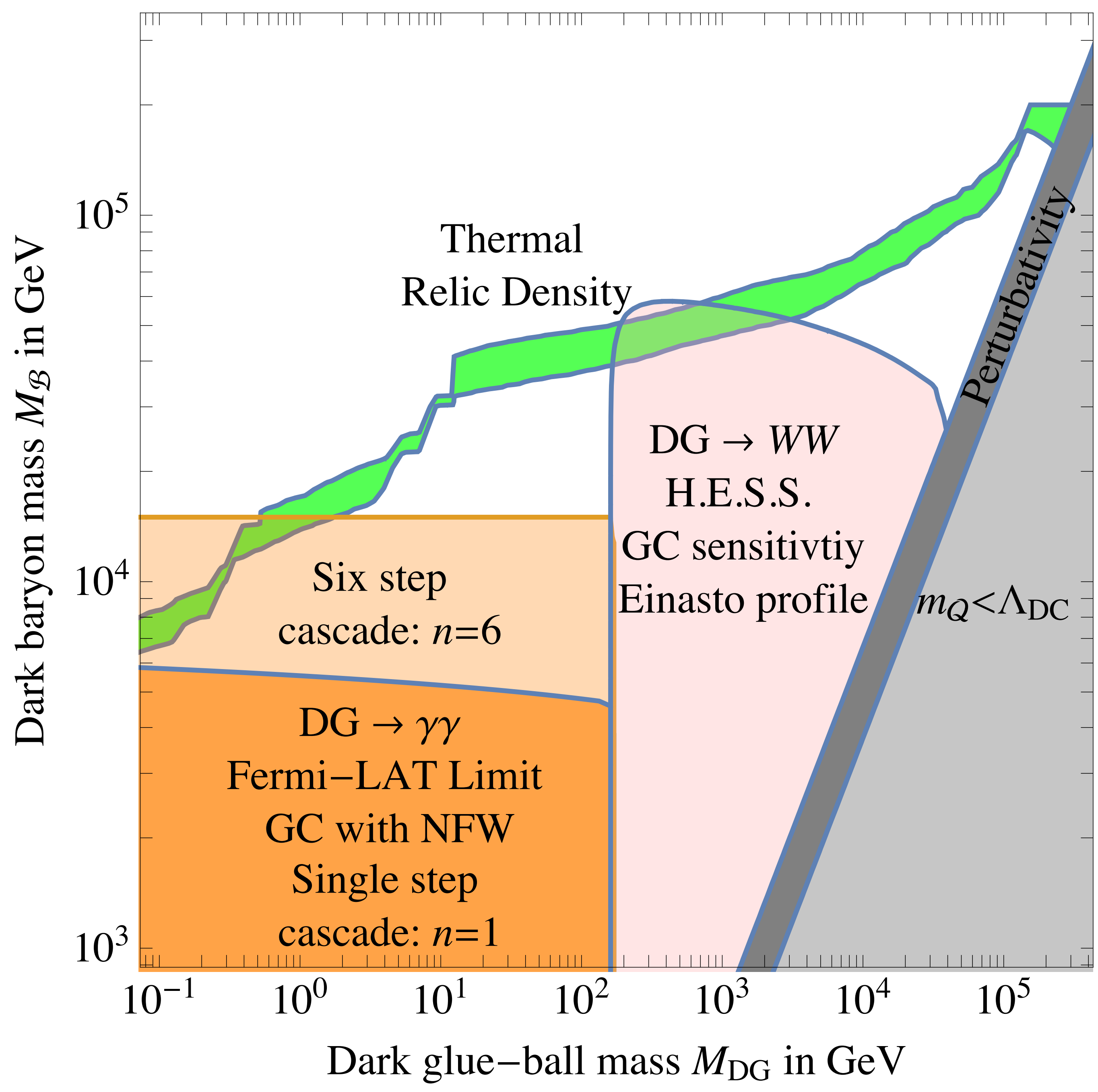


Figure 5: *Examples of dark condensation for* $N_{DC} = 3$ *(left),* 4 *(middle) and* 5 *(right). Dark quarks* $\mathcal{Q}$ *(anti-quarks* $\bar{\mathcal{Q}}$*) are denoted as red (blue) dots, placed at random positions. We assume that each DM particle combines with its dark nearest neighbour, forming either unstable* $\mathcal{Q}\bar{\mathcal{Q}}$ *dark mesons (gray lines) or stable* $\mathcal{Q}^{N_{DC}}$ *dark baryons (red regions) and* $\bar{\mathcal{Q}}^{N_{DC}}$ *dark anti-baryons (blue regions).*

Freeze-out and Confinement



The power of H.E.S.S.

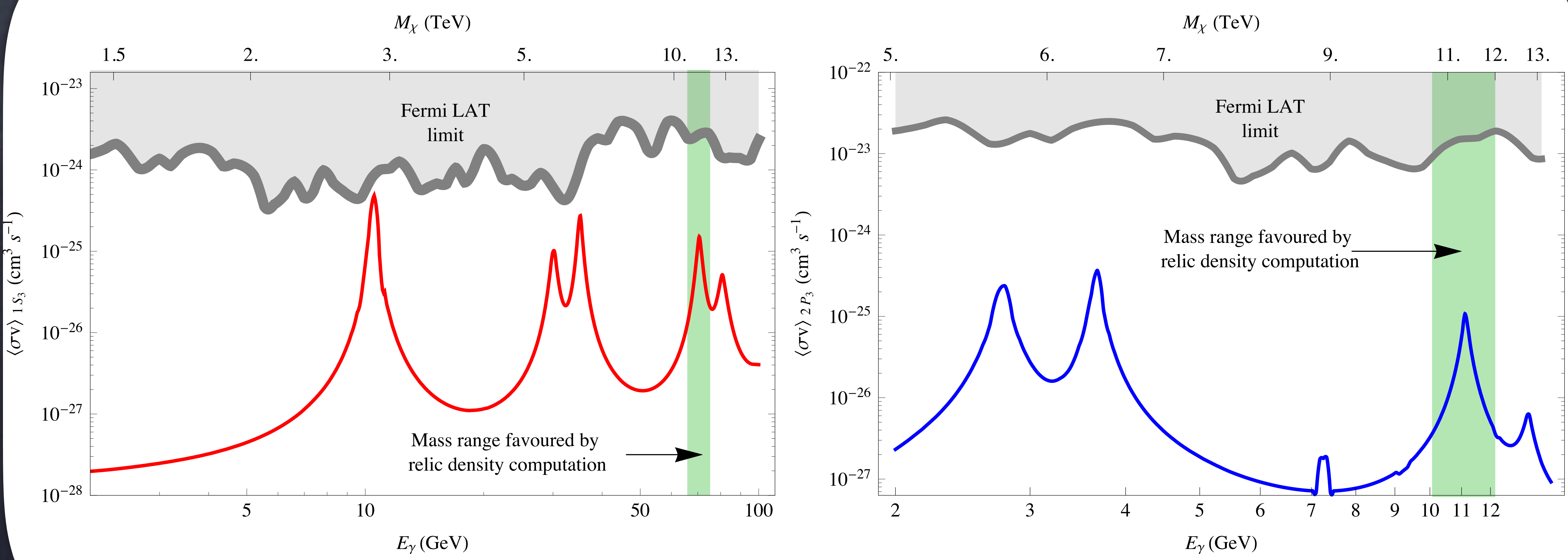


Summary

- For heavy Dark Matter mit SM mediator BSF unavoidable and important
- Low energy and High precision <-> High energy gamma rays
- Stable Dark Matter bound states may be related to new symmetries
- New and Strong Signals possible, we need to explore the entire space of the thermal relic

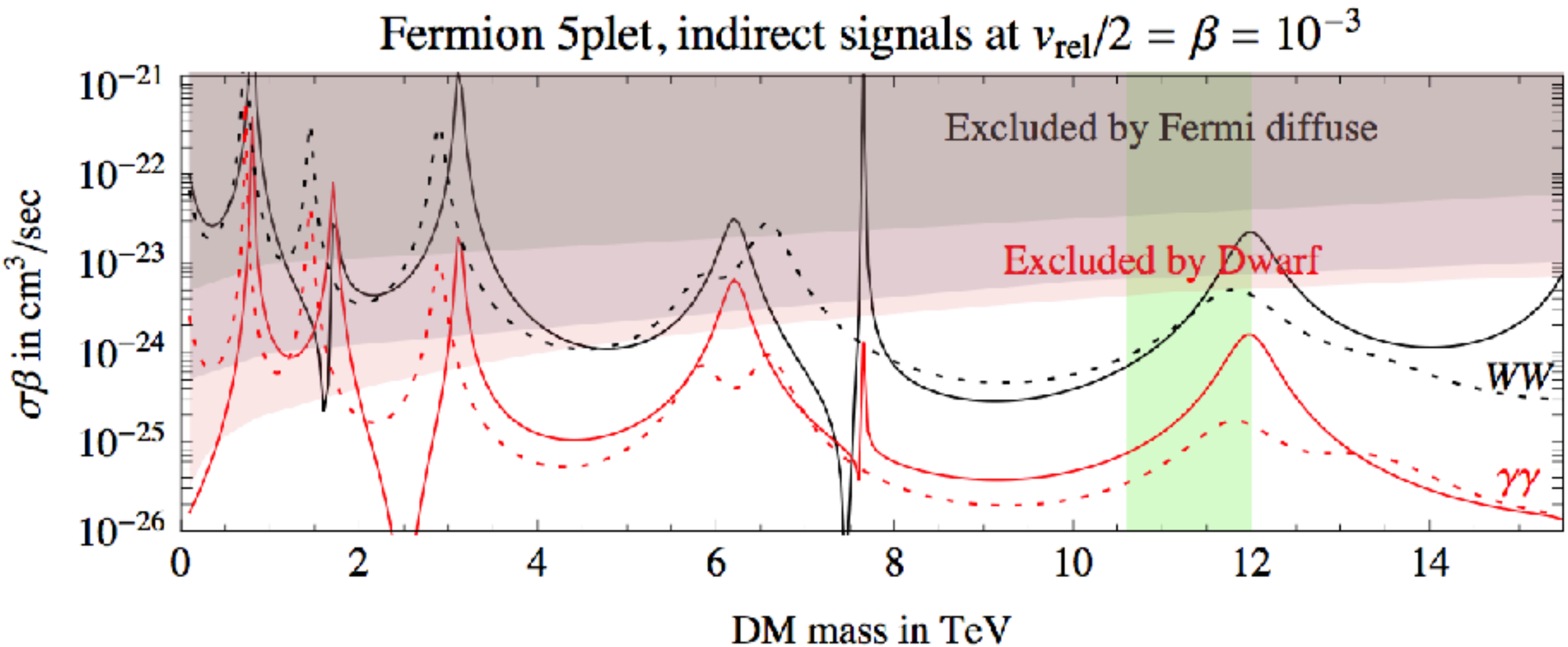
Thank you!

Probing the Heavy

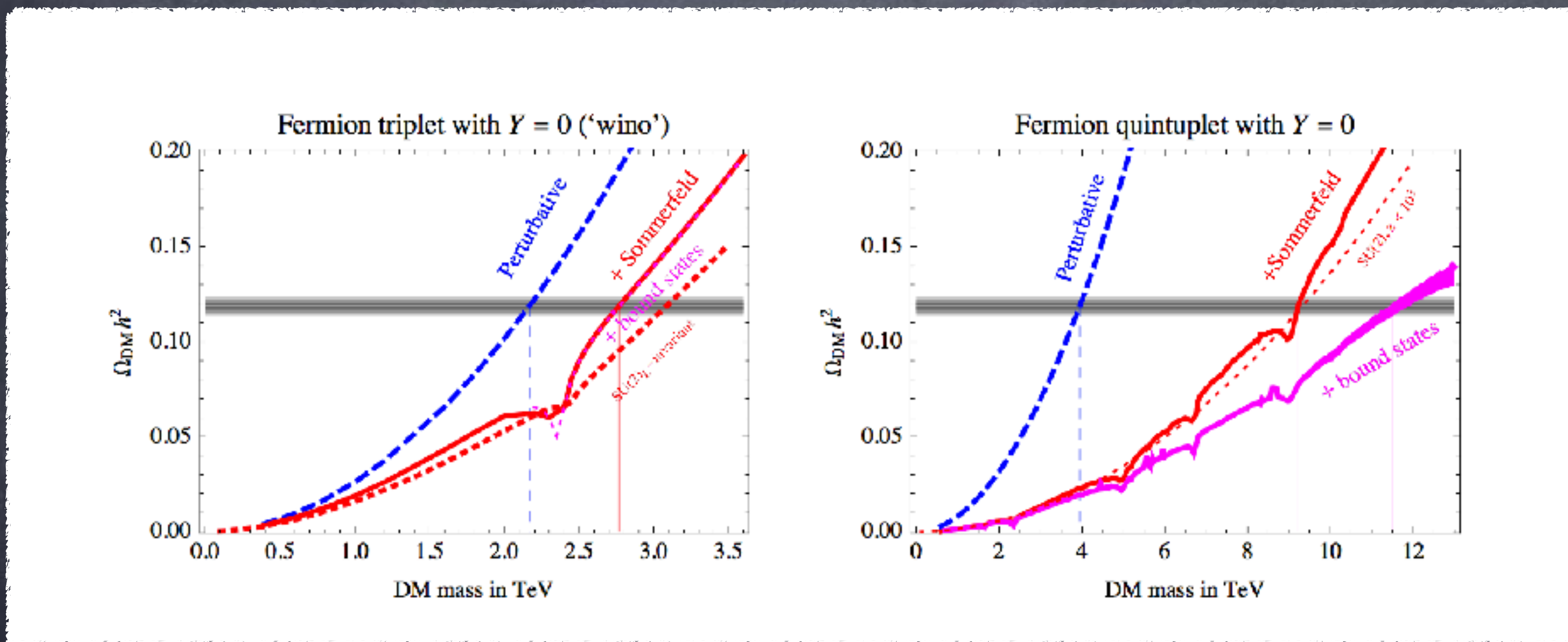


The cross sections confronted with Fermi LAT searches

Minimal Dark Matter



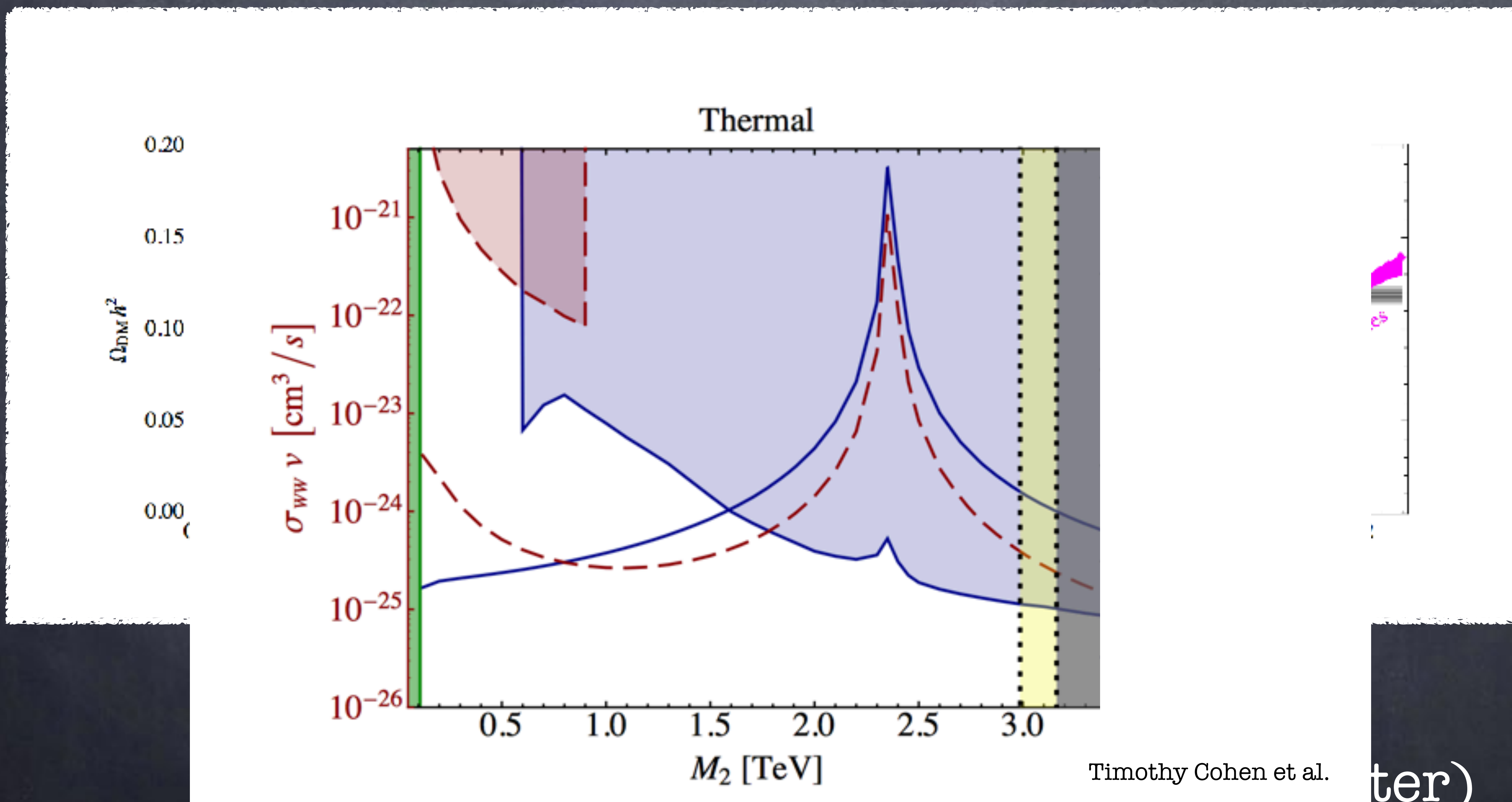
Applications: The gauge portal



The Triplet
(Wino)

The Quintuplet
(Minimal Dark Matter)

Applications: The gauge portal



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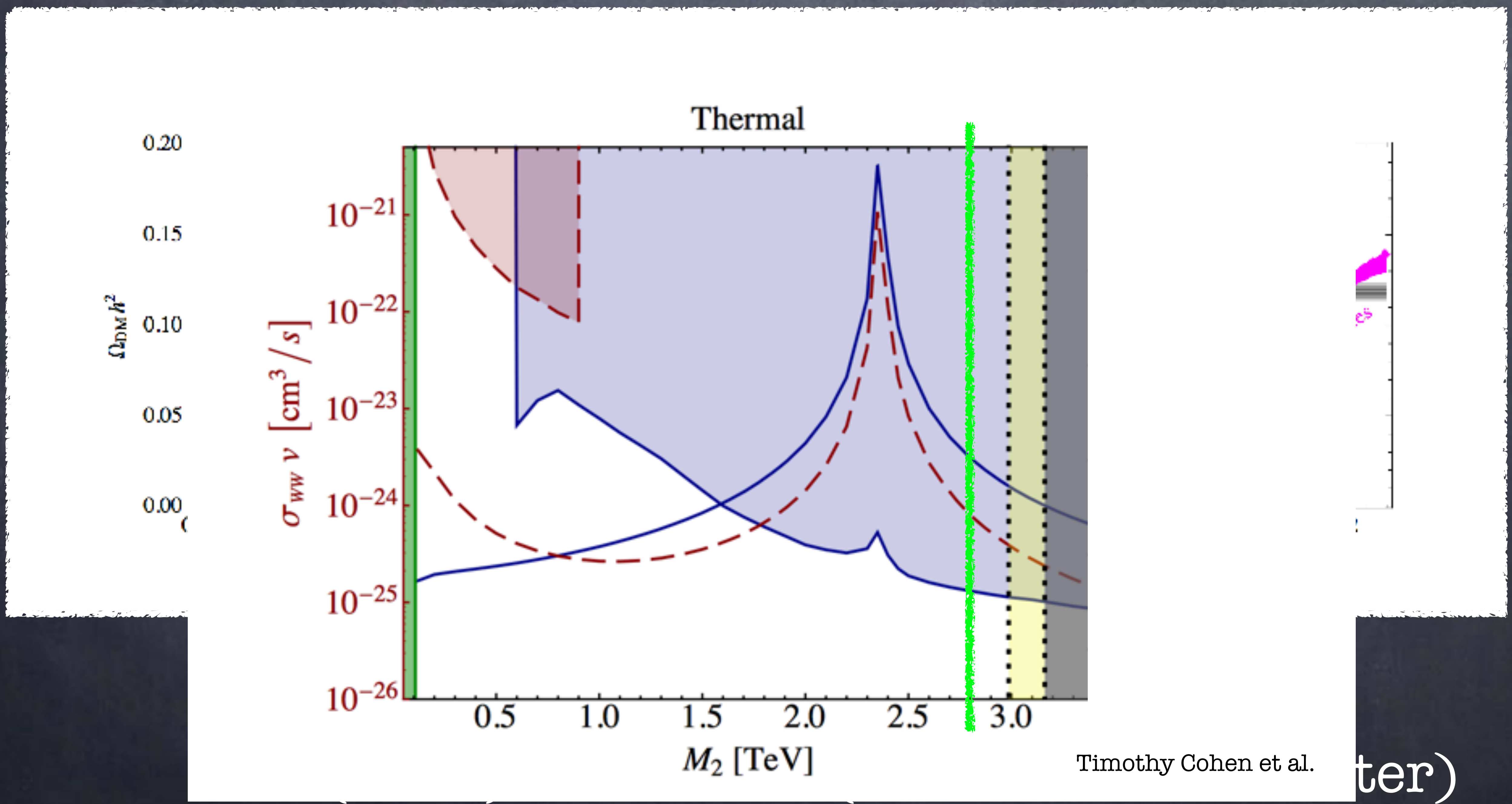
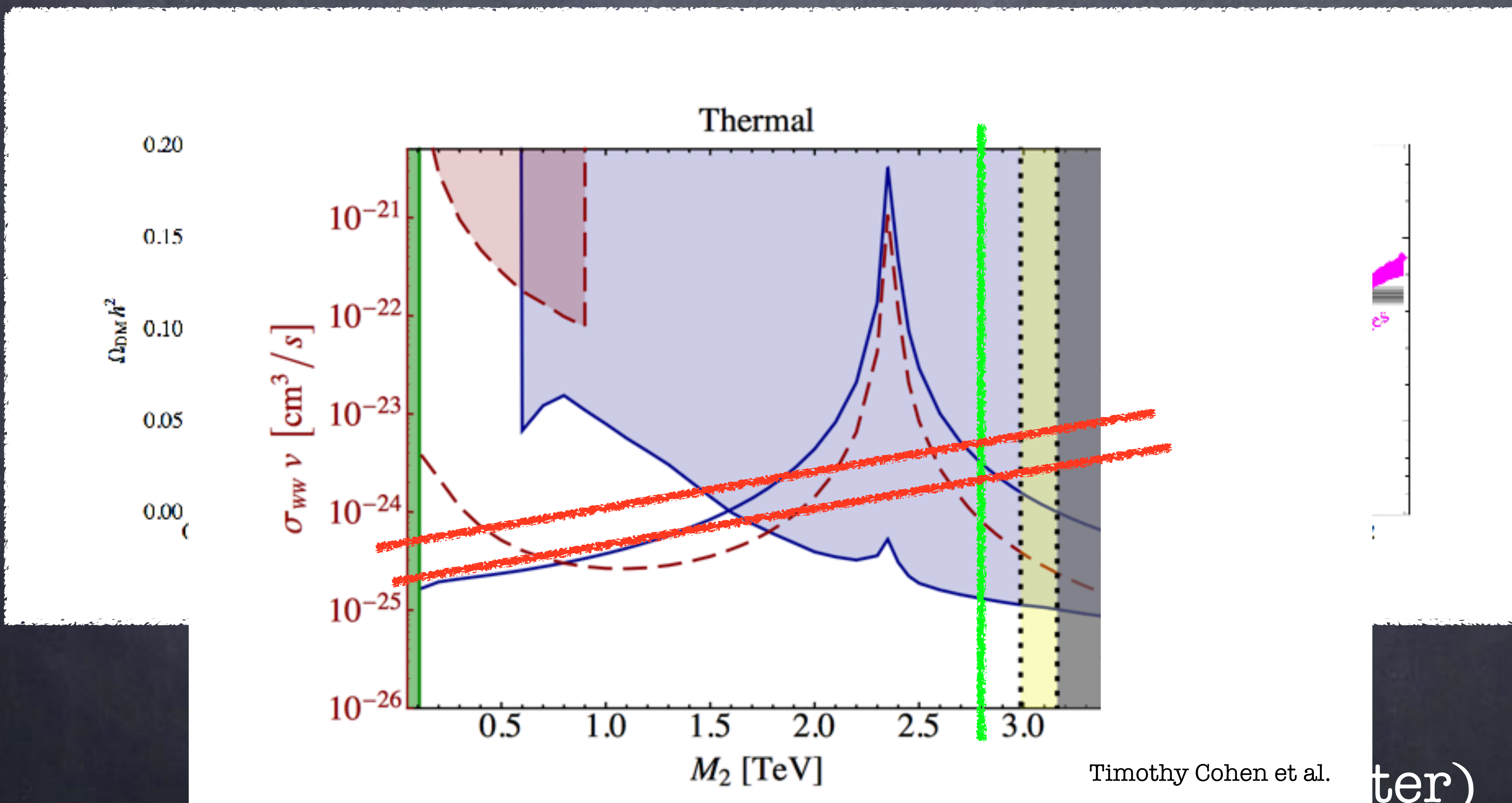
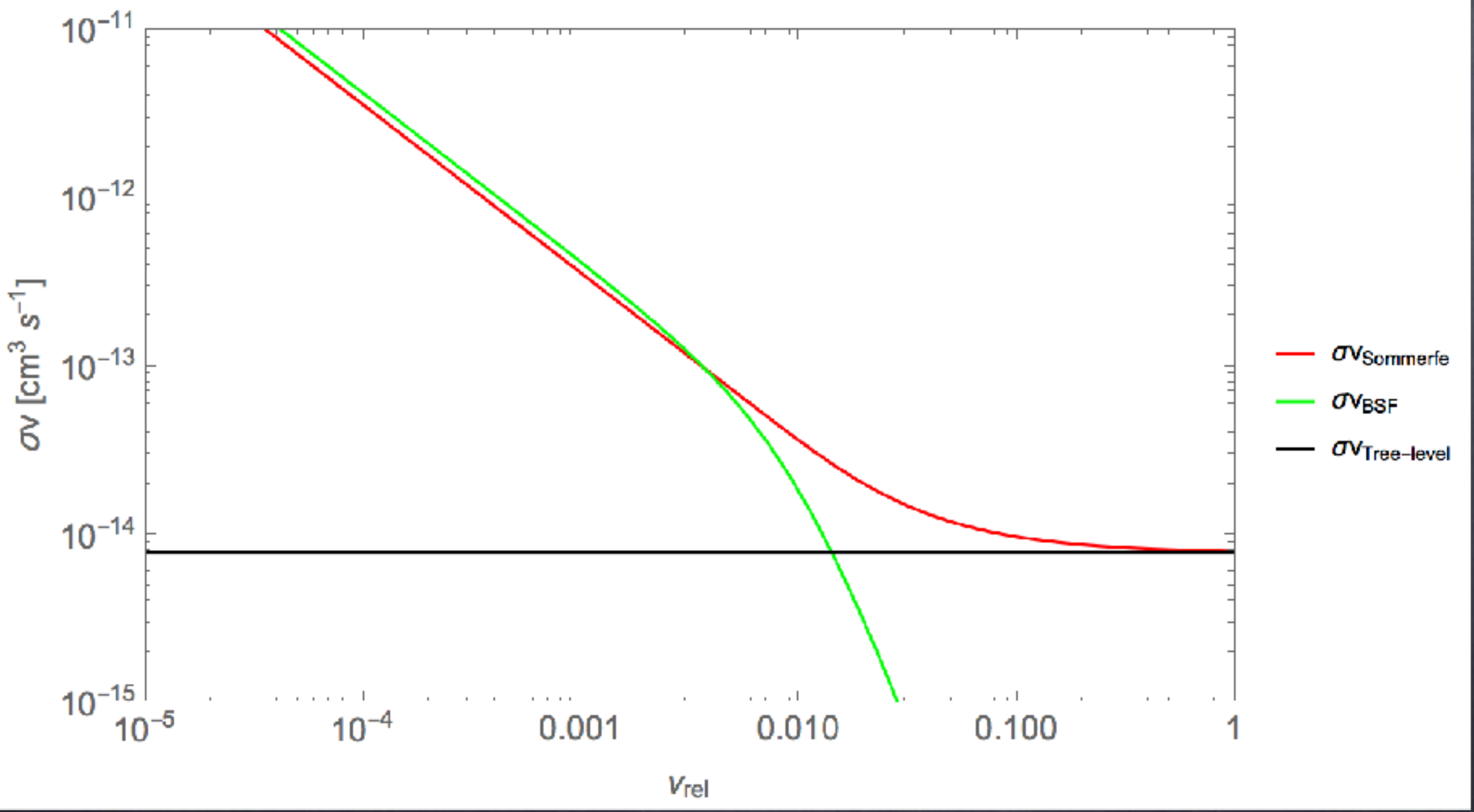
Applications: The gauge portal



Applications: The gauge portal



QED annihilation



Bound State Analytic

$$(\sigma v_{\rm rel})^{n=1,\ell=0}_{\rm bsf} = \sigma_0 \lambda_i (\lambda_f \zeta)^5 \frac{2S+1}{g_\chi^2} \frac{2^{11}\pi(1+\zeta^2\lambda_i^2) e^{-4\zeta\lambda_i \operatorname{arccot}(\zeta\lambda_f)}}{3(1+\zeta^2\lambda_f^2)^3 (1-e^{-2\pi\zeta\lambda_i})} \times \sum_{aMM'} \left| C_{\mathcal{J}}^{aMM'} + \frac{1}{\lambda_f} C_{\mathcal{T}}^{aMM'} \right|^2 \tag{51}$$

where $\sigma_0 = \pi\alpha^2/M_\chi^2$. For the bound states with $n=2$ and $\ell = \{0,1\}$ we get

$$(\sigma v_{\rm rel})^{n=2,\ell=0}_{\rm bsf} = \sigma_0 \lambda_i \lambda_f^5 \frac{2S+1}{g_\chi^2} \frac{2^{14}\pi\zeta^5 (\zeta^2\lambda_i^2+1)\, e^{-4\zeta\lambda_i \operatorname{arccot}(\zeta\lambda_f/2)}}{3\left(\zeta^2\lambda_f^2+4\right)^5 (1-e^{-2\pi\zeta\lambda_i})} \tag{52}$$

$$\times \sum_{aMM'} \left| C_{\mathcal{J}}^{aMM'} \left(\zeta^2\lambda_f(\lambda_f - 2\lambda_i) - 4\right) + C_{\mathcal{T}}^{aMM'} \left(\zeta^2 (3\lambda_f - 4\lambda_i) - \frac{4}{\lambda_f}\right) \right|^2,$$

$$(\sigma v_{\rm rel})^{n=2,\ell=1}_{\rm bsf} = \sigma_0 \lambda_i \lambda_f^5 \frac{2S+1}{g_\chi^2} \frac{2^{12}\pi\alpha\zeta^7 e^{-4\zeta\lambda_i \operatorname{arccot}(\zeta\lambda_f/2)}}{9\left(\zeta^2\lambda_f^2+4\right)^5 (1-e^{-2\pi\zeta\lambda_i})} \times$$

$$\times \sum_{aMM'} \left[\left| C_{\mathcal{J}}^{aMM'} \left(\lambda_f(\zeta^2\lambda_i(3\lambda_f - 4\lambda_i) + 8) - 12\lambda_i\right) + C_{\mathcal{T}}^{aMM'} \left(\zeta^2(-3\lambda_f^2 + 12\lambda_f\lambda_i - 8\lambda_i^2) + 4\right) \right|^2 + \right.$$

$$\left. + 2^5(\zeta^2\lambda_i^2+1)(\zeta^2\lambda_i^2+4) \left| C_{\mathcal{J}}^{aMM'} \lambda_f + 2 C_{\mathcal{T}}^{aMM'} \right|^2 \right]. \tag{53}$$

$M_V = 0$

$$(\sigma v_{\rm rel})^{n=1,\ell=0}_{\rm bsf} = \sigma_0 \frac{2S+1}{g_\chi^2} \frac{2^{11}\pi}{3} \sum_{aMM'} \left| C_{\mathcal{J}}^{aMM'} + \frac{1}{\lambda_f} C_{\mathcal{T}}^{aMM'} \right|^2 \times \begin{cases} \dfrac{\lambda_i^3 \alpha}{\lambda_f v_{\rm rel}} e^{-4\lambda_i/\lambda_f} & v_{\rm rel} \ll \lambda_{i,f}\alpha \\ \dfrac{\lambda_f^5 \alpha^4}{2\pi v_{\rm rel}^4} & v_{\rm rel} \gg \lambda_{i,f}\alpha \end{cases}$$

Bound State Analytic

$M_V = 0$

$$(\sigma v_{\rm rel})^{n=1,\ell=0}_{\rm bsf} = \sigma_0 \lambda_i (\lambda_f \zeta)^5 \frac{2S+1}{g_\chi^2} \frac{2^{11}\pi(1+\zeta^2\lambda_i^2)e^{-4\zeta\lambda_i \operatorname{arccot}(\zeta\lambda_f)}}{3(1+\zeta^2\lambda_f^2)^3(1-e^{-2\pi\zeta\lambda_i})} \times \sum_{aMM'} \left| C_{\mathcal{J}}^{aMM'} + \frac{1}{\lambda_f} C_{\mathcal{T}}^{aMM'} \right|^2 \tag{51}$$

where $\sigma_0 = \pi\alpha^2/M_\chi^2$. For the bound states with $n=2$ and $\ell = \{0,1\}$ we get

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$$\times \sum_{aMM'} \left| C_{\mathcal{J}}^{aMM'}\left(\zeta^2\lambda_f(\lambda_f-2\lambda_i)-4\right) + C_{\mathcal{T}}^{aMM'}\left(\zeta^2(3\lambda_f-4\lambda_i)-\frac{4}{\lambda_f}\right)\right|^2,$$

$$(\sigma v_{\rm rel})^{n=2,\ell=1}_{\rm bsf} = \sigma_0 \lambda_i \lambda_f^5 \frac{2S+1}{g_\chi^2} \frac{2^{12}\pi\alpha\zeta^7 e^{-4\zeta\lambda_i \operatorname{arccot}(\zeta\lambda_f/2)}}{9\left(\zeta^2\lambda_f^2+4\right)^5\left(1-e^{-2\pi\zeta\lambda_i}\right)} \times$$

$$\times \sum_{aMM'} \left[\left| C_{\mathcal{J}}^{aMM'}\left(\lambda_f(\zeta^2\lambda_i(3\lambda_f-4\lambda_i)+8)-12\lambda_i\right) + C_{\mathcal{T}}^{aMM'}\left(\zeta^2(-3\lambda_f^2+12\lambda_f\lambda_i-8\lambda_i^2)+4\right)\right|^2 + \right.$$

$$\left. + 2^5(\zeta^2\lambda_i^2+1)(\zeta^2\lambda_i^2+4)\left| C_{\mathcal{J}}^{aMM'}\lambda_f + 2C_{\mathcal{T}}^{aMM'}\right|^2 \right]. \tag{53}$$

$$(\sigma v_{\rm rel})^{n=1,\ell=0}_{\rm bsf} = \sigma_0 \frac{2S+1}{g_\chi^2} \frac{2^{11}\pi}{3} \sum_{aMM'} \left| C_{\mathcal{J}}^{aMM'} + \frac{1}{\lambda_f} C_{\mathcal{T}}^{aMM'} \right|^2 \times \begin{cases} \dfrac{\lambda_i^3\alpha}{\lambda_f v_{\rm rel}} e^{-4\lambda_i/\lambda_f} & v_{\rm rel} \ll \lambda_{i,f}\alpha \\ \dfrac{\lambda_f^5\alpha^4}{2\pi v_{\rm rel}^4} & v_{\rm rel} \gg \lambda_{i,f}\alpha \end{cases}$$