The Trigonometric Quantum Spectral Curve

Solving the spectral problem of the η-deformed superstring theory

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Based on work with
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Universität Hamburg
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Desy Theory workshop 2017

Lifting energy degeneracies

Goal: understanding a quantum mechanical system through its spectrum

- Coinciding energy eigenvalues often signal the presence of symmetry
- Lifting the degeneracy can help discover new symmetries/features

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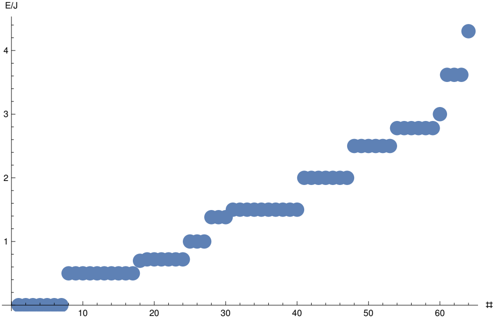
Example: Heisenberg XXX spin chain

Problem: Finding the precise relation between energies and solutions to its Bethe equations



$$H_{\mathrm{XXX}} = -J\sum_{j=1}^{L}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \quad \sigma_j^z\sigma_{j+1}^z\right)$$

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Example: Heisenberg XXZ spin chain

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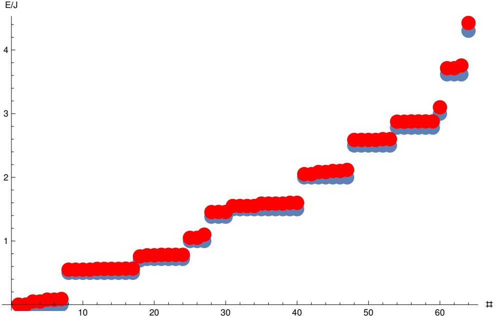


$$H_{\mathrm{XXZ}} = -J \sum_{j=1}^{L} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right)$$

Studying deformations proved very useful

Motivation: AdS/CFT correspondence

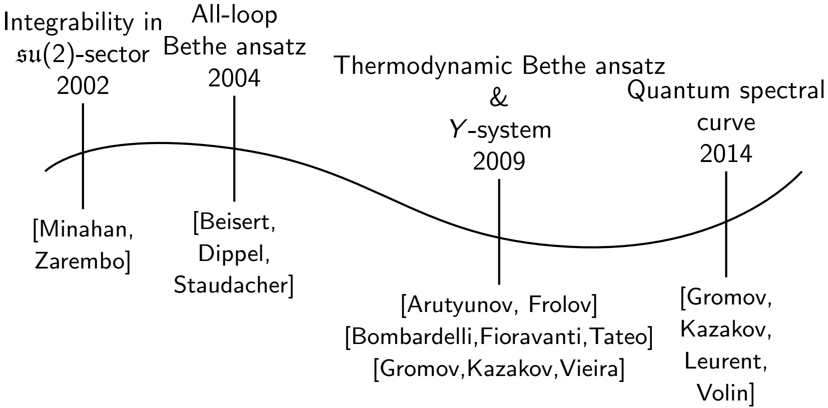
free type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$	$\Longleftrightarrow$	planar $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory	[Maldacena 1999]

- strong/weak duality
- To test the hypothesis one can study *the spectral problem*

string energies $\Longleftrightarrow$ scaling dimensions

very accessible due to integrability

Timeline of the spectral problem for $\mathcal{N} = 4$-SYM-theory



The quantum spectral curve

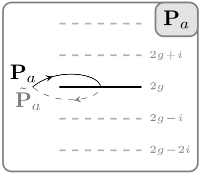
- reduces the problem to a simple set of equations
- allows to analyze the complete spectrum perturbatively up to very high order (14 loops)

Quantum Spectral Curve

Unknown functions: $\mathbf{P}_a, \mathbf{P}^a$, $\mu_{ab} = -\mu_{ba}$, $a, b = 1, \cdots, 4$.

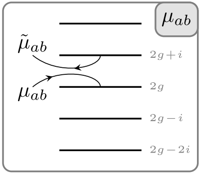
$\mathbf{P}\mu$-system:

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a \tilde{\mathbf{P}}_b - \mathbf{P}_b \tilde{\mathbf{P}}_a,$$
$$\tilde{\mathbf{P}}_a = \mu_{ab} \mathbf{P}^b,$$
$$\mathbf{P}_a \mathbf{P}^a = 0, \quad \mathrm{Pf}(\mu) = 1.$$



The 6 charges parametrizing PSU(2,2|4) multiplets occur in the asymptotics: as (for $u \in \mathbb{R}$) $u \to \infty$

$$\mathbf{P}_a \simeq A_a u^{-\hat{\lambda}_a},$$
$$\mu_{12} \simeq u^{\Delta - J}.$$



adapted from

[Gromov, Kazakov, Leurent, Volin 2014]

Quantum Spectral Curve

Triumphs:

- Perturbative analytical and numerical solution of the spectrum of $\mathcal{N} = 4$ [Marboe,Volin 2014-2017],[Gromov, Levkovich-Maslyuk, Sizov 2015]
- Access to QCD pomeron [Gromov, Levkovich-Maslyuk,2014],[Alfimov, Gromov, Kazakov 2015]

Questions:

- How does the quantum spectral curve fit into the integrability framework?
- Is there any hidden structure present in the resulting anomalous dimensions?
- Can we use the quantum spectral curve for different observables or theories? Other (non-local) field theories, non-commutative geometries, etc.?

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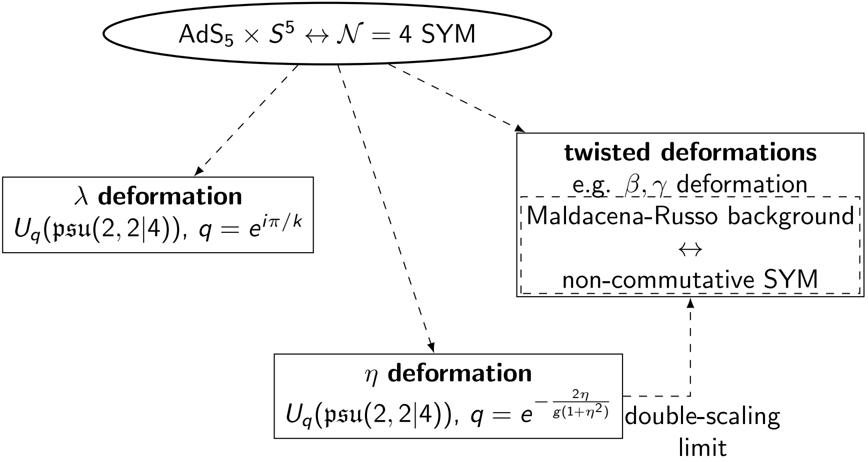
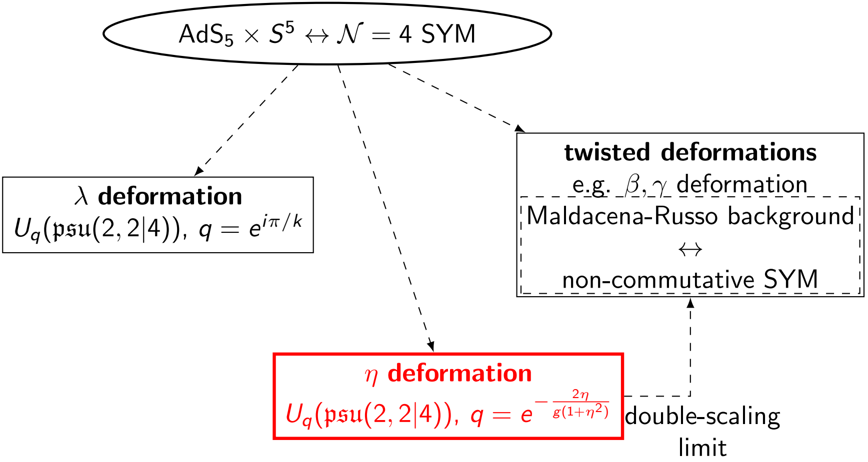
Integrable super string deformations



Integrable super string deformations



Properties

This model

- reduces to the famous $\mathrm{Ad}S_5 \times S^5$ model as $\eta \to 0$
- remains to be classically integrable [Delduc, Magrot and Vicedo, 2013]
- is not supersymmetric
- has κ symmetry, but does not admit embedding in a consistent supergravity theory [Arutyunov, Frolov, Hoare, Roiban, Tseytlin, 2015][Arutyunov, Borsato, Frolov, 2016]

Roadmap: building the η-deformed QSC

S-matrix
[Beisert,Koroteev 2008]

→

mirror Thermodynamic Bethe Ansatz
[Arutyunov, van Tongeren, de Leeuw 2014]

↓ Finding analytic properties of Y functions on the cylinder

analytic Y-system

← requires spectral theory for periodic functions

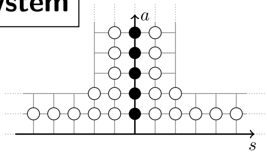
analytic T-system

→ due to presence of Hirota equation

QSC$_\eta$

Finding right boundary conditions is hard problem

Hirota: $T^+_{a,s}T^-_{a,s} = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}$



Algebraic simplifications of the spectral problem

- **TBA equations:** infinitely many coupled integral equations for unknown periodic functions $Y_{a,s}$

$$\log(Y_\bullet) = \sum_{(a,s)} \log(1 + Y_{a,s}^{m_\bullet}) \star K^{\bullet,(a,s)}$$

- Y **system:** infinitely many finite-difference equations

$$Y_{a,s}^+ Y_{a,s}^- = \frac{(1+Y_{a,s+1})(1+Y_{a,s-1})}{(1+1/Y_{a+1,s})(1+1/Y_{a-1,s})}, \quad f^\pm(u) := f(u \pm icn)$$

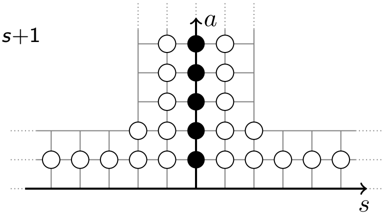
- T **system/Hirota equation:**

$$T_{a,s}^+ T_{a,s}^- = T_{a-1,s} T_{a+1,s} + T_{a,s-1} T_{a,s+1}$$

by reparametrizing

$$Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}$$

Y/T **hook**



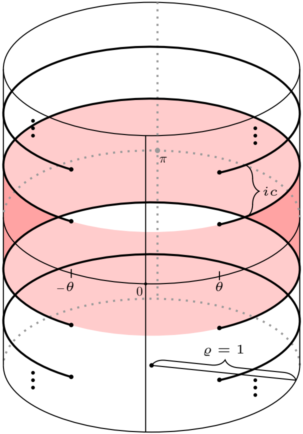
Transferring analytic properties

- A string energy corresponds to a **particular** solution of the equations
- Specify this solution by a set of **analyticity data**

Analyticity data:

- *TBA equations:* driving terms, difficult to construct
- *Y/T system:* discontinuity relations and asymptotics

Y analyticity



$\mathbf{P}\mu$ system

Algebraically

$$\hat{\mathbb{T}}_{1,s} = \mathbf{P}_1^{[+s]}\mathbf{P}_2^{[-s]} - \mathbf{P}_1^{[-s]}\mathbf{P}_2^{[+s]} \qquad \text{for } s > 0,$$
$$\hat{\mathbb{T}}_{1,s} = \mathbf{P}^{4[+s]}\mathbf{P}^{3[-s]} - \mathbf{P}^{4[-s]}\mathbf{P}^{3[+s]} \qquad \text{for } s < 0$$

and $\mu_{12} := \mathbf{T}_{0,1}^{1/2}$ parametrize all T functions.

Nicely repackaging the analyticity data by introducing more μ_{ab} and $\mathbf{P}_a, \mathbf{P}^a$, leads to the $\mathbf{P}\mu$-system: for $a, b = 1, \cdots, 4$

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a, \quad \tilde{\mathbf{P}}_a = \mu_{ab}\mathbf{P}^b, \quad \mathbf{P}_a\mathbf{P}^a = 0, \quad \mathrm{Pf}(\mu) = 1$$

with all functions 2π (anti-)periodic.

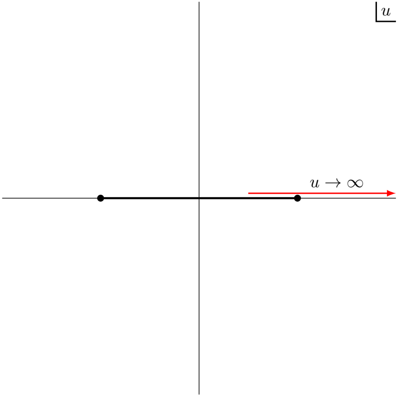
Asymptotics

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a \tilde{\mathbf{P}}_b - \mathbf{P}_b \tilde{\mathbf{P}}_a, \quad \tilde{\mathbf{P}}_a = \mu_{ab} \mathbf{P}^b, \quad \mathbf{P}_a \mathbf{P}^a = 0, \quad \mathrm{Pf}(\mu) = 1$$

Which solutions should we pick?

- Look at asymptotics of $\mathbf{P}$ and μ_{ab}

Undeformed case:



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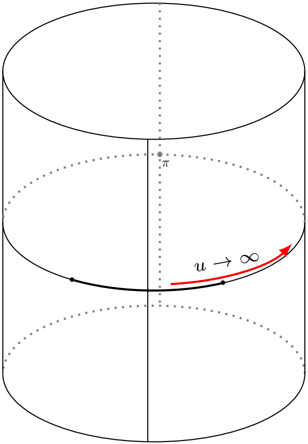

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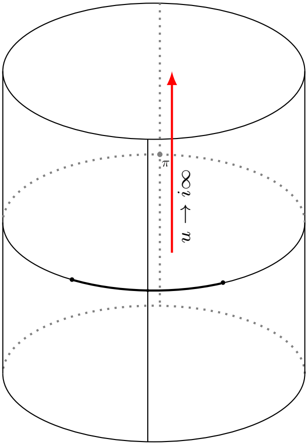

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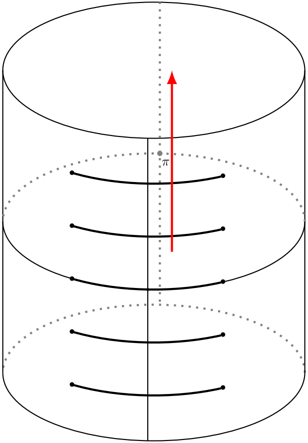

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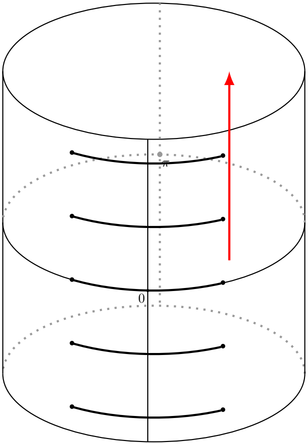

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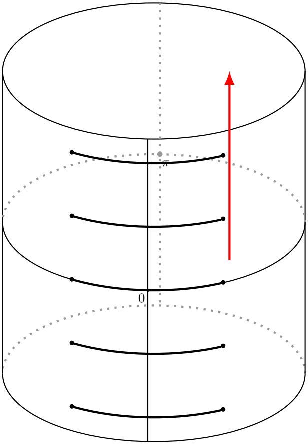

Asymptotics from TBA

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$$\frac{\mu_{12}^{[+2]}}{\mu_{12}} = \frac{\hat{\mathbb{T}}_{10}\hat{\mathbb{T}}_{23}}{\hat{\mathbb{T}}_{01}\hat{\mathbb{T}}_{32}}$$

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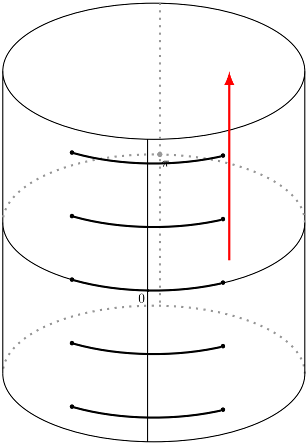

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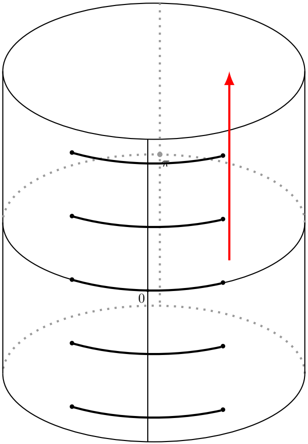

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$$\frac{\mu_{12}^{[+2]}}{\mu_{12}} = \exp\left(\log\left(1 + Y_{P,0}\right) \star K^{Py}\right)$$

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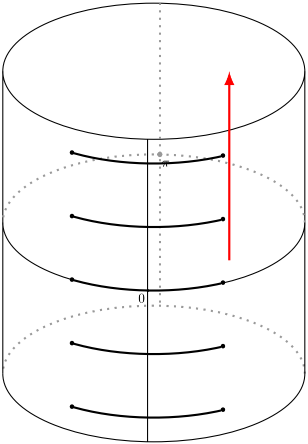

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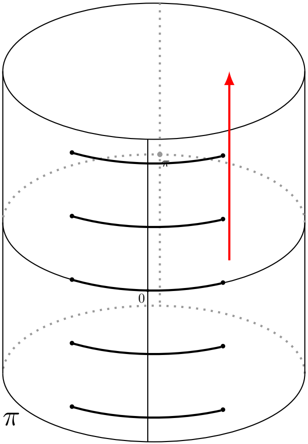
$$\frac{\mu_{12}^{[+2]}}{\mu_{12}} = \exp\left(\log\left(1 + Y_{P,0}\right) \star K^{Py}\right)$$
$$\simeq \exp\left(cE + \mathcal{O}\left(e^{iu}\right)\right)$$

$$\mu_{12} \simeq e^{-iu\frac{E}{2}} \text{ as } u \to i\infty, \text{ with } \theta < |\mathrm{Re}(u)| < \pi$$

Deformed case?:



$\mathbf{P}_a$-asymptotics

Postulating

$$\mathbf{P}_a \simeq A_a e^{iu\tilde{M}_a/2} \text{ as } u \to i\infty, \text{ with } \theta < |\mathrm{Re}(u)| < \pi$$

we use

- consistency of the $\mathbf{P}\mu$-system
- weak coupling comparison

to constrain the $\tilde{M}_a$ and A_a:

$$\tilde{M} = \frac{1}{2}\{J_1 + J_2 - J_3 + 2, J_1 - J_2 + J_3, -J_1 + J_2 + J_3, -J_1 - J_2 - J_3 - 2\}$$

A_a depend on all 6 charges

$\mathbf{P}_a$-asymptotics

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A_a depend on all 6 charges

Sidenote:

- Deriving the dual $\mathbf{Q}\omega$-system is straightforward
- The $\mathbf{Q}$ carry the other 3 quantum numbers in their asymptotics
- These can be embedded in the $GL(4|4)$-QQ-system

η-deformed Quantum Spectral Curve

$\mathbf{P}\mu$-system: for $a, b = 1, \cdots, 4$

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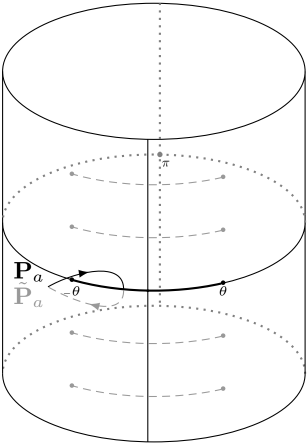
The 6 charges parametrising $\mathrm{PSU}_q(2,2|4)$-multiplets occur in the asymptotics: as $z = e^{-iu/2} \to \infty$

$$\mathbf{P}_a \simeq A_a z^{-\tilde{M}_a},$$
$$\mu_{12} \simeq z^E.$$



Conclusions

We have constructed a trigonometric QSC!

- Algebraically it is identical to the undeformed QSC
- But its analytical properties are very different

So we see that

- the rational-to-trigonometric concept applies to the QSC

Conclusions

We have constructed a trigonometric QSC!

- Algebraically it is identical to the undeformed QSC
- But its analytical properties are very different
- the rational-to-trigonometric concept applies to the QSC

Future directions:

- Generalize the analytic perturbative algorithm to the trigonometric case
- Compute energies, e.g. "*q*-deformed Konishi"
- Use mirror duality to analyze the relationship between spectral data and thermodynamics
- Thermodynamics of the $AdS_5 \times S^5$-string
- Treat the root-of-unity case
- Find an elliptic QSC?

The $\mathrm{AdS}_5 \times S^5$ world-sheet theory

Let $\mathfrak{g} \in \mathrm{SU}(2,2|4)$ and define the current

$$A_\alpha = -\mathfrak{g}^{-1}\partial_\alpha \mathfrak{g} = A_\alpha^{(0)} + A_\alpha^{(2)} + A_\alpha^{(1)} + A_\alpha^{(3)}$$

with a decomposition under the $\mathbb{Z}_4$-grading of $\mathfrak{su}(2,2|4)$.

Langrangian density [Metsaev and Tseytlin, 1998][Bena et al., 2003]]

$$\begin{aligned}\mathcal{L} &= -g(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\mathrm{str}\left(A_\alpha \mathrm{d} A_\beta\right) \\ &= -\frac{g}{2}\left(\underbrace{\gamma^{\alpha\beta}\mathrm{str}\left(A_\alpha^{(2)}A_\beta^{(2)}\right)}_{\text{kinetic term}} + \underbrace{\kappa\epsilon^{\alpha\beta}\mathrm{str}\left(A_\alpha^{(1)}A_\beta^{(3)}\right)}_{\text{Wess-Zumino term}}\right)\end{aligned}$$

$$\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}, \quad \mathrm{d} = P_1 + 2P_2 - P_3$$

Symmetry algebra: centrally extended $\mathfrak{psu}(2,2|4)$

The $\text{AdS}_5 \times S^5$ world-sheet theory

Bosonic Polyakov action of the NLSM: with $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$

$$S^{\text{b}} = -\frac{1}{2} g \int d\sigma d\tau \gamma^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN}$$

Target space metric:

$$\begin{aligned} ds^2_{\text{AdS}_5} &= -\left(1+\rho^2\right) dt^2 + \frac{d\rho^2}{(1+\rho^2)} \\ &+ \rho^2 \left(d\zeta^2 + \cos^2\zeta d\psi_1^2\right) + \rho^2 \sin^2\zeta d\psi_2^2 \\ ds^2_{\text{S}^5} &= \left(1-r^2\right) d\phi^2 + \frac{dr^2}{(1-r^2)} \\ &+ r^2 \left(d\xi^2 + \cos^2\xi d\phi_1^2\right) + r^2 \sin^2\xi d\phi_2^2 \end{aligned}$$

The $\left(\mathrm{AdS}_5 \times S^5\right)_\eta$ world-sheet theory

Bosonic Polyakov action of the NLSM: with $\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}$

$$S^{\mathrm{b}} = -\frac{1}{2}\left(\frac{1+\eta^2}{1-\eta^2}g\right)\int d\sigma d\tau \left(\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN} - \epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N B_{MN}\right)$$

[Delduc, Magrot and Vicedo, 2013]

Target space metric:

$$ds^2_{(\mathrm{AdS}_5)_\eta} = -\frac{1+\rho^2}{1-\varkappa^2\rho^2}dt^2 + \frac{d\rho^2}{(1+\rho^2)(1-\varkappa^2\rho^2)} + \frac{\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta}\left(d\zeta^2 + \cos^2\zeta\, d\psi_1^2\right) + \rho^2\sin^2\zeta\, d\psi_2^2$$

$$ds^2_{(\mathrm{S}^5)_\eta} = \frac{1-r^2}{1+\varkappa^2\rho^2}d\phi^2 + \frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} + \frac{r^2}{1+\varkappa^2 r^4\sin^2\xi}\left(d\xi^2 + \cos^2\xi\, d\phi_1^2\right) + r^2\sin^2\xi\, d\phi_2^2$$

with $\varkappa = \frac{2\eta}{1-\eta^2}$ and $\eta \in [0, 1[$.

η-deforming the $\mathrm{AdS}_5 \times S^5$ world-sheet theory

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with a decomposition under the $\mathbb{Z}_4$-grading of $\mathfrak{su}(2,2|4)$.

Langrangian density [Delduc, Magrot and Vicedo, 2013]

$$\mathcal{L} = -\frac{g}{4}(1+\eta^2)(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta})\mathrm{str}\left(A_\alpha \mathrm{d}_\eta \circ \frac{1}{1-\eta R_{\mathfrak{g}} \circ \mathrm{d}_\eta} A_\beta\right)$$

$$\gamma^{\alpha\beta} = h^{\alpha\beta}\sqrt{-h}, \quad \mathrm{d}_\eta = P_1 + \frac{2}{1-\eta^2}P_2 - P_3$$

Symmetry algebra: centrally extended $U_{q(\eta)}\left(\mathfrak{psu}(2,2|4)\right)$

$R_{\mathfrak{g}}(M) = \mathfrak{g}^{-1}R\left(\mathfrak{g}M\mathfrak{g}^{-1}\right)\mathfrak{g}$ where R solves the *modified classical Yang-Baxter equation*:
$[R(M), R(N)] - R\left([R(M), N] + [M, R(N)]\right) = [M, N]$ for all $M, N \in \mathrm{SU}(2,2|4)$

Polyakov action

When restricted to the bosonic sector, the Polyakov action becomes

$$S^{\mathrm{b}} = -\frac{1}{2}\left(\frac{1+\eta^2}{1-\eta^2}g\right)\int d\sigma d\tau\left(\gamma^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN} - \epsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N B_{MN}\right),$$

with $(\varkappa = \frac{2\eta}{1-\eta^2})$

$$\begin{aligned} ds^2_{(\mathrm{AdS}_5)_\eta} &= -\frac{1+\rho^2}{1-\varkappa^2\rho^2}dt^2 + \frac{d\rho^2}{(1+\rho^2)(1-\varkappa^2\rho^2)} \\ &+ \frac{\rho^2}{1+\varkappa^2\rho^4\sin^2\zeta}\left(d\zeta^2+\cos^2\zeta d\psi_1^2\right) + \rho^2\sin^2\zeta d\psi_2^2 \\ ds^2_{(\mathrm{S}^5)_\eta} &= \frac{1-r^2}{1+\varkappa^2\rho^2}d\phi^2 + \frac{dr^2}{(1-r^2)(1+\varkappa^2 r^2)} \\ &+ \frac{r^2}{1+\varkappa^2 r^4\sin^2\xi}\left(d\xi^2+\cos^2\xi d\phi_1^2\right) + r^2\sin^2\xi d\phi_2^2 \end{aligned}$$

Taking $\eta \to 0$ reduces to the undeformed action and target space metric.

Discontinuity relations

$$\left[\log Y^{(\alpha)}_{1|w}\right]_{\pm 1}(u) = \left[L^{(\alpha)}_{-}\right]_0(u) \qquad L_\chi = \log(1+1/Y_\chi),$$

$$\left[\log Y^{(\alpha)}_{1|vw}\right]_{\pm 1}(u) = \left[\Lambda^{(\alpha)}_{-}\right]_0(u), \qquad \Lambda_\chi = \log(1+Y_\chi),$$

$$\left[\log \frac{Y_-}{Y_+}\right]_{\pm 2N}(u) = -\sum_{P=1}^{N}\left[\Lambda_P\right]_{\pm(2N-P)}(u) \text{ for } N \geq 1$$

$$\left[\Delta\right]_{\pm 2N}(u) = \pm\sum_{\alpha}\left(\left[L^{(\alpha)}_{\mp}\right]_{\pm 2N}(u) + \sum_{M=1}^{N}\left[L^{(\alpha)}_{M|vw}\right]_{\pm(2N-M)}(u) + \left[\log Y^{(\alpha)}_{-}\right]_0(u)\right),$$

where

$$\Delta(u) = \begin{cases} \check{\Delta}(u) & \text{if } \mathrm{Im}(u) > 0 \\ \tilde{\check{\Delta}}(u) & \text{if } \mathrm{Im}(u) < 0 \end{cases}$$

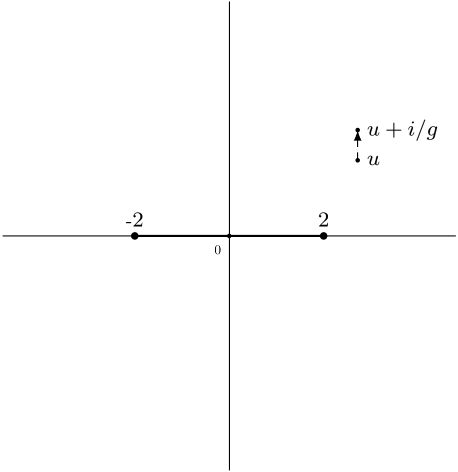
$$\check{\Delta}(u) := \left[\log Y_1\right](u)$$

$$\Delta(iu+\epsilon) - \Delta(iu-\epsilon) = 2\pi L i, \text{ for } u \in \mathbb{R}$$

Properties of the undeformed S-matrix

One can find the S-matrix from symmetry assuming integrability on the quantum level

- parametrized by shifts of the Zhukovsky coordinate $x^{\pm}(u) = x\,(u \pm i/g)$.
- The branch point locations ± 2

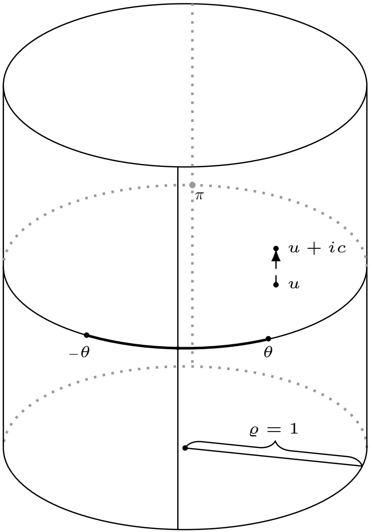


Properties of the η-deformed S-matrix

One can find the S-matrix from symmetry assuming integrability on the quantum level [Beisert, Koroteev, 2008]:

- parametrized by shifts of the Zhukovsky coordinate $x^{\pm}(u) = x\left(u \pm ic(\eta, g)\right)$.
- The branch point locations $\pm\theta$ depend on η and g
- exhibits mirror duality
- Undeformed limit:

$$\lim_{c \to 0} x(gcu) = x_{\text{und}}(u)$$



Building the η-deformed QSC

"Going from rational to trigonometric"

Short representations of $\mathfrak{psu}(2|2)^{\oplus 2}$ are parametrized via $x^{\pm} := x(u \pm i/2)$ with

$$x(u) = \frac{1}{2}\left(\frac{u}{g} + i\sqrt{4 - \frac{u^2}{g^2}}\right)$$

- solves Zhukovsky identity

$$\frac{u}{g} = x + \frac{1}{x}$$

- 1 square root branch cut between $\pm 2g$
- behaves as $x(u) \simeq u$ as $u \to \infty$.

Building the η-deformed QSC

"Going from rational to trigonometric"

Short representations of $\mathfrak{psu}_q(2|2)^{\oplus 2}$ are parametrized via $x^{\pm} = x(u \pm ic)$ with

$$x(u) = -i \csc\theta \left(e^{iu} - \cos\theta + \left(1 + e^{iu}\right) \sqrt{\frac{\cos(u) - \cos(\theta)}{\cos(u) + 1}} \right),$$

with $\theta = 2 \arcsin(h \sinh c))$. Note $h = h(g, q)$ and $c = c(g, q)$.

- solves deformed Zhukovsky identity

$$e^{ui} = \frac{x + \frac{1}{x} + \xi + \xi^{-1}}{\xi^{-1} - \xi} \text{ with } \xi = i \tan(\theta/2),$$

- 1 square root branch cut between $\pm\theta$ on the strip with $|\mathrm{Re}(u)| \leq \pi$
- behaves as $x(u) \simeq e^{iu}$ as $u \to -i\infty$.

Solving difference equations: rational case

Let g be upper-half-plane analytic.
Question: solve for f in

$$f^+ - f^- = g.$$

Formal solution:

$$f = \sum_{n=1}^{\infty} g^{[2n-1]}$$

If $\lim_{u\to\infty} g(u) = 0$ in the uhp

$$g(u) = -\frac{1}{2\pi i}\int_{\mathbb{R}} dv \frac{\rho_g(v)}{u-v} \text{ in the uph,}$$

with spectral density

$$\rho_g(u) = 2\lim_{\epsilon\to 0}\mathrm{Re}(g(u+i\epsilon)).$$

Summing

$$\sum_{n=0}^{\infty}\left(\frac{1}{v+2icn} - \frac{1}{2ic(n+1)}\right) \sim \psi\left(\frac{iv}{2c}\right)$$

yields the regularized solution:

$$f \sim \int_{\mathbb{R}} dv \psi^+(v-u)\rho_g(v) \text{ in the uph}$$

Solving difference equations: periodic case

Let g be upper-half-plane analytic.
Question: solve for f in

$$f^+ - f^- = g. \qquad \text{Formal solution: } f_\mathsf{F} = \sum_{n=1}^{\infty} g^{[2n-1]}.$$

If $\lim_{u\to i\infty} g(u) \in \mathbb{R}$,

$$g(u) = -\frac{1}{2\pi i}\int_{-\pi}^{\pi} dv \frac{\rho_g(v)}{\tan(u-v)} \text{ in the uhp}$$

with spectral density

$$\rho_g(u) = 2\lim_{\epsilon\to 0}\mathrm{Re}(g(u+i\epsilon)).$$

In our formal solution:

$$f_\mathsf{F} \sim \int_{-\pi}^{\pi} dv \sum_{n=1}^{\infty} \frac{1}{tan(u-v+2icn)}\rho_g(v).$$

Summing

$$\sum_{n=0}^{\infty}\left(\frac{1}{\tan(v+2icn)} - \frac{1}{\tan(2ic(n+1))}\right) \sim$$

$\sim \Psi_c :=$ combination of q-polygammas.

yields the regularized solution:

$$f \sim \int_{-\pi}^{\pi} dv \Psi_c^+(u-v)\rho_g(v) \text{ in the uhp.}$$

$$A_{a_0} A^{a_0} = 2 \frac{\prod_j \sinh\left(\frac{\tilde{M}_{a_0} - \hat{M}_j}{2}\right)}{\prod_{b \neq a_0} \sinh\left(\frac{\tilde{M}_{a_0} - \tilde{M}_b}{2}\right)}. \quad (1)$$

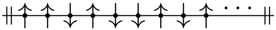
x-functions

$$x_m(u) = \frac{i}{\csc\theta}\left(e^{-iu} - (1 - e^{-iu})\sqrt{\frac{\cos u - \cos\theta}{\cos u - 1}}\right) \tag{2}$$

Motivation: finding integrability

On the CFT-side:

- $\mathfrak{su}(2)$ subsector:

$Tr\,(ZZ\varphi Z\varphi\varphi Z\varphi Z\cdots)$ $\sim$  [Minahan, Zarembo 2002]

dilatation operator D $\sim$ spin chain hamiltonian

- Perturbative approach:

$$D(\xi) = J + \sum_{k=1}^{\infty} H_k \xi^k \qquad \xi = \lambda / J^2$$

- $H_1 = H_{\mathrm{XXX}} \Rightarrow$ Integrability (Bethe ansatz)

On the string side:

- Classical world-sheet theory was found to be an integrable σ-model. [Bena et al. 2004]