Holographic lattice field theories

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As befits a twenty-minute talk, let's start with a simple question:

What is a quantum field theory?

"Lots and lots of harmonic oscillators, coupled together anharmonically, but not too strongly."

—A. M. Polyakov (light paraphrase)

At root, a quantum field theory is a theory of

- many identical local degrees of freedom,
- parameterized by a geometric space,
- coupled together in a local and homogeneous way.

There are lots of additional possible ingredients, but these are the key ones. (This is why spin systems can often be described by field theories, at least in some range of parameters.)

More technical working definition:

A QFT is a quantum theory whose degrees of freedom are functions* on some underlying space X.

These functions represent measurements (observables) that can be made independently everywhere in X.

The interactions of the theory are encoded in an *action functional:*

$$S : \mathscr{F}(X) \to \mathbb{R}.$$

(At this point, X could be anything: a manifold, a lattice, a graph, a set...)

*connections, tensor fields, sections of other bundles, ...

QFTs (and especially "toy" models like *topological* field theories) have also been of interest in pure math.

As one example, when X is a smooth four-manifold, Donaldson† defined new and sophisticated smooth invariants by considering (very roughly speaking) the behavior of particular field theories defined on X.

Many ideas have been imported from physics into mathematics across this bridge.

What about the reverse?

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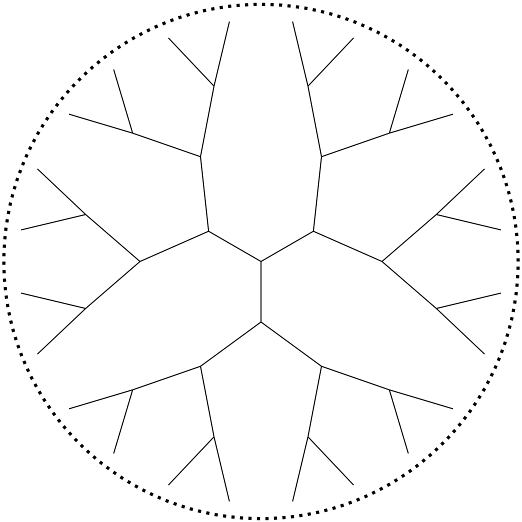
†Donaldson, *J. Diff. Geom.* 18.2 (1983); Witten, *Comm. Math. Phys.* 117.3 (1988)

A (rather vague and conceptual) question that is of great interest in physics lately:

How is the *geometry* of X encoded in—or how does it *emerge* from—the (not intrinsically geometric) data of the field theory: its Hilbert space, entanglement structure, time evolution, and so on?

I'm going to discuss (the very beginnings of) a perspective on this that borrows some ideas from pure math. The idea is to start to address this question by varying X.



To foreshadow a bit, here's an example of an interesting X...

To get started more methodically, let's flesh out the picture a bit by asking about the kinds of *structures* that X (and, correspondingly, the theory on it) might have.

What are some important examples?

— *Locality:*
X has a notion of *distance*, *measure*, or *causal structure,* which is respected by the interactions in the theory.

Often, this means something like

$$S[\phi] = \int_X \mathscr{L}[\phi(x)].$$

— *Symmetry:*
X may have symmetries (implemented by the action of a group G on X). The theory may or may not respect the action of these symmetries on the fields $\mathscr{F}(X)$.

What symmetries might we require of X?

The basic physical example is affine space, $X = \mathbb{R}^n$. It has many symmetries, but perhaps the most important one in QFT is the action of the *Poincaré group.*

Poincaré invariance embodies the requirement that the corresponding physics in X be *homogeneous* and *isotropic*.

Of course, $X = \mathbb{R}^n$ has even more symmetries, which theories may preserve (or not) in interesting fashion:

- *discrete symmetries* (P, T, et cetera. . .)
- *scale invariance.* (Broken scale invariance is renormalization group theory).
- *conformal invariance,* or local scale invariance. Scale-invariant theories are usually conformal.

— *One last piece of structure:*

If X has a notion of (mutually commuting) translation symmetries, I might further ask that there is a complete basis of eigenfunctions $\phi_k \in \mathscr{F}(X)$, diagonalizing those translations. Here k takes values in the joint spectrum of the translation operators, which I'll denote $X^\vee$.

This amounts to saying that there is a notion of mode expansion, or equivalently, of the Fourier transform.

On $\mathbb{R}^n$, $X = X^\vee$, but this isn't necessarily true: in lattice models, for example, $\mathbb{Z}^\vee = S^1$ (the "Brillouin zone").

I might also ask for a notion of "size" on $X^\vee$ (generalizing the length of a vector).

Once I have this, together with a notion of measure, translation symmetry, and a mode expansion, I have enough to write down a free theory of a real field on X:

$$S[\phi] = \int_{X^\vee} \phi(-k)\left(|k|^2 + m^2\right)\phi(k) + \cdots$$

And once I can do this, I'm really in familiar territory...

Key point: The more of this structure X has, the more a theory on it looks like your favorite typical QFT.

Two ways to make Conf(n)*-invariant Euclidean theories:*

- Pick $X = S^n$ (or $\mathbb{R}^n$), and look for conformal theories: fixed points of renormalization group flow.
- Pick $X = H^{n+1}$ (hyperbolic space one dimension higher), and take any field theory!

Isometries of H^{n+1} (analogues of Poincaré symmetry) are given by the group $G = \mathrm{Conf}(n)$.

In fact, $S^n = \partial H^{n+1}$, and a metric on H induces a conformal structure on its boundary...

This is (one) starting point of the *AdS/CFT correspondence*.[‡]

Central idea: (Certain) conformal theories on S^n are equivalent to field theories (with gravity) on H^{n+1}.

From the boundary perspective, the extra coordinate plays the role of a renormalization-group scale.

A newer entry in the dictionary: geometric features of the bulk are encoded in the entanglement structure of boundary states.

[‡] Maldacena, in *AIP Conf. Proc.* CONF-981170, 484.1 (1999); Witten, *ATMP* 2 (1998); Gubser, Klebanov, & Polyakov, *Phys. Lett. B* 428.1 (1998)

There has been much interest recently in *discrete* models of holography; most of these are variants of *tensor network* constructions, designed to reproduce Ryu-Takayanagi.

However, discretization usually breaks symmetries, which seem to be a key part of the story. Moreover, tensor networks aren't field theories, at least according to my definition... (they don't tend to have Hamiltonians; just produce vacuum state).

Can we make a discrete analogue of the story I've been telling?

Just need to come up with a discrete X, with properties that mimic those of H^{n+1}...[§]

[§] Manin & Marcolli, *ATMP* 5 (2001)

Key idea from mathematics:

Most of the structures on $\mathbb{R}^n$ exist because it's an affine space over a field (which, confusingly, now means a collection of abstract "numbers" for which addition, subtraction, multiplication, and division all work as expected).

At least as far as algebraic structures are concerned, affine spaces over fields all behave similarly. We can simply replace one field by another...

Where do fields come from?

$\mathbb{R}$ is defined by starting with the rational numbers, and "filling in the holes" (completing with respect to a notion of distance).

We will use fields called $\mathbb{Q}_p$, that are defined by an identical procedure—but with respect to a *different* notion of distance:

$$x = p^{\nu}(a/b) \quad (a, b \perp p) \qquad \Longrightarrow \; |x|_p = p^{-\nu}.$$

Note that the norm takes on only a discrete set of values.

These are the only possible completions of $\mathbb{Q}$!

$\mathbb{Q}_p$ *has all the structures I catalogued before:*

- There is an obvious translation symmetry on the affine space $\mathbb{Q}_p^n$.
- There are scaling symmetries as well.
- There is a unique translation-invariant integration measure dx on $\mathbb{Q}_p$ (additive Haar measure).
- The space of well-behaved (locally constant) functions on $\mathbb{Q}_p$ is spanned by eigenfunctions of translation, which take the form
$$\chi_p(kx) = e^{2\pi i \{kx\}_p}.$$
- $\mathbb{Q}_p$ is Fourier-self-dual: $k \in \mathbb{Q}_p^\vee \cong \mathbb{Q}_p$.
- There's a notion of size, namely $|\cdot|_p$.

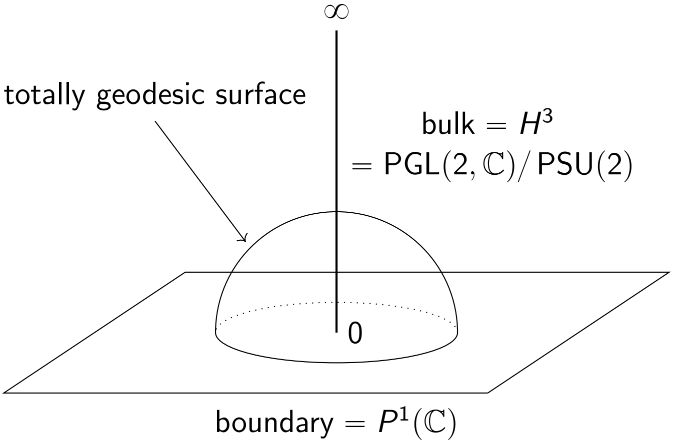
In low dimensions, the relevant symmetries for AdS/CFT can also be formulated algebraically!

$\mathrm{Conf}(2) = \mathrm{PGL}(2, \mathbb{C})$ acts on the boundary, $S^2 = P^1(\mathbb{C})$, by *Möbius transformations:*

$$z \mapsto \frac{az + b}{cz + d}.$$

H^3 is the quotient of the isometry group, $\mathrm{PGL}(2, \mathbb{C})$, by its maximal compact subgroup $\mathrm{PSU}(2)$.

To illustrate:



Replacing $\mathbb{C}$ by $\mathbb{Q}_p$ produces a discrete bulk space that is an analogue of H^3!

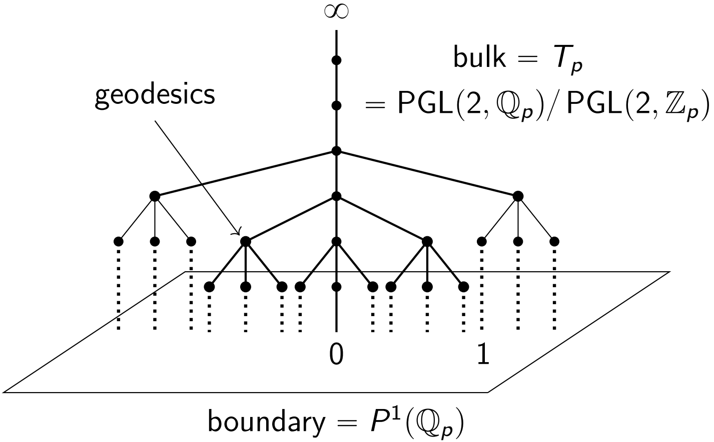
This space is the *Bruhat–Tits tree:*

$$T_p = \mathsf{PGL}(2, \mathbb{Q}_p)/\,\mathsf{PGL}(2, \mathbb{Z}_p),$$

an infinite tree of uniform valence $p + 1$.

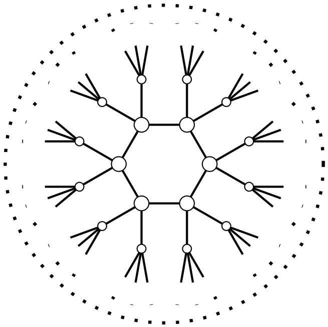
It has many properties characteristic of hyperbolic space: the perimeter of a circle is exponential in the radius; geodesic triangles are slim (in fact, they all look like the letter Y).

To illustrate ($p = 3$):



Just as before, $\mathrm{PGL}(2, \mathbb{Q}_p)$ acts by isometries on the tree, and by Möbius transformations on its boundary.

We can even obtain "black holes" in the same way, as quotients of this geometry by certain free subgroups:



The p-adic BTZ black hole (pictured for $p = 3$).

Then it's off to the races:
One can now try to understand the simplest instances of holography: for instance, free bulk scalar fields propagating without backreaction.

The Klein–Gordon equation and its plane-wave solutions:¶

$$\triangle\phi = \sum_{v'} (\phi(v') - \phi(v)) = m^2\phi, \quad \phi_\kappa(v) = p^{\kappa\langle v,x\rangle}.$$

$\langle v, x\rangle$ is the distance from v to the boundary point x, regularized to be zero at the (arbitrary) center vertex C.

The corresponding mass eigenvalue is

$$m_\kappa^2 = p^\kappa + p^{1-\kappa} - (p+1).$$

(Thus, the BF bound is $m_\kappa^2 \geqslant -(\sqrt{p}-1)^2$.)

¶Zabrodin, *CMP* 123.3 (1989); Heydeman, Marcolli, IAS, and Stoica, `arXiv:1605.07639`

Just as in ordinary AdS/CFT, these solutions provide a bulk/boundary Green's function:

$$\phi(v) = \frac{p}{p+1} \int_{\mathbb{Q}_p} d\mu_0(x)\, \phi_0(x) p^{\langle v,x \rangle}.$$

Bulk fields of mass m_κ couple to boundary operators of conformal dimension κ:[‖]

$$\langle \mathcal{O}_\kappa(x) \mathcal{O}_\kappa(y) \rangle \sim \frac{1}{|x-y|_p^{2\kappa}}.$$

If the boundary field ϕ_0 is a single mode (additive character of $\mathbb{Q}_p$), it stops contributing to the reconstruction of bulk physics abruptly, at a height determined by its wavelength.

[‖] For p-adic CFT, see Melzer, *Int. J. Mod. Phys. A* 4.18 (1989)

In many cases, p-adic analogues of familiar field theory models (such as the $O(N)$ model) can be defined straightforwardly. Computations in these theories exhibit universal answers, independent of which space the theory is defined on!

As an example, leading-order anomalous dimensions in $O(N)^{\aleph}$ for the operators ϕ and ϕ^2:

$$\gamma_{\phi,\phi^2} = \mathrm{Res}_{\delta=0}\, g_{\phi,\phi^2}(\delta) + O(1/N^2)$$

where the functions g_{ϕ,ϕ^2} are given by

$$g_{\phi}(\delta) = \frac{1}{N}\frac{B(n-s,\delta-s)}{B(n-s,n-s)},$$
$$g_{\phi^2}(\delta) = -\frac{2}{N}\frac{B(n-s,\delta-s)}{B(n-s,n-s)} + \frac{1}{N}\frac{B(\delta,\delta)}{B(n-s,n-s)}\left(2\frac{B(n-s,n-2s)}{B(n-s,n-s)} - 1\right).$$

$^{\aleph}$Gubser, Jepsen, Parikh, & Trundy, `arXiv:1703.04202`

These results apply equally well in every case, assuming the special functions involved are defined uniformly!

Let the *local zeta function* be defined following Tate's thesis as

$$\zeta_p(s) = \frac{1}{1 - p^{-s}}, \quad \zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2).$$

Then gamma and beta functions are defined for $\mathbb{R}^n$ or $\mathbb{Q}_p^n$ by the relations

$$\Gamma_p(s) = \frac{\zeta_p(s)}{\zeta_p(n - s)}, \quad B(t_1, t_2) = \frac{\Gamma_p(t_1)\Gamma_p(t_2)}{\Gamma_p(t_1 + t_2)},$$

where p is a prime or ∞.

Grassmann variables + sign characters = fermions:

We considered analogues of Klebanov-Tarnopolsky models (variants of SYK):[ϒ]

$$S_{\text{free}} = \int d\omega \, \frac{1}{2} \phi^{abc}(-\omega) |\omega|_p^s \, \text{sgn}(\omega) \phi^{abc}(\omega)$$

$$S_{\text{int}} = \int dt \, \phi^{abc} \phi^{ab'c'} \phi^{a'bc'} \phi^{a'b'c}$$

Here, the field is either commuting or anticommuting; pairs of indices are contracted either with δ or with a fixed antisymmetric matrix; and the sign character may be either "odd" or "even."

Exactly one specific collection of choices leads to consistent behavior in the IR for each X!

[ϒ] Gubser, Heydeman, Jepsen, Parikh, IAS, Stoica, & Trundy, `arXiv:1707.01087`

In the limit of large N and weak coupling, with g^2N^3 fixed, the leading-order Schwinger-Dyson equation is

$$G = F + \sigma_\Omega(g^2N^3)G \star G^3 \star F.$$

Solve in the IR to obtain universal limiting behavior:

$$G(t) = b\frac{\mathrm{sgn}(t)}{|t|^{1/2}}, \quad |t| \gg (g^2N^3)^{1/(2-4s)}$$

where

$$\frac{1}{b^4g^2N^3} = -\sigma_\Omega\Gamma(\pi_{-1/2,\mathrm{sgn}})\Gamma(\pi_{1/2,\mathrm{sgn}}).$$

Scaling in the IR limit is completely independent of the spectral parameter of the UV theory!

For fermionic theories with “direction-dependent” characters, one can do even better: it is possible to explicitly solve the Schwinger-Dyson equation for behavior at all scales, interpolating between the UV and the (universal) IR.

If $F(t) = f(|t|)\,\mathrm{sgn}(t)$ (and similarly for G), then

$$g = f - \frac{g^2 N^3}{p} |t|^2 g^4 f.$$

What about analogues of gravity?

One might try making the edge lengths dynamical.[ℷ] A plausible action comes from a notion of *Ricci curvature* for graphs.[††] On a tree-like graph, it reduces to

$$\kappa_{xy} = \frac{b_{xy}}{d_x}\left(b_{xy} - \sum b_{xx_i}\right) + \frac{b_{xy}}{d_y}\left(b_{xy} - \sum b_{yy_i}\right)$$

Linearized equations of motion are massless!

On-shell action (after regularization by an analogue of the Gibbons-Hawking-York boundary term) is *topological*.

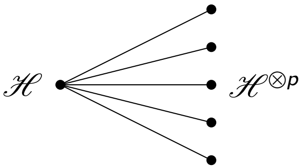
[ℷ] Gubser, Heydeman, Jepsen, Marcolli, Parikh, IAS, Stoica, & Trundy, JHEP 06 (2017) 157, `arXiv:1612.09580`

[††] Lin, Lu, & Yau, *Tohoku Math. J. 2nd ser.* 63.4 (2011); see also Ollivier

One might also try to strengthen the connection to ordinary tensor networks.[‡‡]

Several plausible directions here: first, one might try to use tensors (quantum codes) connected to curves over finite fields, and connect to algebraic structure of the tree.

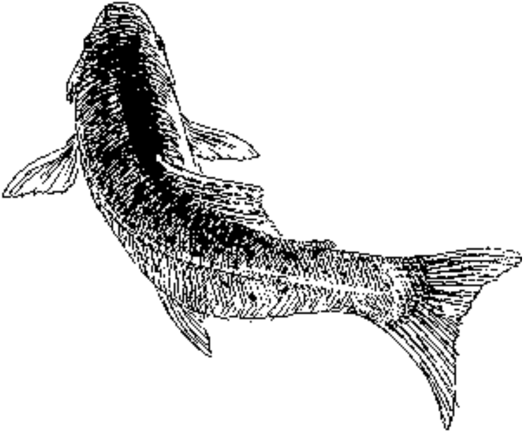
Another: the path integral of our theory *is* already, in some sense, a tensor network! (It's built by concatenating many copies of the same linear map...)



[‡‡] Heydeman, Marcolli, IAS, and Stoica, work in progress

Other future directions:

- Entanglement entropy?
- Further connections to exact renormalization group? Statistical mechanics models defined on Cayley trees?
- Use models to do complicated calculations in real AdS/CFT, via universality or adelic relations?
- Can one define invariants of p-adic spaces using field theories?



Thanks!

Drawing courtesy of Robert Savannah, U.S. Fish and Wildlife Service