Gaugino Condensation and Holomorphic BF Theory

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Supersymmetric gauge theories are toy models for theories of physical interest such as QCD.

Difficult questions such as the mass gap in Yang-Mills theory can be profitably attacked using tools from supersymmetry.

More technical motivation comes from trying to reproduce black hole entropy.

The growth of the number of BPS operators in $\mathcal{N} = 4$ SYM is expected to reproduce the entropy of supersymmetric black holes in $AdS_5 \times S^5$.

Counting BPS operators is a notoriously hard problem!

Can we study a simpler 'toy' problem that might teach some lessons?

Pure $\mathcal{N} = 1$ supersymmetric gauge theory with a simple gauge group G is widely believed to have $h^\vee$ massive vacua, where $h^\vee$ is the dual Coxeter number of the Lie algebra corresponding to G.

The vacua are distinguished by the phase of the gaugino condensate.

The theory has a classical $U(1)$ R-symmetry, the gaugino number, which is broken to a discrete $\mathbb{Z}_{2h^\vee}$ subgroup by instantons.

Common lore is that this theory undergoes spontaneous chiral symmetry breaking, with $\mathbb{Z}_{2h^\vee}$ spontaneously broken to $\mathbb{Z}_2$ by gaugino condensation.

The vacua are distinguished by the phase of the gaugino condensate.

There are $h^\vee$ massive vacua and the gaugino condensate satisfies

$$\frac{1}{16\pi^2}\langle \operatorname{Tr} \lambda^2 \rangle = \omega \Lambda^3$$

where ω is an $h^\vee$-th root of unity.

The Witten index of the theory is $\operatorname{Tr}(-1)^F = h^\vee$, where each vacua contributes one to the index.

The appearance of the dual Coxeter number as the Witten index suggests the appearance of affine Lie algebras.

One way to see their appearance is by compactifying the theory on a circle.

The affine Toda potential appears as the superpotential of the resulting three-dimensional low energy effective theory.
[Katz, Vafa '96] [Davies et. al. 2000]

By compactifying on a torus, the low-energy effective theory is described by a sigma model with target given by the moduli space of flat G-connections.

For $G = SU(n)$ the moduli space is simply $\mathbb{CP}^{n-1}$.

For general G the moduli space of flat G connections on T^2 is $\mathbb{WCP}(\{a_i\})$ where $\{a_i\}$ are the Dynkin indices of the affine Lie group corresponding to G [Friedman, Morgan, Witten].

$\mathbb{CP}^n$ Analogy

The $\mathbb{CP}^{n-1}$ non-linear sigma model has a $U(1)$ vector R-symmetry, but the classical $U(1)$ axial symmetry is broken to $\mathbb{Z}_{2n}$.

The discrete $\mathbb{Z}_{2n}$ axial R-symmetry is spontaneously broken to $\mathbb{Z}_2$ resulting in n-massive vacua.

The χ_y genus of $\mathbb{CP}^{n-1}$ is

$$y^{-n/2}(1 + y + \dots y^{n-1}) = y^{-n/2}(1 - y^n)/(1 - y)$$

vanishes at n-th roots of unity.

Similarly the flavored Witten index $\mathrm{Tr}(-1)^F y^J$ where J only makes sense for y a primitive n-root of unity.

The flavored Witten index corresponding to the sigma models for other G vanishes at primitive $h^\vee-$th roots of unity [Witten 2000].

It is natural to wonder if something similar happens in pure $\mathcal{N} = 1$ supersymmetric gauge theory.

The gauge invariant operators contributing to the index are found as follows. First we consider the single letter index satisfying

$$\{Q, \mathcal{O}\} = \{Q^\dagger, \mathcal{O}\} = 0.$$

The component fields of the vector multiplet contributing to the index are λ_α and $\overline{f}_{\dot{2}\dot{2}}$ where $\alpha = 1, 2$.

Furthermore, there is a relation from the equations of motion

$$\partial_{22}\lambda_1 = \partial_{12}\lambda_2.$$

The local operators can be constructed as follows. Build all possible words from the single letters λ and $\overline{f}$ and then projecting onto gauge invariant words. The index is then given by

$$I(t,x,h) = \int_G d\mu(g) \exp\left(\sum_{n=1}^{\infty} i(t^n, x^n, h^n, g^n)\right)$$

Table: The index of pure $\mathcal{N} = 1$ gauge theory

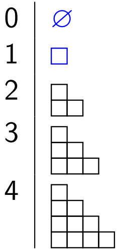
G	$\mathcal{I}_G(T)$
A_{N-1}	$(T^N)_\infty^N (T)_\infty^{-1}$
B_N	$(T^{4N-2})_\infty (T^{2N-1})_\infty^{N-1} (T^2)_\infty (T)_\infty^{-1}$
C_N	$(T^{2N+2})_\infty^{N-1} (T^{N+1})_\infty (T^2)_\infty (T)_\infty^{-1}$
D_N	$(T^{2N-2})_\infty^{N+1} (T^{N-1})_\infty^{-1} (T^2)_\infty (T)_\infty^{-1}$
E_6	$(T^{12})_\infty^7 (T^6)_\infty^{-1} (T^4)_\infty^{-1} (T^3)_\infty (T^2)_\infty (T)_\infty^{-1}$
E_7	$(T^{18})_\infty^8 (T^9)_\infty^{-1} (T^6)_\infty^{-1} (T^3)_\infty (T^2)_\infty (T)_\infty^{-1}$
E_8	$(T^{30})_\infty^9 (T^{15})_\infty^{-1} (T^{10})_\infty^{-1} (T^6)_\infty^{-1} (T^5)_\infty (T^3)_\infty (T^2)_\infty (T)_\infty^{-1}$
F_4	$(T^{18})_\infty^2 (T^9)_\infty^2 (T^6)_\infty^{-1} (T^3)_\infty (T^2)_\infty (T)_\infty^{-1}$
G_2	$(T^{12})_\infty (T^6)_\infty^{-1} (T^4)_\infty (T^3)_\infty (T^2)_\infty (T)_\infty^{-1}$

where $T = pq$.

k-core Partitions

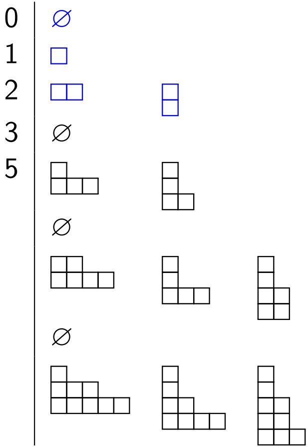
- For $G = A_{k-1}$ the index has a simple expression in terms of *k-core* partitions.
- These are partitions such that no hook length is divisible by k.

$SU(2)$ and 2-core partitions



$$I(A_1) = 1 + t^3 + t^6 + t^{10} + \dots$$

$SU(3)$ and 3-core partitions



$$I(A_2) = 1 + t + 2t^2 + 2t^4 + t^5 + 2t^6 + t^8 + \dots$$

Sum over Young Tableaux is similar to Nekrasov instanton partition function on ALE space.
[Dijkgraaf/Sulkowski, Bonelli/Maruyoshi/Tanzini, Fujii/Manabe, ...]

Surprisingly, a similar relation is known in the context "Pursuing the double affine Grassmannian" by Braverman and Finkelberg.

E_8 and the Monster

Recall

$$I(E_8) = (T^{30})^9_\infty (T^{15})^{-1}_\infty (T^{10})^{-1}_\infty (T^6)^{-1}_\infty (T^5)_\infty (T^3)_\infty (T^2)_\infty (T)^{-1}_\infty$$

Dropping some factors from the Cartan,

$$\chi = (T^{30})^1_\infty (T^{15})^{-1}_\infty (T^{10})^{-1}_\infty (T^6)^{-1}_\infty (T^5)_\infty (T^3)_\infty (T^2)_\infty (T)^{-1}_\infty$$

is the inverse of a "fake" McKay-Thompson series for the Monster group.

Does the Monster group play a role in gauge theory?

Is there a similar relation for E_6, E_7 and the largest Fischer and baby Monster groups? [He-McKay]

$\mathcal{N} = 2$ theory and Schur limit

Interestingly, independence of p and q means we can take the limit $p \to q$ and get the same result.

$$i(t, x, h, g) = \frac{p}{1-p} + \frac{q}{1-q} \to \frac{2q}{1-q}$$

This is the Schur limit of the $\mathcal{N} = 2$ superconformal index.
The same expression for the Schur index in non-superconformal theories computes the BPS Monodromy.
[Iqbal-Vafa, Cordova-Shao, S. Cecotti, J. Song, C. Vafa, W. Yan]

Holomorphic BF theory has a $(0,1)$-form $A \in \Omega^{0,1}(\mathbb{C}^2, \mathfrak{g})$ and a $(2,0)$-form $B \in \Omega^{2,0}(\mathbb{C}^2, \mathfrak{g})$.

The action is

$$S(A, B) = \int \langle B, \left(\overline{\partial} A + \frac{1}{2}[A, A] \right) \rangle_{\mathfrak{g}}.$$

It is equivalent to a "holomorphic twist" of pure $\mathcal{N} = 1$ theory. [Baulieu-Tanzini, Costello]

The equations of motion are

$$\overline{\partial}A + \frac{1}{2}[A, A] = 0$$
$$\overline{\partial}B + [A, B] = 0.$$

The first equation is the Mauer-Cartan equation and describes the deformation theory of G-bundles.

Recall that the fields contributing to the index of the vector multiplet were λ_α and $\overline{f}_{\dot{2}\dot{2}}$.

These correspond to the fields A and B in holomorphic BF theory.

Following [Grant, Grassi, Kim, Minwalla] we introduce fields corresponding to the supersymmetric letters.

$$\lambda^m(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (z^{\dot{\alpha}})^n (z^{\dot{\beta}} \lambda_{\dot{\beta}})$$
$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (z^{\dot{\alpha}})^n f$$

The action of the supersymmetry generator Q on these fields is

$$\{Q, \lambda(z)\} = -\frac{i}{2} \{\lambda(z), \lambda(z)\}$$
$$[Q, f(z)] = -i[\lambda(z), f(z)]$$

These are precisely the equations of motion of the holomorphic BF theory.

Thank you!

Taking this into account,

$$\partial_{12}^n \partial_{22}^m |\lambda_1\rangle,\ \partial_{22}^m |\lambda_2\rangle\ \partial_{12}^n \partial_{22}^m |\bar{f}_{\dot{2}\dot{2}}\rangle, \qquad n, m = 0, 1, 2 \dots$$

form a basis for the single-letter operators.

Their index is given by the supertrace:

$$\begin{aligned} i(t,x,h,g) &= \mathrm{str}(t^R x^{2J_3}) \\ &= -\frac{tx}{(1-tx)(1-tx^{-1})} - \frac{tx^{-1}}{(1-tx)(1-tx^{-1})} \\ &+ \frac{t^2}{(1-tx)(1-tx^{-1})} \\ &= \frac{2t^2 - t\chi_2(x)}{(1-tx)(1-tx^{-1})} \end{aligned}$$

where $\chi_2(x) = x + x^{-1}$

Explicitly,

$$I(A_N) = (p;p)^N (q;q)^N \frac{1}{(N+1)!} \times \int \prod_{j=1}^{N} \frac{dz_j}{2\pi i z_j} \prod_{1\leq i<j<N+1} \frac{1}{\Gamma(z_i/z_j, z_j/z_i; p, q)}\Big|_{\prod_{j=1}^{N+1} z_j = 1}$$

where

$$\frac{1}{\Gamma(z, z^{-1}; p, q)} = \theta(z;q)\theta(z^{-1};p)$$

$$\theta(z;q) = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} x^n$$