Yangian Symmetry of Scalar Fishnet Graphs

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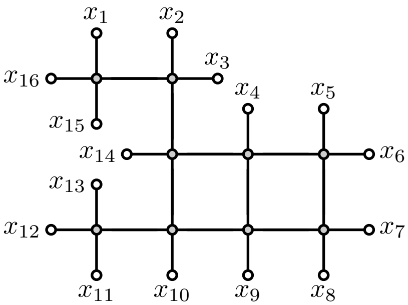
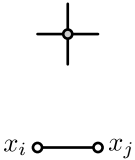
based on

D. Chicherin, V. Kazakov, F. Loebbert, D.M. and D. Zhong [1704.01967, 1708.00007, to appear]

Scalar Fishnet Feynman Graphs

We consider fishnet graphs of the type:





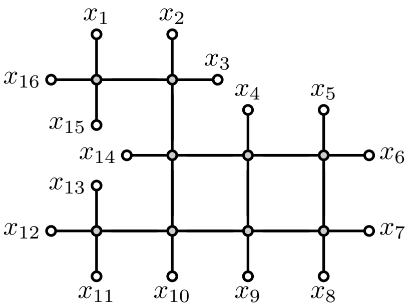
$$\int \mathrm{d}^4 x_i$$

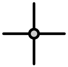
$$\frac{1}{x_{ij}^2}$$

Fishnet graphs have many nice features:

- IR and UV finite
- Fishnet Feynman integrals play a role in many QFT's
- Conformal symmetry
- Can be interpreted as integrable vertex models [Zamolodchikov '80]

This talk: We add Yangian Symmetry to this list!





$$\int \mathrm{d}^4 x_i$$

$$\frac{1}{x_{ij}^2}$$

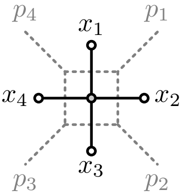
Fishnet graphs are in one-to-one correspondence with correlators in an integrable bi-scalar QFT (limit of $\mathcal{N}=4$ SYM) [Gürdoğan, Kazakov '15]

$$\mathcal{L} = \tfrac{N_c}{2}\,\mathrm{tr}\big(\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + 2\xi^2\, \phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2\big)$$

$$|F_n\rangle = \langle \mathrm{Tr}(\chi_1(x_1)\ldots\chi_n(x_n))\rangle$$

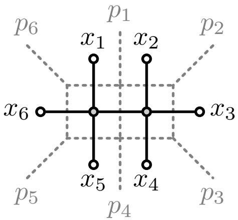
where $\chi_i(x_i) \in \{\phi_1(x_i), \phi_2(x_i), \phi_1^\dagger(x_i), \phi_2^\dagger(x_i)\}$.

The (Double) Cross Integral



$$|F_4\rangle = \int \frac{\mathrm{d}^4 x_0}{x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2} = \frac{\phi(u,v)}{x_{13}^2 x_{24}^2}$$

where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ are the conformal cross ratios. [Ussyukina, Davydychev '93]



$$|F_6\rangle = \int \frac{\mathrm{d}^4 x_0\, \mathrm{d}^4 x_{0'}}{x_{10}^2 x_{50}^2 x_{60}^2 x_{00'}^2 x_{20'}^2 x_{30'}^2 x_{40'}^2}$$

$$= \text{unkown}$$

Integrals can be related to momentum space integrals by introducing the dual coordinates:

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu$$

Conformal Symmetry

Conformal algebra $\mathfrak{so}(2,4)$:

$$\mathrm{D}^{\Delta} = -ix_\mu \partial^\mu - i\Delta \qquad \mathrm{L}_{\mu\nu} = ix_\mu \partial_\nu - ix_\nu \partial_\mu$$

$$\mathrm{P}_\mu = -i\partial_\mu \qquad \mathrm{K}_\mu^{\Delta} = ix^2 \partial_\mu - 2ix_\mu x^\nu \partial_\nu - 2i\Delta x_\mu$$

Its convenient to study the inversion $\mathrm{K}_\mu^{\Delta=0} = \mathrm{I}\,\mathrm{P}_\mu\,\mathrm{I}$

$$\mathrm{I}[x^\mu] = \frac{x^\mu}{x^2} \qquad \mathrm{I}[x_{ij}^{-2}] = \frac{x_i^2 x_j^2}{x_{ij}^2} \qquad \mathrm{I}[\mathrm{d}^4 x_0] = \frac{\mathrm{d}^4 x_0}{x_0^8}$$

The (double) cross integral transforms under inversion as

$$\mathrm{I}[|F_4\rangle] = x_1^2 x_2^2 x_3^2 x_4^2 |F_4\rangle \qquad \mathrm{I}[|F_6\rangle] = x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_6^2 |F_6\rangle$$

Fishnet integrals are conformal:

$$\sum_i \mathrm{K}_i^{\mu,\Delta=1} |F_4\rangle = 0 \qquad \sum_i \mathrm{K}_i^{\mu,\Delta=1} |F_6\rangle = 0$$

The Yangian Algebra

The Yangian algebra $Y[\mathfrak{g}]$ of a semi-simple Lie algebra $\mathfrak{g}$ is the enveloping algebra spanned by

Level-0: $\{\mathrm{J}^\kappa\}$ Level-1: $\{\widehat{\mathrm{J}}^\kappa\}$

satisfying the following axioms

(1) Lie Algebra $\left[\mathrm{J}^\kappa, \mathrm{J}^\rho\right] = f^{\kappa\rho}{}_\delta\, \mathrm{J}^\delta$

(2) Level-one Relation $\left[\mathrm{J}^\kappa, \widehat{\mathrm{J}}^\rho\right] = f^{\kappa\rho}{}_\delta\, \widehat{\mathrm{J}}^\delta$

(3) Serre Relation $\left[\widehat{\mathrm{J}}^{(\kappa}, \left[\widehat{\mathrm{J}}^\rho, \mathrm{J}^{\delta)}\right]\right] = \frac{1}{24}\, c^{\kappa\rho\delta}{}_{\sigma\omega\gamma} \left\{\mathrm{J}^\sigma, \mathrm{J}^\omega, \mathrm{J}^\gamma\right\}$

$Y(\mathfrak{g})$ is an infinite-dimensional, graded algebra, i.e. $\left[\widehat{\mathrm{J}}^\kappa, \widehat{\mathrm{J}}^\rho\right] = f^{\kappa\rho}{}_\delta\, \widehat{\widehat{\mathrm{J}}}^\delta + X^{\kappa\rho}$.

Coproduct:

$$\Delta(\mathrm{J}^\kappa) = \mathrm{J}^\kappa \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{J}^\kappa \qquad \Delta(\widehat{\mathrm{J}}^\kappa) = \widehat{\mathrm{J}}^\kappa \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{\mathrm{J}}^\kappa + \frac{1}{2}\, f^\kappa{}_{\rho\delta}\, \mathrm{J}^\delta \otimes \mathrm{J}^\rho$$

Tensor space representation:

$$\mathrm{J}^\kappa = \sum_i \mathrm{J}_i^\kappa \qquad \widehat{\mathrm{J}}^\kappa = \sum_i c_i\, \mathrm{J}_i^\kappa + f^\kappa{}_{\rho\delta} \sum_{i<j} \mathrm{J}_i^\delta\, \mathrm{J}_j^\rho$$

Yangian Symmetry

Level-one Momentum Generator:

$$\widehat{\mathrm{P}}^{\mu}_{\mathrm{bi}} = -\tfrac{i}{2}\sum_{j<k}\big[(\mathrm{L}^{\mu\nu}_j + \eta^{\mu\nu}\mathrm{D}^{\Delta=1}_j)\mathrm{P}_{k,\nu} - (j \leftrightarrow k)\big]$$

Applied to the cross integral:

$$\widehat{\mathrm{P}}^{\mu}_{\mathrm{bi}}\,|F_4\rangle = \left(\mathrm{P}^{\mu}_2 + 2\mathrm{P}^{\mu}_3 + 3\mathrm{P}^{\mu}_4\right)|F_4\rangle$$

Thus, we have

$$\widehat{\mathrm{P}}^{\mu}_{F_4}\,|F_4\rangle = (\widehat{\mathrm{P}}^{\mu}_{\mathrm{bi}} - \mathrm{P}^{\mu}_2 - 2\mathrm{P}^{\mu}_3 - 3\mathrm{P}^{\mu}_4)\,|F_4\rangle = 0$$

Similarly, one can show that $\widehat{\mathrm{P}}^{\mu}_{F_6}|F_6\rangle = 0$. The Yangian symmetry implies that scalar fishnet graphs satisfy PDE's. For example, for the cross integral one finds

$$\left(1 + 3u\partial_u + (3v-1)\partial_v + u^2\partial_u^2 + (v-1)v\partial_v^2 + 2vu\partial_u\partial_v\right)\phi(u,v) = 0$$

where u and v are the conformal cross ratios.

The Conformal Lax Operator

The Lax operator is defined as

$$\mathrm{L}_{\alpha\beta}\{u,\Delta\} = u\delta_{\alpha\beta} + \tfrac{1}{2}\mathrm{S}^{ab}_{\alpha\beta}\,\mathrm{J}^{\Delta}_{ab}$$

The conformal Lax operator that we use reads [Chicherin, Derkachov, Isaev '12]

$$\mathrm{L}(u_+,u_-) = \begin{pmatrix} u_+\cdot\mathbf{1} - \mathbf{p}\cdot\mathbf{x} & \mathbf{p} \\ \mathbf{x}\cdot\mathbf{p}\cdot\mathbf{x} + (u_+ - u_-)\cdot\mathbf{x} & u_-\cdot\mathbf{1} + \mathbf{x}\cdot\mathbf{p} \end{pmatrix}$$

where

$$\mathbf{x} \equiv -i\overline{\boldsymbol{\sigma}}^{\mu}x_{\mu} \qquad \mathbf{p} \equiv -\tfrac{i}{2}\boldsymbol{\sigma}^{\mu}\partial_{\mu} \qquad u_+ \equiv u + \tfrac{1}{2}(\Delta - 4) \qquad u_- \equiv u - \tfrac{1}{2}\Delta$$

Lax operator has nice properties:

- Intertwining relation

$$x_{12}^{-2}\,\mathrm{L}_1(u,\#)\,\mathrm{L}_2(\star,u+1) = \mathrm{L}_1(u+1,\#)\,\mathrm{L}_2(\star,u)\,x_{12}^{-2}$$

- Vacuum action

$$\mathrm{L}(u,u+2)\cdot 1 = (u+2)\mathbb{1} \qquad \mathrm{L}^T(u+2,u)\cdot 1 = (u+2)\mathbb{1}$$

Yangian Symmetry - RTT Formulation

Inhomogeneous monodromy matrix:

$$\mathrm{T}_n(u, \{\delta_i^\pm\}) := \mathrm{L}_n[\delta_n^+, \delta_n^-] \dots \mathrm{L}_2[\delta_2^+, \delta_2^-]\mathrm{L}_1[\delta_1^+, \delta_1^-]$$

where $\mathrm{L}_i[\delta_i^+, \delta_i^-] := \mathrm{L}_i(u + \delta_i^+, u + \delta_i^-)$.

Yangian symmetry	$\longleftrightarrow$	Eigenvalue relations for the monodromy matrix

$$\mathrm{T}_n(u, \{\delta_i^\pm\})|F_n\rangle = \lambda(u, \{\delta_i^\pm\})|F_n\rangle \mathbb{1}$$

The monodromy packages all the Yagian generators

$$\mathrm{T}_n(u, \{\delta_i^\pm\}) - \lambda(u, \{\delta_i^\pm\})\mathbb{1} \simeq \mathbb{1} + u\,\mathrm{J} + u^2\,\widehat{\mathrm{J}} + \dots$$

Yangian algebra encoded in RLL-relation:

$$\mathrm{R}_{12}(u-v)\mathrm{L}_1(u_+, u_-)\mathrm{L}_2(v_+, v_-) = \mathrm{L}_1(v_+, v_-)\mathrm{L}_2(u_+, u_-)\mathrm{R}_{12}(u-v)$$

The Cross Integral Reloaded

Eigenvalue relation for the cross integral:

$$\mathrm{L}_4[4,5]\mathrm{L}_3[3,4]\mathrm{L}_2[2,3]\mathrm{L}_1[1,2]\,|F_4\rangle = [3][4]^2[5]\,|F_4\rangle\mathbb{1}$$

where $[\delta_k]$ is shorthand for $(u+\delta_k)$.

First we extend the monodromy by inserting an identity operator

$$[2]^{-1}\int \mathrm{d}^4x_0\,\mathrm{L}_4[4,5]\mathrm{L}_3[3,4]\mathrm{L}_2[2,3]\mathrm{L}_1[1,2](\mathrm{L}_0^T[2,0]\cdot 1)\,x_{10}^{-2}x_{20}^{-2}x_{30}^{-2}x_{40}^{-2}$$
$$=[2]^{-1}\int \mathrm{d}^4x_0\,\mathrm{L}_4[4,5]\mathrm{L}_3[3,4]\mathrm{L}_2[2,3]\mathrm{L}_1[1,2]\mathrm{L}_0[2,0]x_{10}^{-2}x_{20}^{-2}x_{30}^{-2}x_{40}^{-2}$$

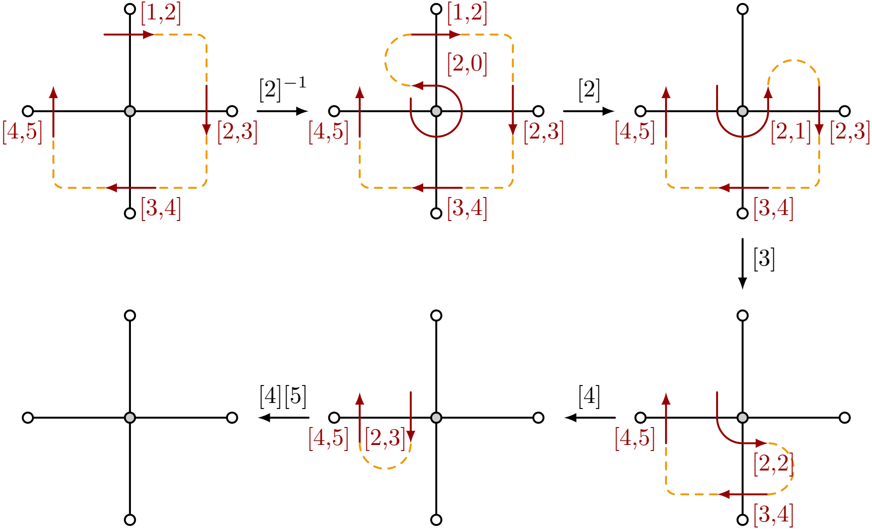
Subsequently, we use the intertwining relation to pull the propagators through

$$\mathrm{L}_1[1,2]\mathrm{L}_0[2,0]\,x_{10}^{-2} = x_{10}^{-2}\,\mathrm{L}_1[0,2]\mathrm{L}_0[2,1] = [2]\,x_{10}^{-2}\,\mathrm{L}_0[2,1]$$

and use the vacuum relation

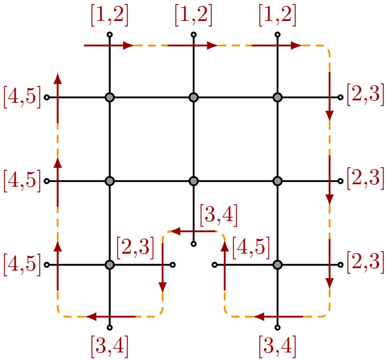
$$\mathrm{L}_1[0,2]\cdot 1 = (u+2)\mathbb{1}$$

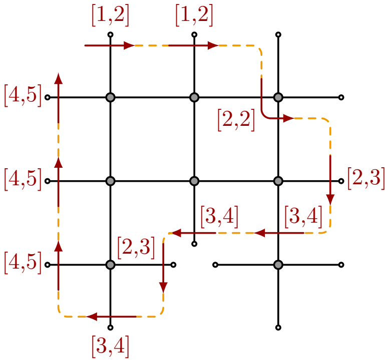
The Cross Integral Graphically



Yangian Symmetry of Generic Fishnet Graphs

We can use similar techniques to show that generic fishnet graphs



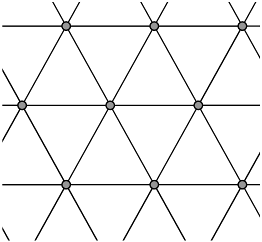


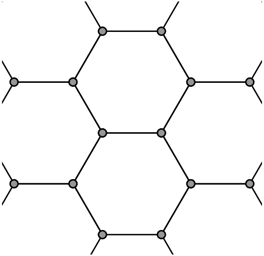
are Yangian invariant:

$$\mathrm{T}_n(u, \{\delta_i^\pm\})|F_n\rangle = \lambda(u, \{\delta_i^\pm\})|F_n\rangle \mathbb{1}$$

Generalized Fishnet Graphs

Fishnet graphs in 3 and 6 dimensions:





$$\mathcal{L}_{3d,\mathrm{int}} = \xi N_c \mathrm{Tr}(Y_1 Y_4^\dagger Y_2 Y_1^\dagger Y_4 Y_2^\dagger)$$

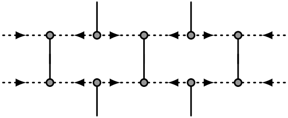
[Caetano, Gurdogan, Kazakov '16]

$$\mathcal{L}_{6d,\mathrm{int}} = N_c \mathrm{Tr}(\xi_1 \phi_1^\dagger \phi_2 \phi_3 + \xi_2 \phi_1 \phi_2^\dagger \phi_3^\dagger)$$

[Mamroud, Torrents '17]

Fishnet graphs with fermions in 4 dimensions:

$$\mathcal{L}_{\phi\psi}^{\mathrm{int}} = N_{\mathrm{c}} \mathrm{Tr}\big(\xi_1^2 \phi_3^\dagger \phi_1^\dagger \phi^3 \phi^1 + \xi_2^2 \phi_2^\dagger \phi_1^\dagger \phi^2 \phi^1 + \sqrt{\xi_1 \xi_2}(\bar{\psi}_1 \phi^1 \bar{\psi}_4 - \psi^1 \phi_1^\dagger \psi^4)\big)$$



Conclusions and Outlook

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- Scalar fishnet Feynman integrals in 4d are Yangian invariant
  - Purely bosonic Yangian $Y[\mathfrak{so}(2,4)]$
  - First appearance of Yangian symmetry at loop level
- Yangian symmetry yields PDE's for these graphs
- Generalized fishnet graphs in 3d, 4d and 6d are Yangian invariant as well

Outlook:

- Can we use the Yangian symmetry to compute/constrain these fishnet integrals?
- Relation to 4-point Fishnet correlator graphs from [Basso,Dixon '17]?
- Consider other limits of $\mathcal{N} = 4$ SYM $\rightarrow$ several graphs at a given order
- Understand on-shell limits in complete detail

Thank you for your attention!

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