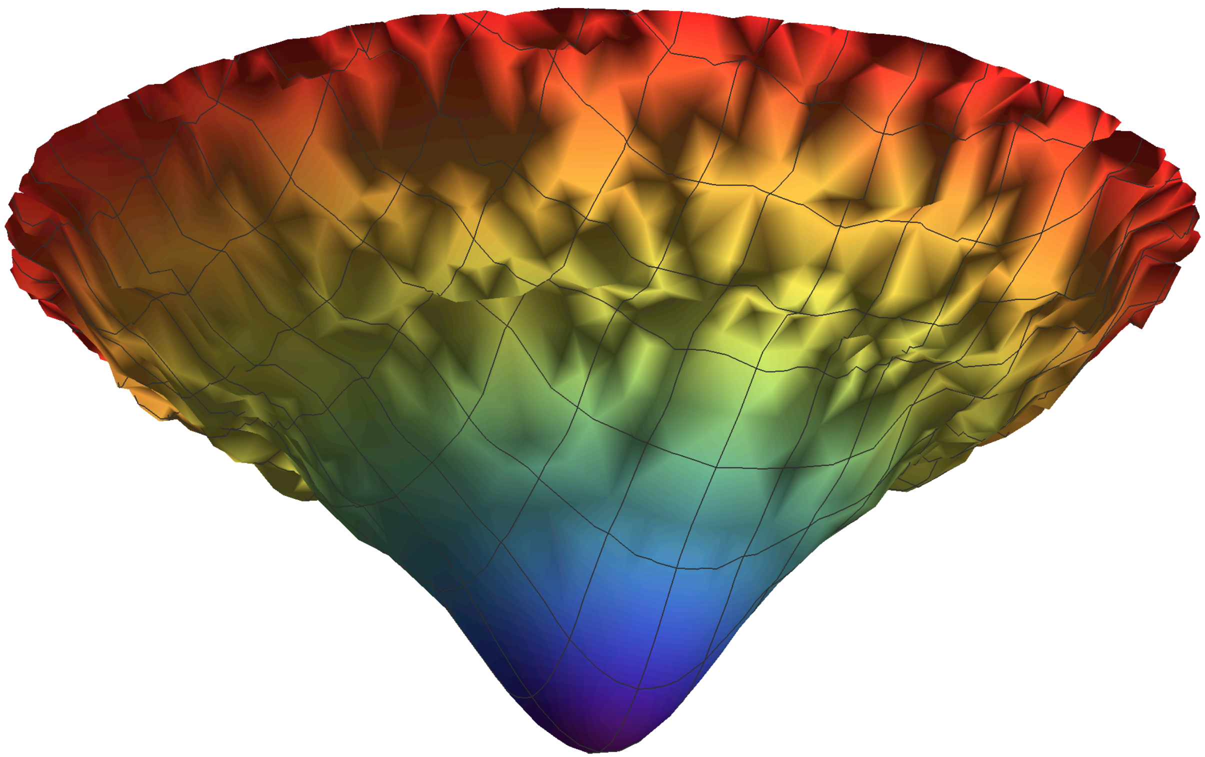
No Smooth Beginning for Spacetime



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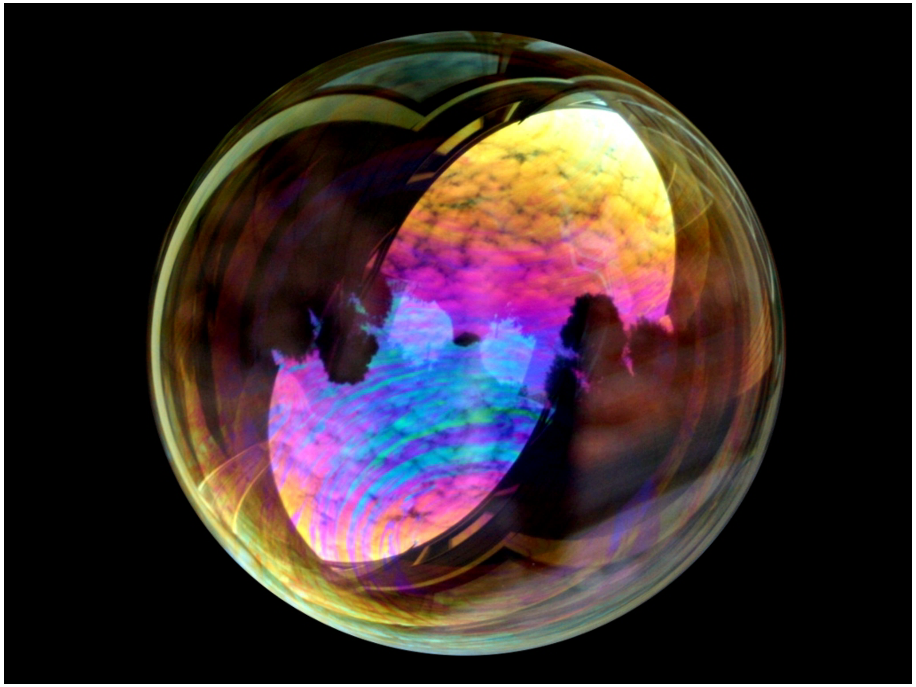
With Job Feldbrugge and Neil Turok: 1703.02076 (PRD), 1705.00192 (accepted as PRL), 1709.03171

- The evolution of the universe strongly depends on initial conditions
- This is true even in attractor models, such as inflation or ekpyrosis

- Independently of such considerations, we might wonder how space and time arose, and how they came to behave classically?
- And can we understand/resolve the big bang?



- There is an old and attractive idea that a closed universe can nucleate out of "nothing", since the total Hamiltonian vanishes, with all total charges zero

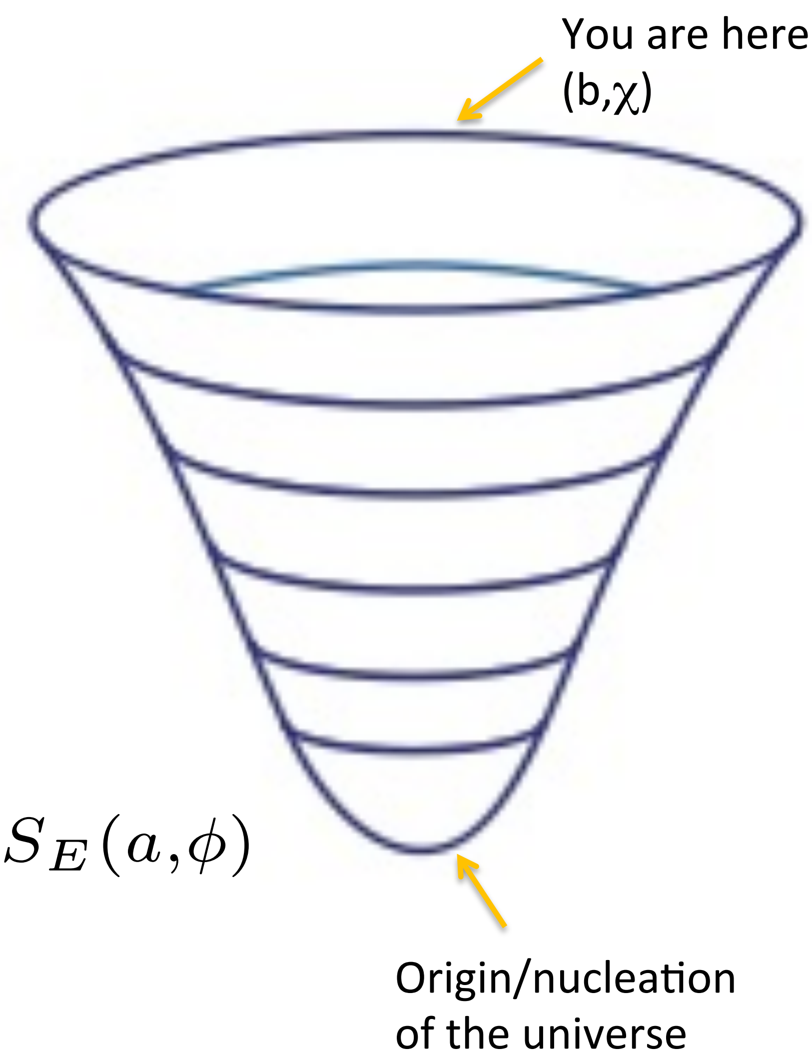
[Lemaître; Tryon;
Brout, Englert & Gunzig]

No-boundary and tunneling proposals

- Hawking (1981): *"There ought to be something very special about the boundary conditions of the universe and what can be more special than the condition that there is no boundary"*
- Tunneling proposal (Vilenkin): creation of the universe seen as a regular tunneling event

$$\Psi(b,\chi) = \int_{\mathcal{C}} \mathcal{D}a\mathcal{D}\phi e^{-S_E(a,\phi)}$$

$$\approx e^{-S_{E,ext}(b,\chi)}$$



Two approaches

- Euclidean
  - In analogy with Wick rotation in QFT it was hoped that this would lead to better convergence
  - However conformal mode problem
- Lorentzian
  - No conformal mode problem
  - Causality can be built in
  - Not clear whether the path integral actually converges

[Hawking, Hartle, Gibbons,...]

[Vilenkin, Teitelboim,...]

$$S = \int \mathrm{d}t N \left(-3a \frac{\dot{a}^2}{N^2} + \frac{1}{2} a^3 \frac{\dot{\phi}^2}{N^2} + \cdots \right)$$

Opposite signs

Picard-Lefschetz theory

- We are interested in oscillatory integrals, whose convergence properties are not clear, in particular the Feynman integral

$$\psi(x_f, x_i) = \int_{\mathcal{C}} \delta x \, e^{iS[x(t)]/\hbar}$$

- View the integrand $\mathcal{I} = iS/\hbar$ as a holomorphic function of $x \in \mathbb{C}$, then we might be able to find an appropriate convergent integration contour

Cf. Wick rotation where a coordinate is continued to the complex plane. But coordinates are not physical in GR, hence it seems preferable to continue the fields to the complex plane

Picard-Lefschetz theory

- Cauchy's theorem tells us that a complex integration contour can be deformed
- Picard-Lefschetz theory tells us how it should be deformed

- A review is provided by E. Witten "Analytic continuation of Chern-Simons theory" (2010)

Picard-Lefschetz theory

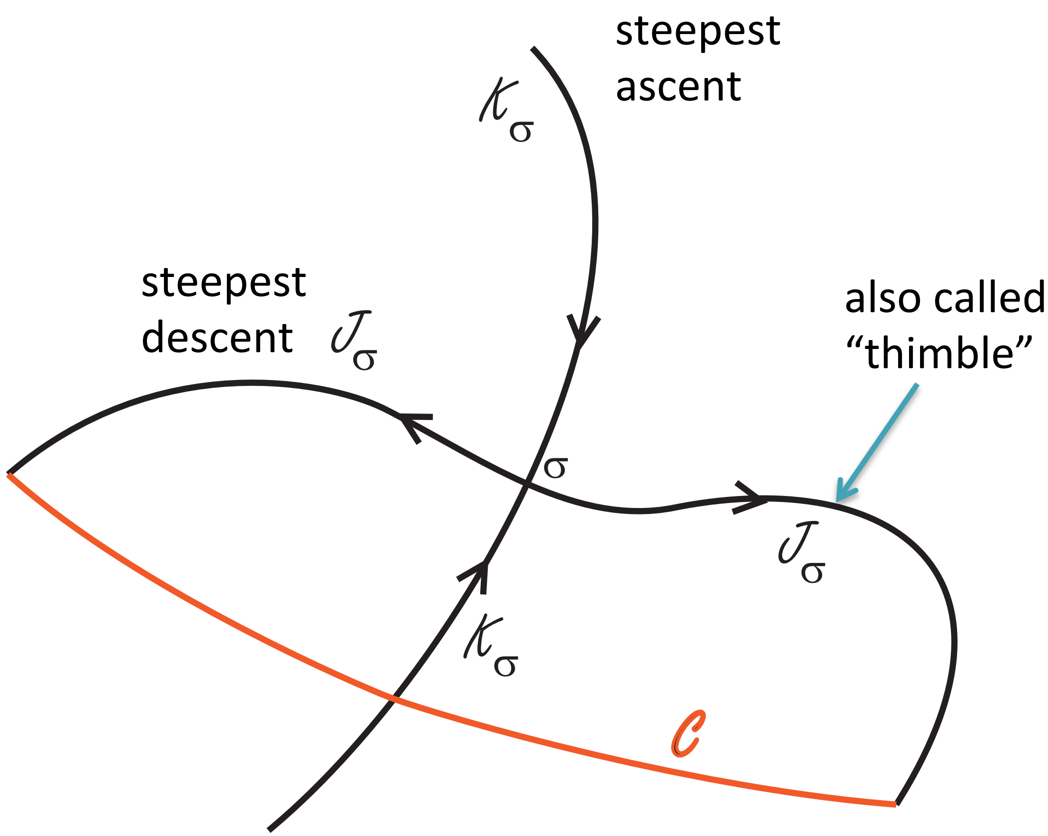
- Define the *Morse function* h as the real part of the integrand, and look at its critical points, which are also critical points of the full integrand

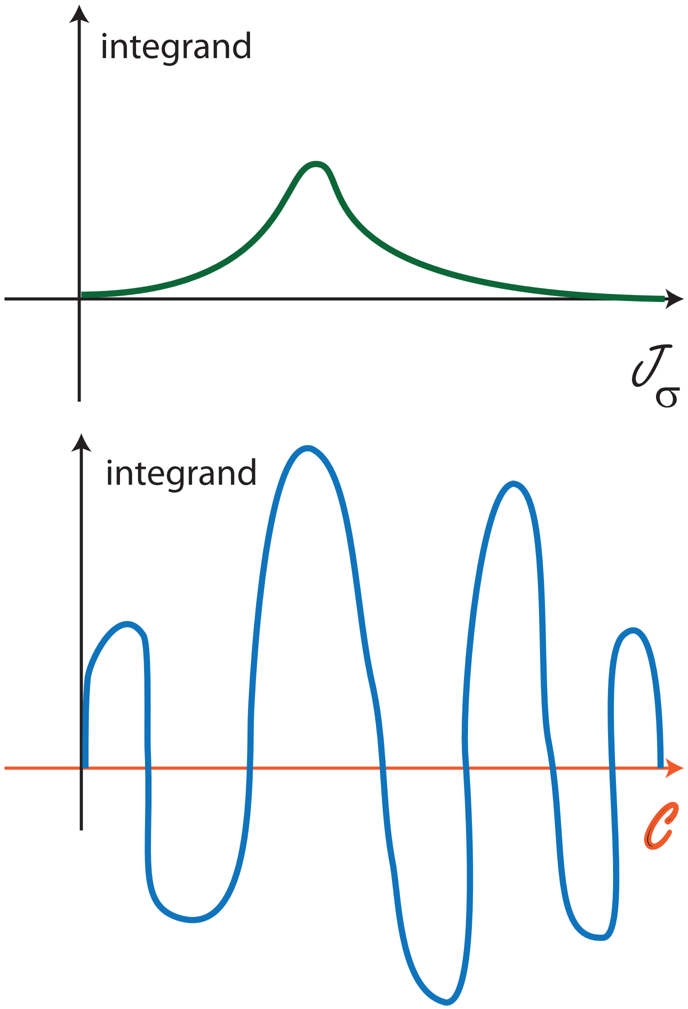
$$e^{\mathcal{I}} = e^{iS} = e^{h+iH} \qquad h, H \in \mathbb{R}$$

$$\partial_x h = \partial_{\bar{x}} h = 0 \iff \partial_x \mathcal{I} = 0$$

- From the critical points, look at lines of H=constant, these are the flows of steepest ascent/descent of h

From conditionally to absolutely convergent





Picard-Lefschetz theory

- Which Lefschetz thimbles contribute? We would like to re-express the original integration contour as a sum over thimbles: $\mathcal{C} = \sum n_\sigma \mathcal{J}_\sigma$

- Upward and downward flows have an intersection pairing: $\langle \mathcal{J}_\sigma , \mathcal{K}_\tau \rangle = \delta_{\sigma\tau}$ which implies that

$$n_\tau = \langle \mathcal{C}, \mathcal{K}_\tau \rangle$$

only saddle points that can be linked to the original integration contour via an upwards flow will contribute

- Final result:

$$\int \mathrm{d}x \, e^{\mathcal{I}} = \sum_\sigma n_\sigma \, e^{i \, Im(\mathcal{I}_\sigma)} \int_{\mathcal{J}_\sigma} e^h$$

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This makes sense: flow down to get from an oscillatory integral, with repeated cancellations, to a non-oscillatory one

- Final result:

$$\int \mathrm{d}x\, e^{\mathcal{I}} = \sum_\sigma n_\sigma\, e^{i\, Im(\mathcal{I}_\sigma)} \int_{\mathcal{J}_\sigma} e^{h}$$

Gravity plus Cosmological Constant

- We will consider the simple system

$$\Psi = \int_{\mathcal{C}} \delta N \, \delta a \, e^{iS(N,a)/\hbar}$$

Can add ghosts and choose constant N gauge in a integral – see e.g. [Teitelboim]

with $$S = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \, (R - 2\Lambda)$$

- For a standard minisuperspace metric

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + a^2 \mathrm{d}\Omega_3^2$$

this is hard to solve

$$S = 2\pi^2 \int \mathrm{d}t N \left(-3a \frac{\dot{a}^2}{N^2} + 3ka - a^3 \Lambda \right)$$

An useful form of the metric

- Technically much simpler to consider

$$\mathrm{d}s^2 = -\frac{N^2}{q(t)}\mathrm{d}t^2 + q(t)\mathrm{d}\Omega_3^2$$

e.g. Halliwell (1988)

since then the action becomes quadratic

$$S = 2\pi^2 \int \mathrm{d}t \left(-\frac{3}{4N}\dot{q}^2 + 3kN - N\Lambda q \right)$$

- Then the integral over $q=a^2$ is simply a Gaussian, and can be done exactly

Path integral for propagator

- We are left with an ordinary integral over the lapse function

$$\Psi = \frac{3\pi i}{2\hbar}\int \frac{\mathrm{d}N}{N^{1/2}} e^{2\pi^2 i S_0/\hbar}$$

Integrate only over N>0
-> causality and no
double counting
See [Teitelboim '80s]

$$S_0 = N^3\frac{\Lambda^2}{36} + N\left(-\frac{\Lambda}{2}(q_0+q_1)+3\right) + \frac{1}{N}\left(-\frac{3}{4}(q_1-q_0)^2\right)$$

$q_0=a_0^2$, initial value of scale factor $q_1=a_1^2$, final value of scale factor

- There are 4 saddle points:

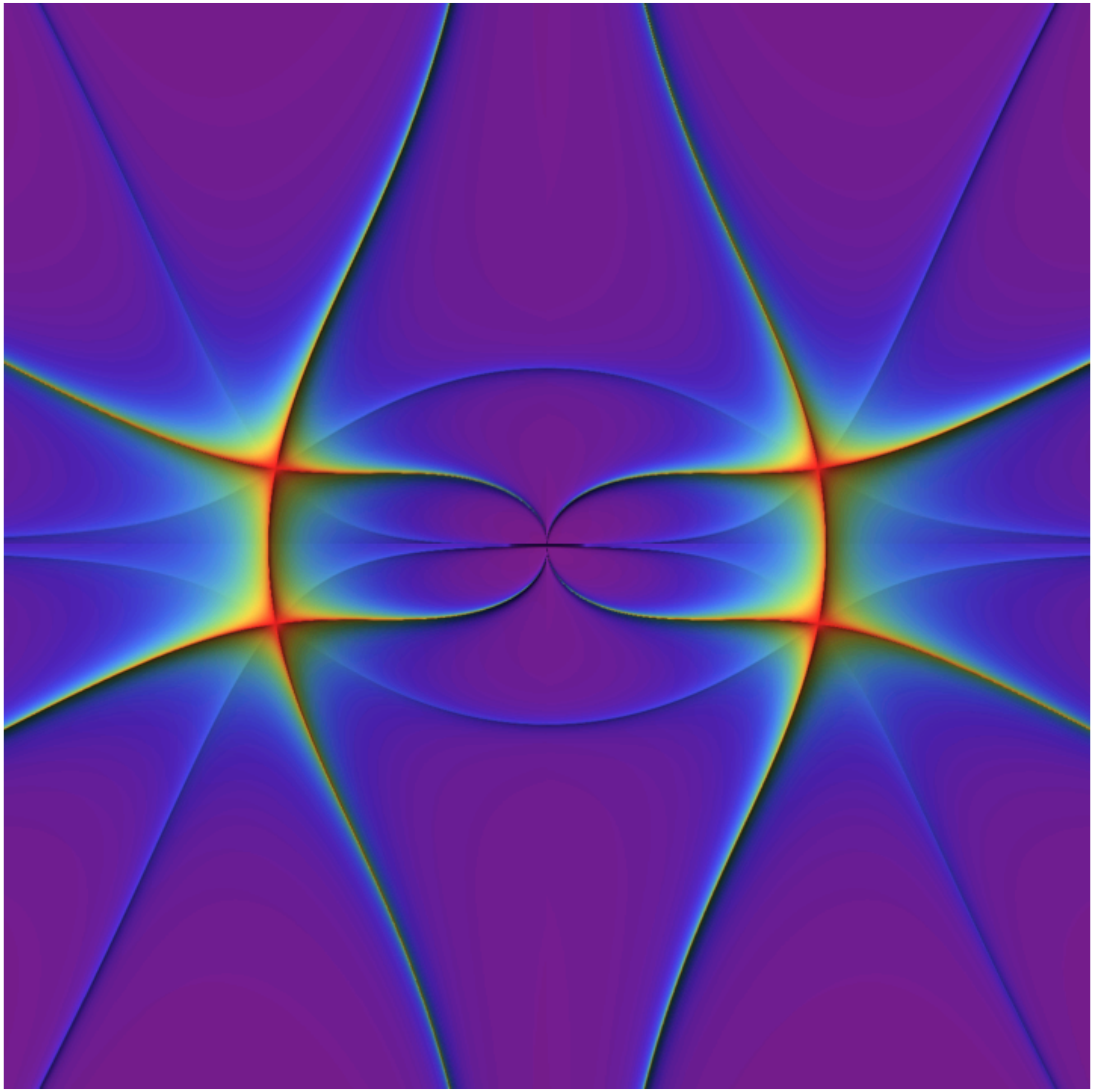
$$N_s = \pm\frac{3}{\Lambda}\left[(\frac{\Lambda}{3}q_0-1)^{1/2} \pm (\frac{\Lambda}{3}q_1-1)^{1/2}\right]$$

- The saddle points will be real/complex depending on the signs of $\Lambda q - 3$
- Now we can apply Picard-Lefschetz theory

No-boundary condition $q_0=0$

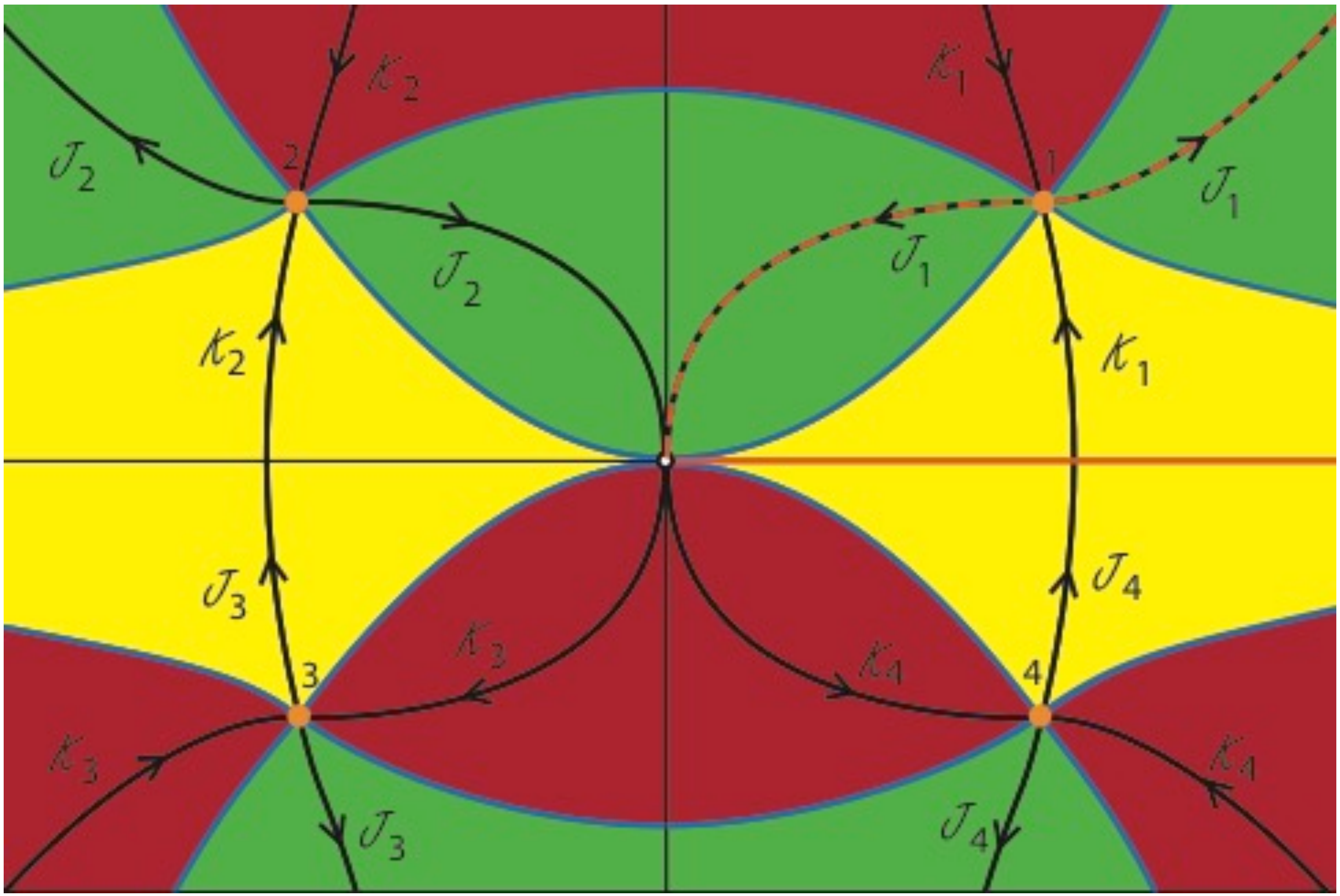
- Propagator from zero scale factor $q_0=0$ to a large final value q_1

- Saddle points are complex



No-boundary conditions

- Upward/downward flows and wedges:



No-boundary conditions

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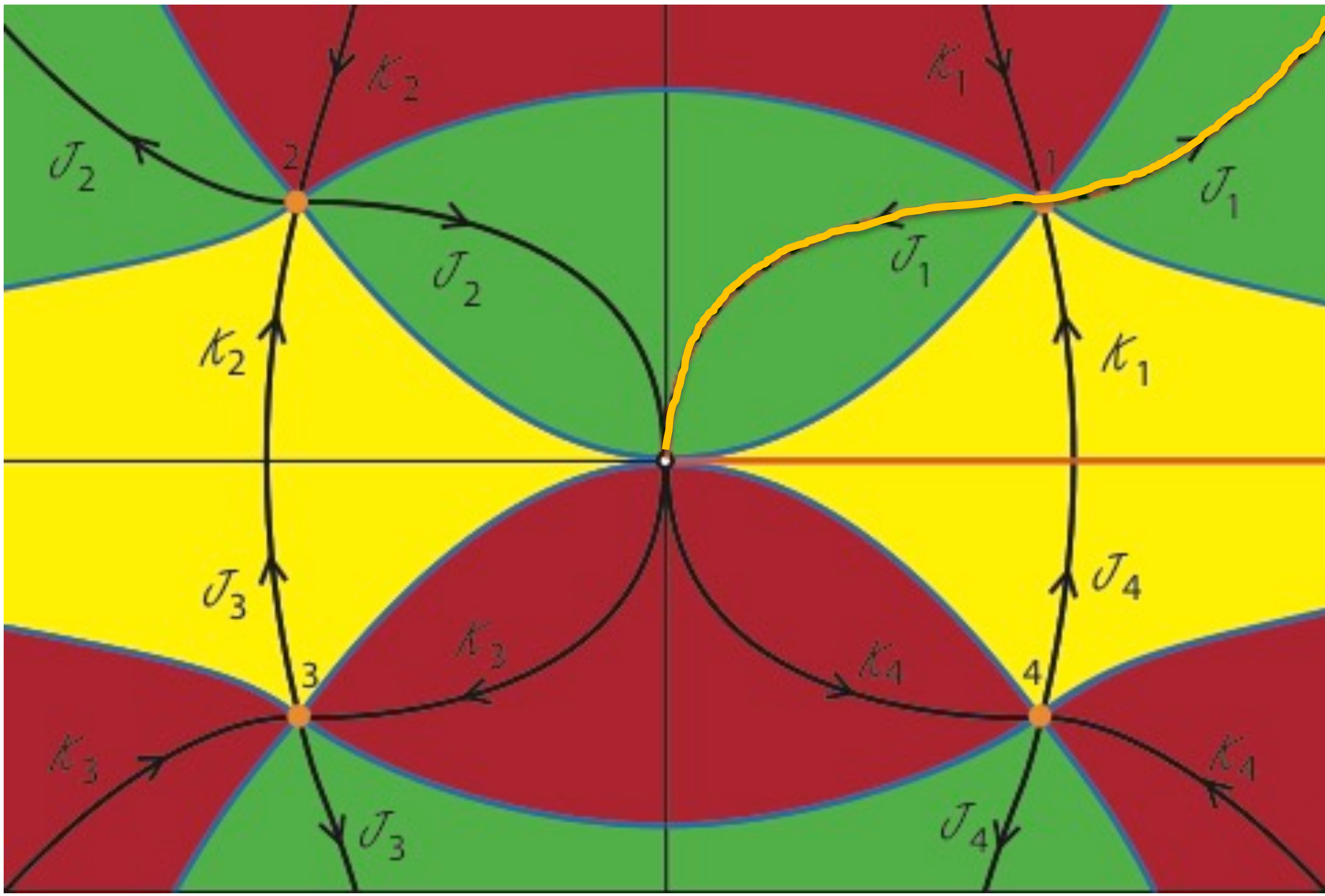
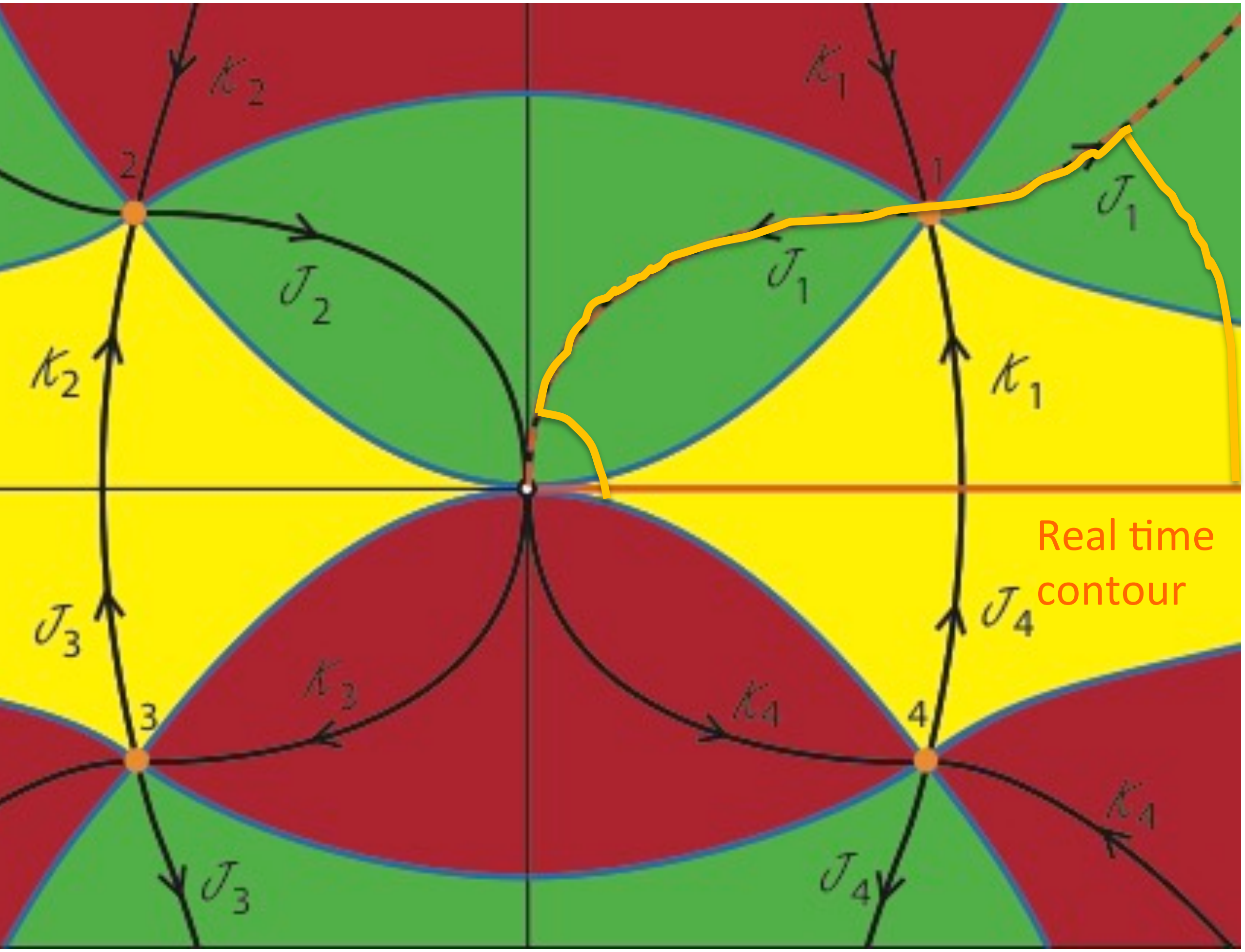


No-boundary conditions

- Convergence near zero/at infinity:



Near $N = 0$:

$$\int \mathrm{d}N e^{-\frac{i}{N}}$$

Near $N = \infty$:

$$\int \mathrm{d}N e^{iN^3}$$

Wavefunction for no-boundary conditions

- The wavefunction is dominated by a single saddle point, yielding

$$\Psi_{nb}(q_1) \approx e^{i\frac{\pi}{4}} \frac{3^{1/4}}{2(\Lambda q_1 - 3)^{1/4}} e^{-12\pi^2/(\hbar\Lambda) - i4\pi^2\sqrt{\frac{\Lambda}{3}}(q_1 - \frac{3}{\Lambda})^{3/2}/\hbar}$$

↑

The weighting is inverse to that advocated by Hartle and Hawking, and is the same as for Vilenkin's tunneling wavefunction

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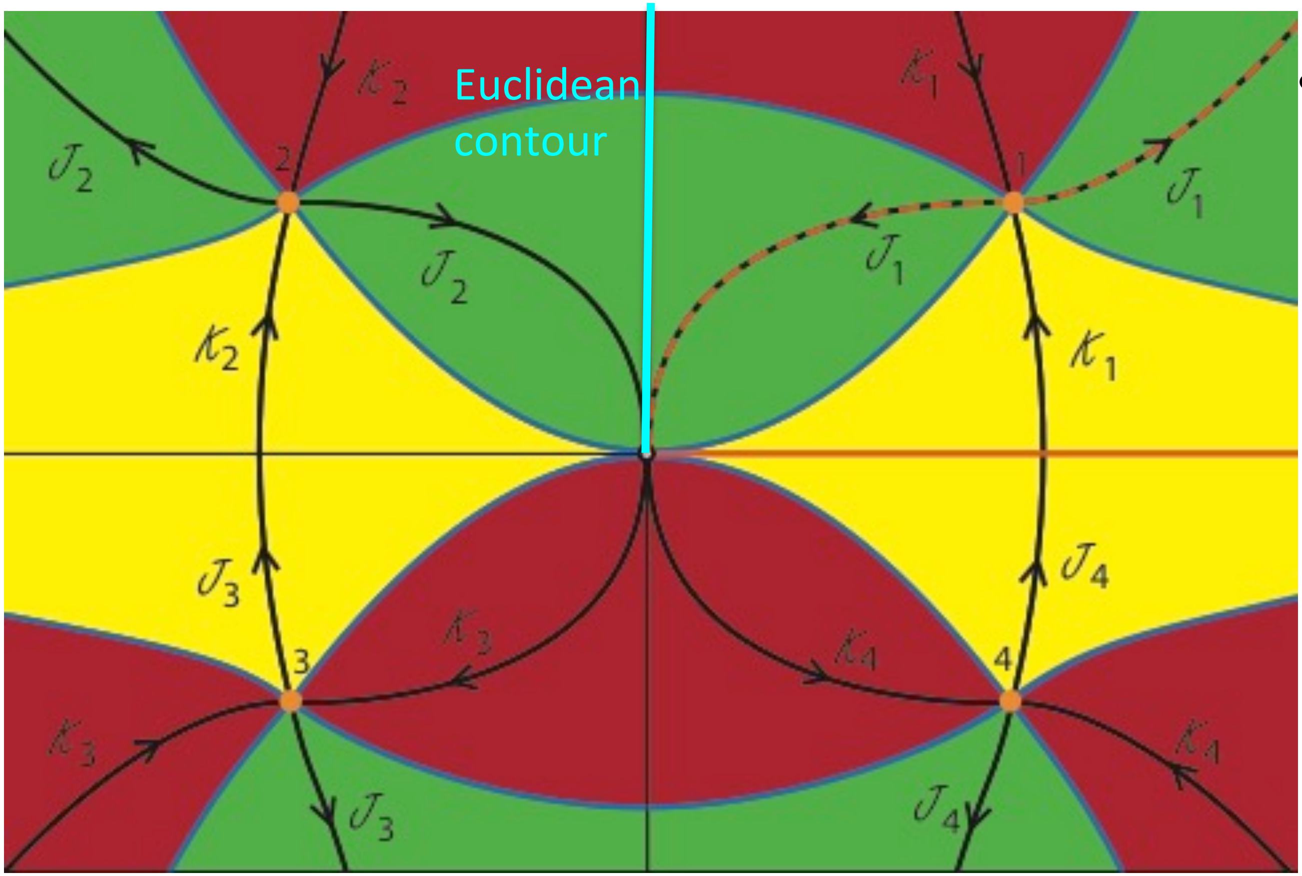
↑

Picard-Lefschetz theory implies that relevant saddle points will always come in with a suppressed amplitude, as they must be linked via an upwards flow to the original integration contour (along which h=0) – this makes sense physically as quantum processes are suppressed (and not enhanced) compared to classical evolution

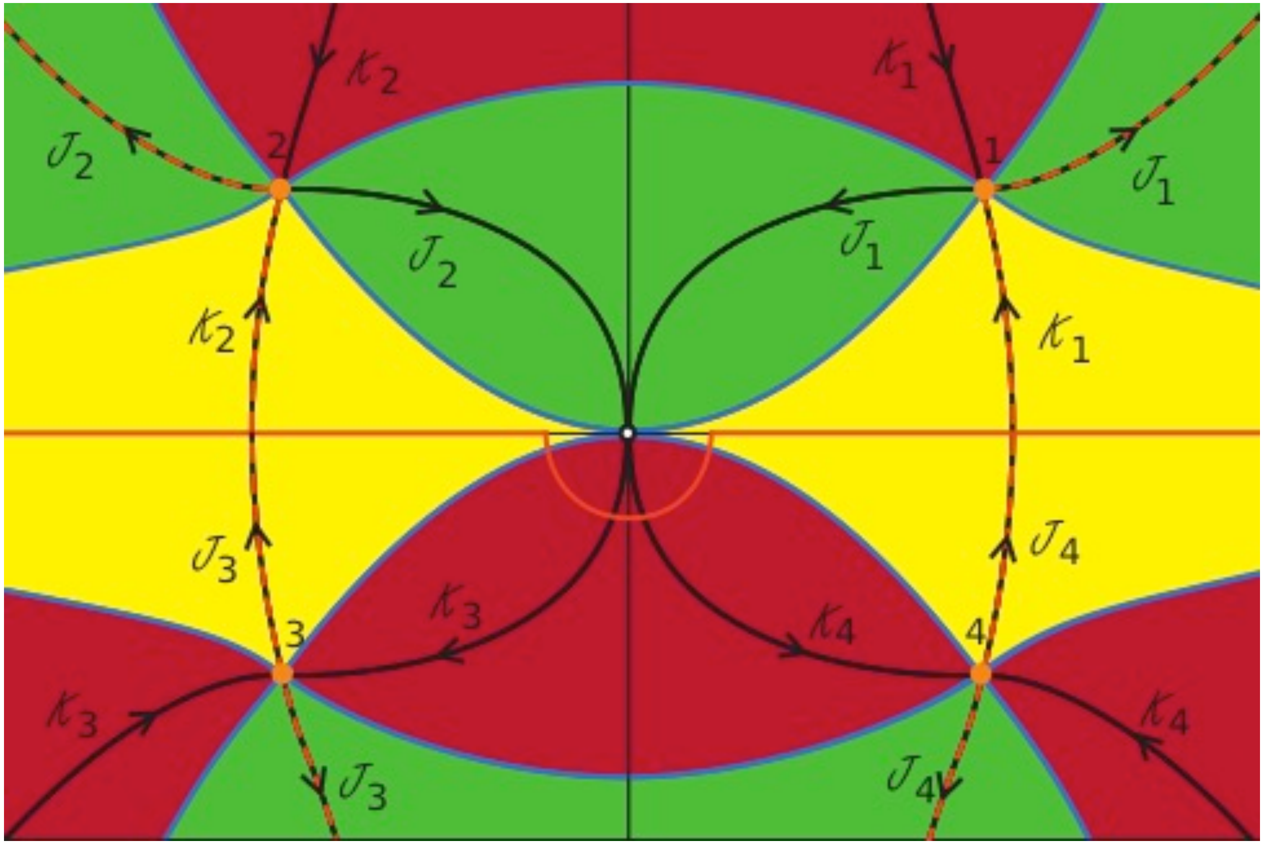
Relation to the Euclidean formulation



- Halliwell pointed out that the Euclidean approach is not complete, since the choice of contour is not specified, and *different results can be obtained by considering different contours*

- Here we see that the Euclidean path integral cannot be approximated by the saddle point method, and is simply not well-defined

Remark on Hartle-Hawking

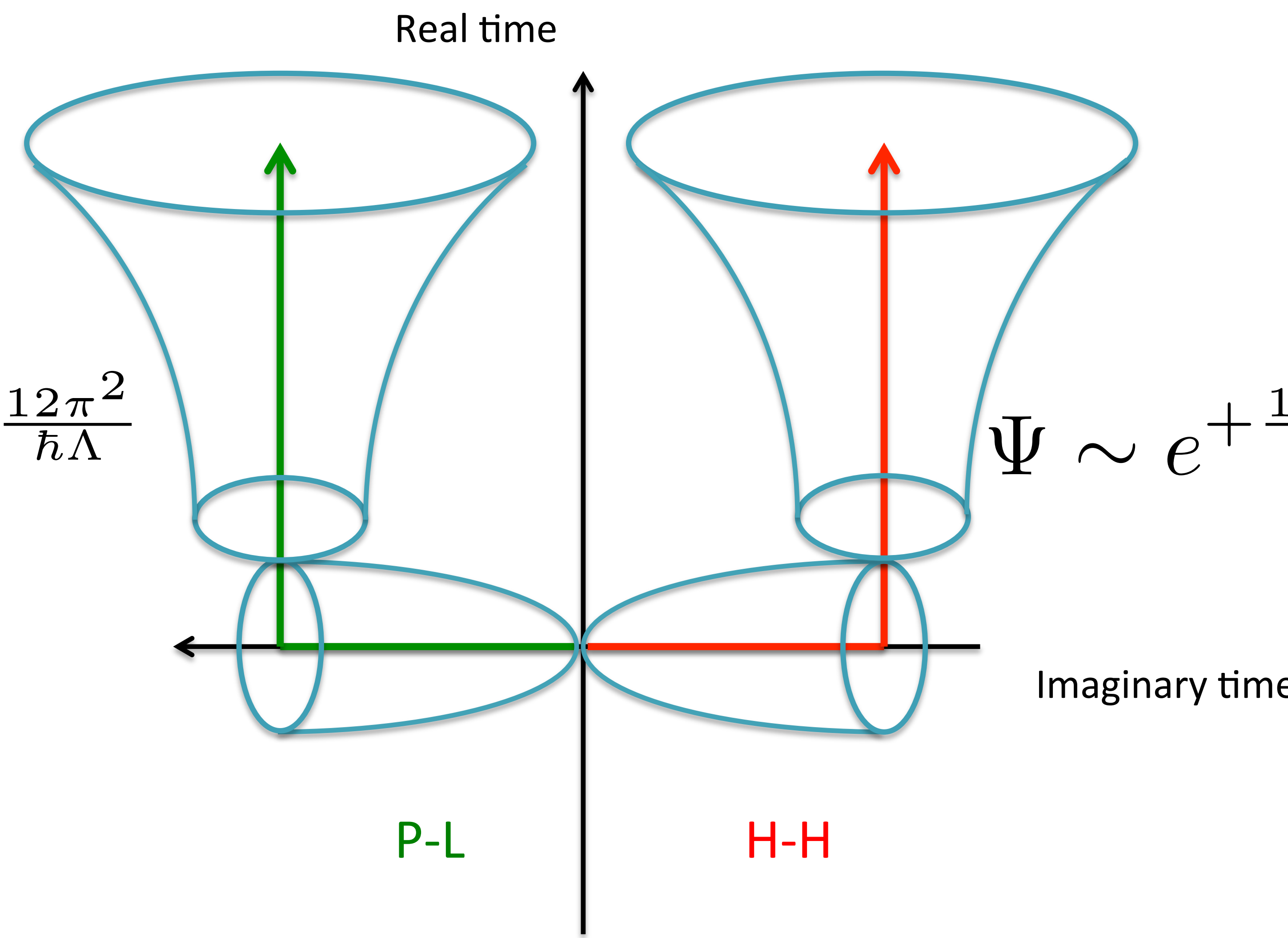


This contour was recently considered by J. Diaz Dorronsoro et al. [1705.05340]

- Hartle and Hawking have conjectured that the relevant saddle points are saddles 3 and 4
- *One can find a convergent contour that would yield this result* (and a real wavefunction), but this is an inherently complex theory

Shape of the saddle points in N

$$\Psi \sim e^{-\frac{12\pi^2}{\hbar\Lambda}}$$



$$\Psi \sim e^{+\frac{12\pi^2}{\hbar\Lambda}}$$

Include Tensor Perturbations

If we add perturbations, the propagator is given by

$$\Psi[q_1, \phi_1; q_0, \phi_0] = \int_{0^+}^{\infty} \mathrm{d}N \int \mathcal{D}q \int \mathcal{D}\phi \, e^{iS[q,\phi,N]/\hbar}$$

with $S = S^{(0)} + S^{(2)}$

where the perturbation action is (e.g. for a gravity wave mode with wavenumber l)

$$S^{(2)} = \frac{1}{2} \int N_s \mathrm{d}t \, \mathrm{d}^3 x \left[q^2 \left(\frac{\dot{\phi}}{N_s} \right)^2 - l(l+2)\phi^2 \right]$$

$$= \frac{1}{2} \left[\frac{q^2}{N_s} \phi \dot{\phi} \right]_0^1 \quad \text{(on-shell)}$$

Include Tensor Perturbations

- In physical time,

$$S^{(2)} = \frac{1}{2}\int N\mathrm{d}t_p\,\mathrm{d}^3x\left[a^3\left(\frac{\phi_{,t_p}}{N}\right)^2 - a\,l(l+2)\phi^2\right]$$

- Solution to the equation of motion (at background saddle point), with $H = \sqrt{\frac{3}{\Lambda}}$

$$\phi = c_1\left(1+\frac{i}{\sinh(Ht_p)}\right)^{\frac{l}{2}}\left(1-\frac{i}{\sinh(Ht_p)}\right)^{-\frac{l+2}{2}}\left(1-\frac{i(l+1)}{\sinh(Ht_p)}\right)$$
$$+c_2\left(1-\frac{i}{\sinh(Ht_p)}\right)^{\frac{l}{2}}\left(1+\frac{i}{\sinh(Ht_p)}\right)^{-\frac{l+2}{2}}\left(1+\frac{i(l+1)}{\sinh(Ht_p)}\right)$$

- For P-L instanton, at South Pole $\sinh(Ht) = -i$
- Then regularity implies $c_2=0$ (now call $c_1 = \phi_1$)

- The action then becomes

$$\Psi \propto e^{\phi_1^2 \frac{l(l+2)}{2\hbar H^2}(-i\sinh(Ht_p)+l+1)}$$

- so that the weighting is given by

$$\left|\Psi_\phi\right| \approx e^{+\phi_1^2 \frac{l(l+1)(l+2)}{2\hbar H^2}}$$

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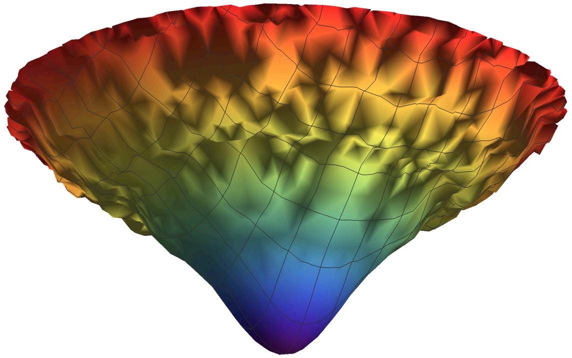


- Thus the perturbations obey an *inverse Gaussian* distribution – the distribution prefers large fluctuations and the model breaks down!

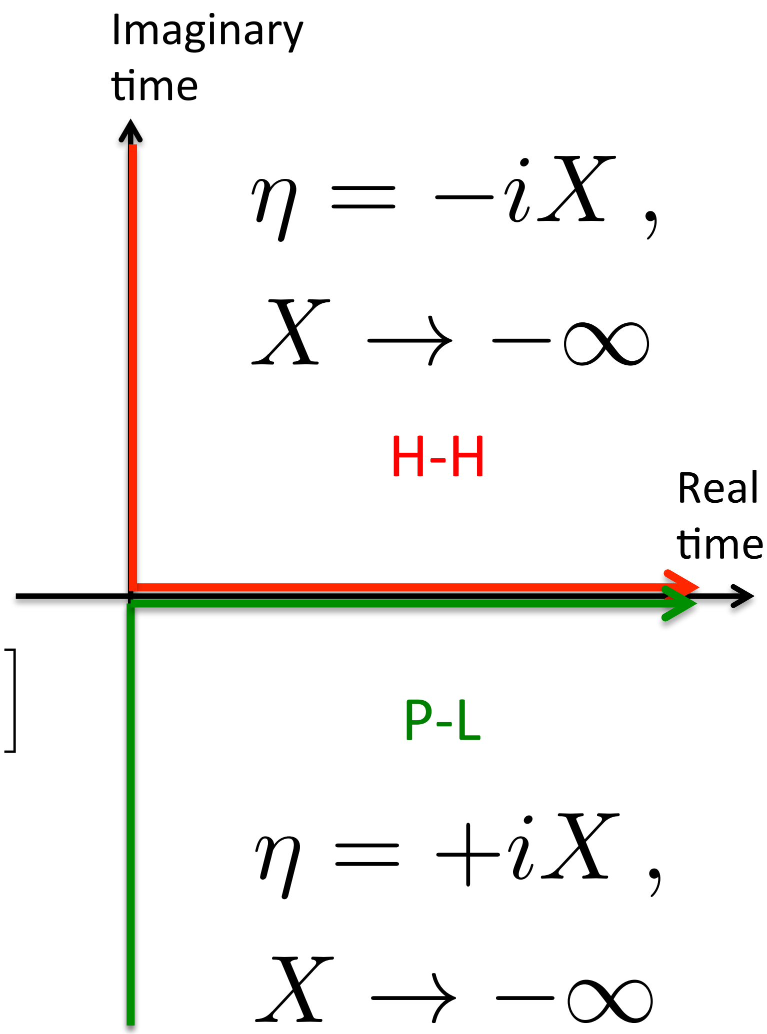
Analogy in terms of Wick rotation

One way to understand this result is to realize that PL theory forces one to choose the "wrong" Wick rotation.

These arguments remain true for more general matter.
In terms of $\chi=\phi\, q^{1/2}$:

$$iS^{(2)} = \pi^2 \int_{-\infty}^{0} dX \left[\chi'^2 + \left(l(l+2) + 1 + \underbrace{\frac{1}{2} q\,(w - \frac{1}{3})\rho}_{\text{remains finite}}\right)\chi^2\right]$$

Then at large wavenumber l our previous arguments still go through.



More rigorously:

- We can in fact find a unique finite action solution (almost) everywhere in the complex N plane reaching the value ϕ_1 at t=1:

$$\phi = \phi_1 \frac{F_\pm(t)}{F_\pm(1)}$$

$$F_\pm(t) = \left(\frac{t}{\Lambda N^2(t-1)+3q_1}\right)^{\alpha_\pm} \left(\Lambda N^2 t + (3q_1 - \Lambda N^2)(\alpha_\pm + 1)\right)$$

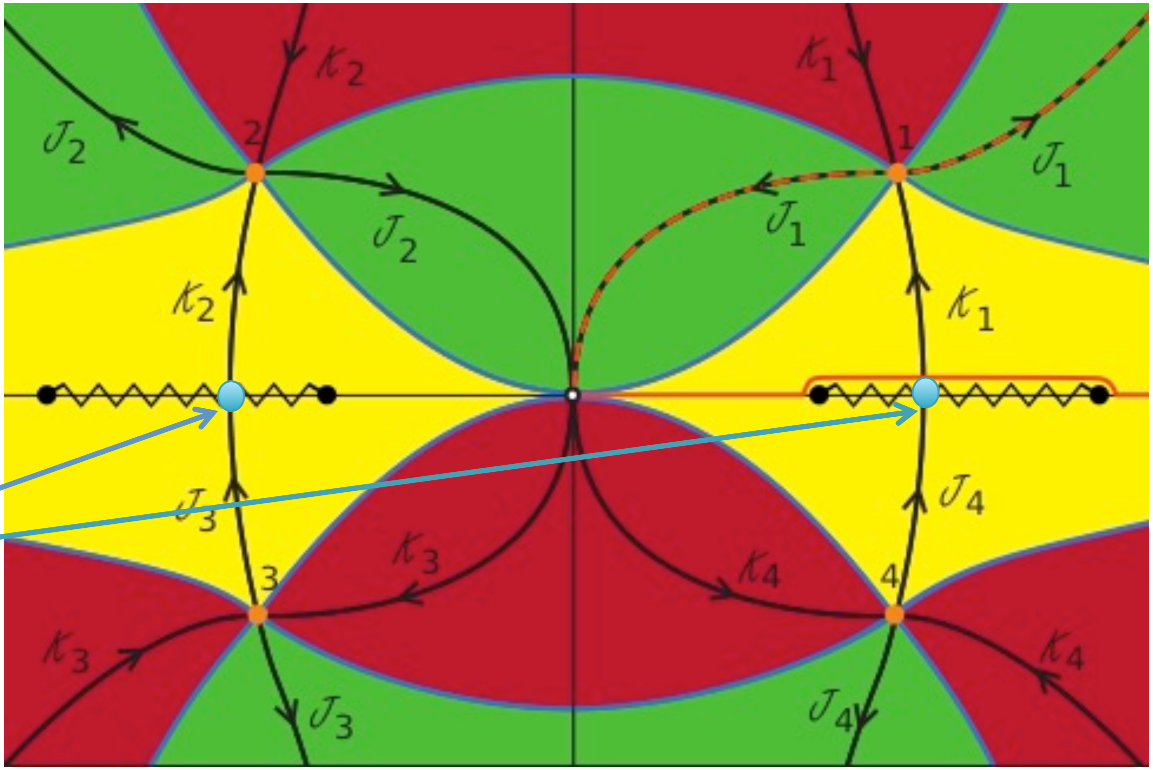
$$\alpha_\pm = -\frac{1}{2} \pm \frac{1}{6q_1 - 2\Lambda N^2}\sqrt{(3q_1 - \Lambda N^2)^2 - 36l(l+2)N^2}$$

- At $N^2 = 3q_1/\Lambda$ there also exists a finite action solution
- There is precisely one finite action solution and one divergent solution at each N (modulo a subtlety to which we now turn)

[Feldbrugge, JLL & Turok, 1709.03171]

Properties of the perturbed action

$$N_\star = \pm\sqrt{\frac{3q_1}{\Lambda}}$$

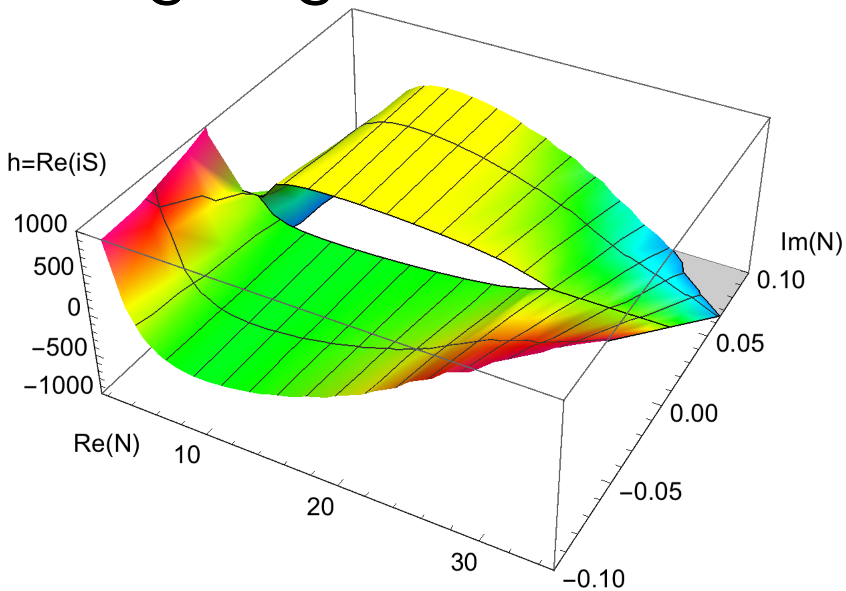


- Perturbed action has a saddle point at N_* (as the perturbations increase the saddle points will move towards the real line)
- There are branch cuts (zigzag lines) on the real N line

Failure of correspondence principle

$$e^{Re(\frac{i}{\hbar}S)} = e^{-\frac{12\pi^2}{\hbar\Lambda}+\frac{1}{2\hbar\Lambda}l(l+1)(l+2)\phi_1^2}$$

- How can we get a weighting larger than 1? The starting action is real, and we must flow down, hence expect a weighting smaller than e^0

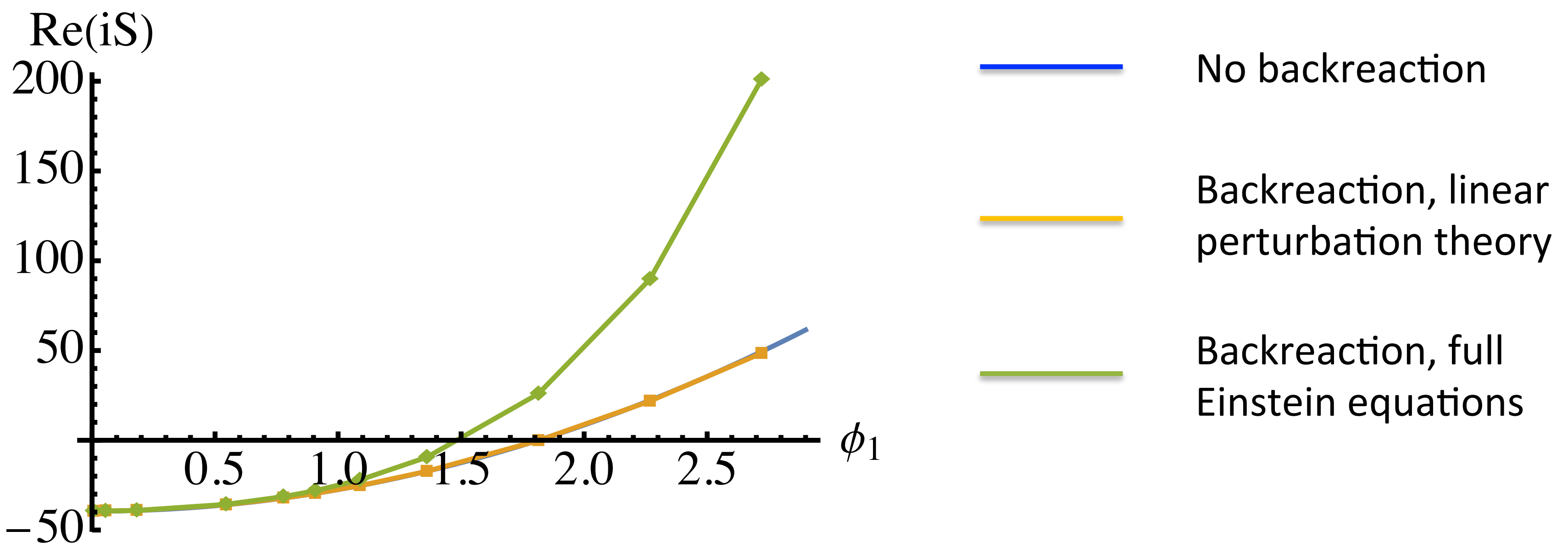


But as we have seen, the action has a jump across the cut on the real line. It is not analytic there, and mathematically the existence of the cut is a clear manifestation of a problem with the no-boundary proposal

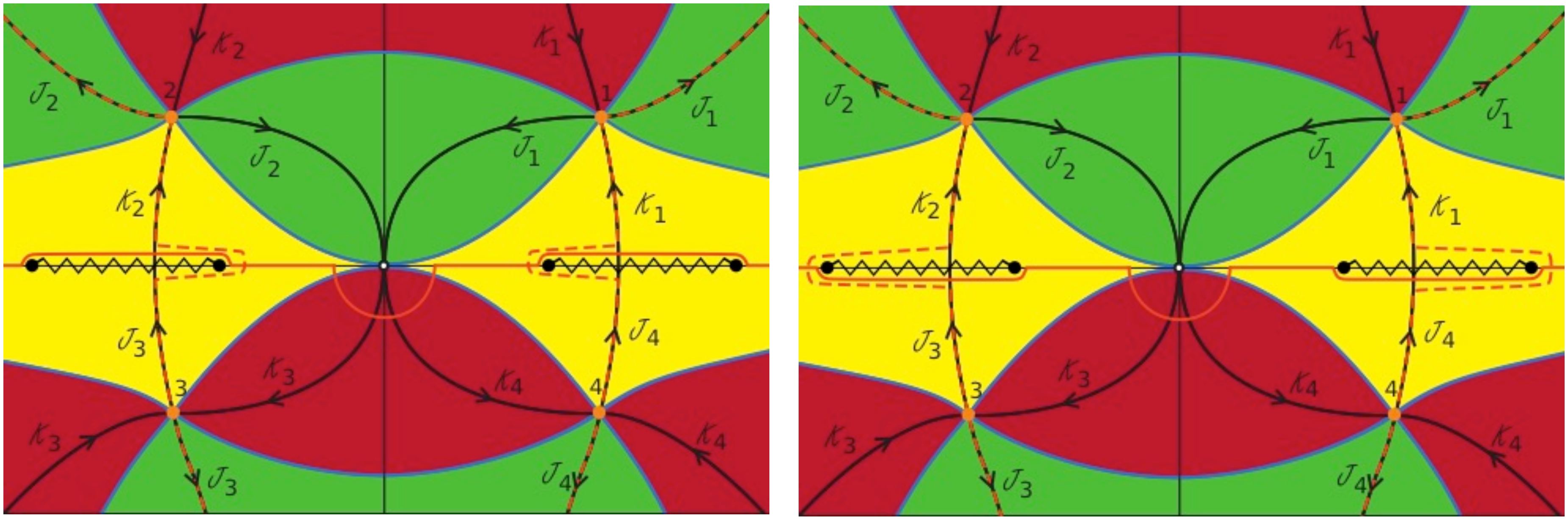
- The correspondence principle, namely that quantum effects should vanish in the limit hbar ->0, fails!

Do our approximations break down?

- Backreaction (i.e. corrections to the scale factor due to the linear perturbations) change the results very little
- Have also checked that the full non-linear l=2 modes show the same qualitative behavior – the instability in fact becomes even stronger

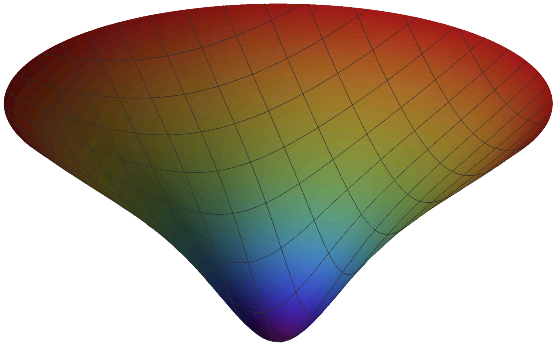


Remark on Diaz Dorronsoro et al.



- The branch cut and the upper saddle points contribute non-perturbatively to the path integral, resulting in the same instability!
- N fact one can prove that *no choice of contour avoids this problem!*

Conclusions



- Background and perturbations must be considered together

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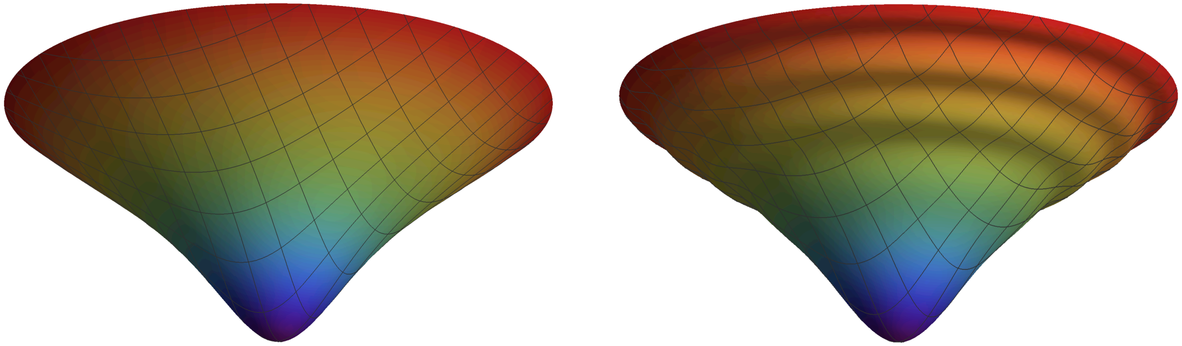

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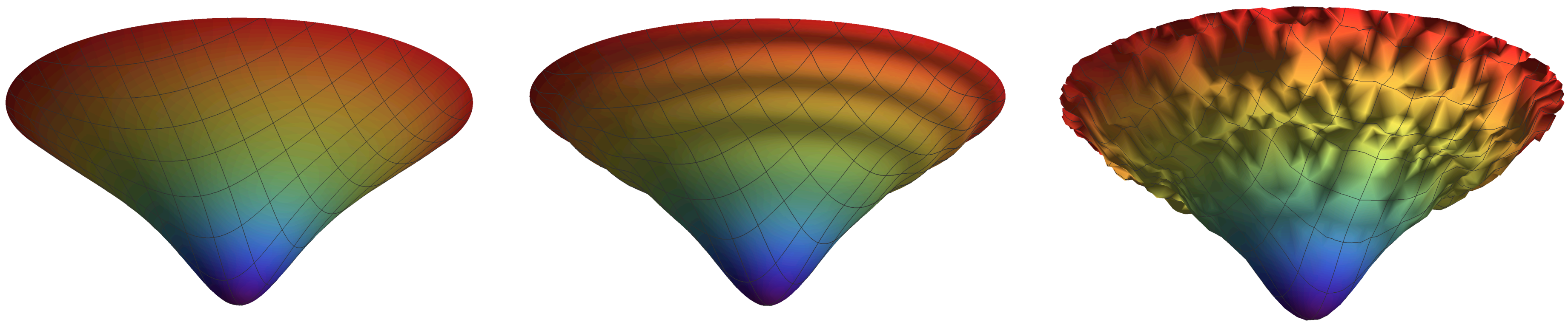

- Background and perturbations must be considered together
- The universe did not have a smooth beginning
- The question of initial conditions is wide open!