Circuit Complexity in QFT

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Ro Jefferson and Rob Myers

Albert Einstein Institute

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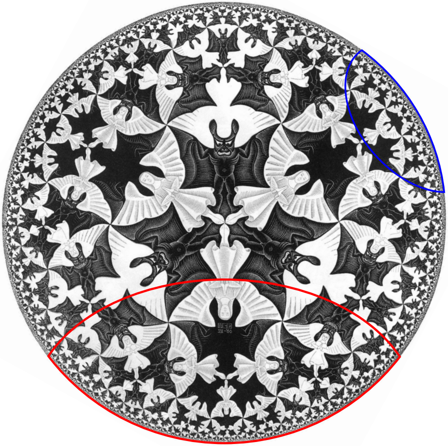
AdS/CFT in a hyperbolic nutshell

- Holographic duality between $(d+1)$-dimensional gravitational theory and d-dimensional CFT
- Question: how to reconstruct the bulk from the boundary?
- Entanglement plays a key role (Ryu-Takayanagi, quantum error correction)
- Problems for non-trivial spacetimes (black holes, holographic shadows)

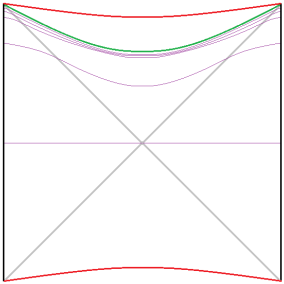


“Entanglement is not enough” (1411.0690)

Consider the thermofield double state as a realization of ER=EPR:

$$|\mathrm{TFD}\rangle = \frac{1}{Z_\beta}\sum_i e^{-\beta E_i/2}\,|i\rangle_L\,|\tilde{i}\rangle_R$$

- Black hole reaches thermal equilibrium quickly, $\sim t_{\mathrm{therm}}$
- Distance along maximal slices increases linearly with time



- $|\mathrm{TFD}\rangle$ continues to evolve for $\sim t_{\mathrm{comp}}$

Holographic complexity

Susskind proposed "holographic complexity" as the CFT quantity that encodes the continued evolution of the ERB.

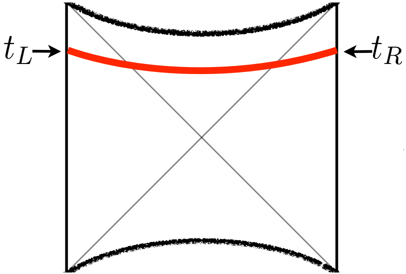
Two proposals for the bulk dual of complexity:

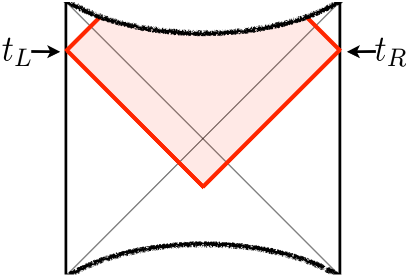
"complexity = volume"

$$\mathcal{C}_V\left(t_L, t_R\right) = \frac{V(t_L, t_R)}{Gl}$$

"complexity = action"

$$\mathcal{C}_A\left(t_L, t_R\right) = \frac{\mathcal{A}}{\pi\hbar}$$





Computational (circuit) complexity

- Goal: construct the optimum circuit for a given task
- Given a reference state $|\psi_0\rangle$, what is the least complex quantum circuit U that produces a given target state $|\psi_1\rangle$?

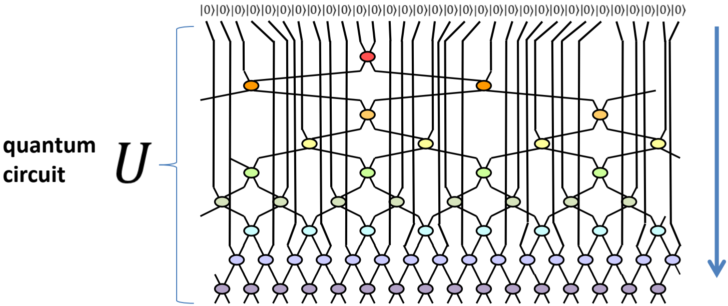
$$|\psi_1\rangle = U\,|\psi_0\rangle$$

- U consists of a sequence of gates Q_i: $U = Q_1 Q_2 \ldots$
- *Circuit complexity* = length of circuit U = number of gates Q_i
- *State complexity* $\mathcal{C}(\psi)$ = complexity of least complex circuit U that generates the state $|\psi\rangle$
- Defined relative to a reference state, $\mathcal{C}(\psi_0) \equiv 0$
- Depends on the set of gates, $\{Q_i\}$

MERA: a quantum circuit

MERA (Mutli-scale Entanglement Renormalization Ansatz) efficiently generates ground-state wavefunction in $d = 2$ critical systems

Vidal 2015:



cMERA describes the ground state of a free scalar field

A free field theory model

Consider a free scalar field as an infinite set of harmonic oscillators:

$$H = \frac{1}{2}\int \mathrm{d}^{d-1}x \left[\pi(x)^2 + \vec{\nabla}\phi(x)^2 + m^2\phi(x)^2\right]$$
$$\to \frac{1}{2}\sum_{\vec{n}}\left\{\frac{p(\vec{n})^2}{\delta^{d-1}} + \delta^{d-1}\left[\frac{1}{\delta^2}\sum_i \left(\phi(\vec{n}) - \phi(\vec{n}-\hat{x}_i)\right)^2 + m^2\phi(\vec{n})^2\right]\right\}$$

Simpler starting point: two oscillators at positions x_1, x_2,

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + \omega^2\left(x_1^2 + x_2^2\right) + \Omega^2\left(x_1 - x_2\right)^2\right]$$
$$= \frac{1}{2}\left(\tilde{p}_+^2 + \tilde{p}_-^2 + \tilde{\omega}_+^2\tilde{x}_+^2 + \tilde{\omega}_-^2\tilde{x}_-^2\right)$$

where $\omega = m$, $\Omega = 1/\delta$, $\tilde{x}_\pm = \frac{1}{\sqrt{2}}\left(x_1 \pm x_2\right)$, $\tilde{\omega}_+^2 = \omega^2$, $\tilde{\omega}_-^2 = \omega^2 + 2\Omega^2$.

Choosing our states

- Target state: ground state oscillators in normal-mode basis $x_\pm$

$$\begin{aligned}\psi_1(\tilde{x}_+,\tilde{x}_-) &= \psi_1(\tilde{x}_+)\psi_1(\tilde{x}_-)\\ &= \frac{(\tilde{\omega}_+\tilde{\omega}_-)^{1/4}}{\sqrt{\pi}}\exp\left[-\frac{1}{2}\left(\tilde{\omega}_+\tilde{x}_+^2+\tilde{\omega}_-\tilde{x}_-^2\right)\right]\end{aligned}$$

Equivalently, in physical coordinates x_1, x_2

$$\psi_1(x_1,x_2) = \frac{\left(\omega_1\omega_2-\beta^2\right)^{1/4}}{\sqrt{\pi}}\exp\left[-\frac{\omega_1}{2}x_1^2-\frac{\omega_2}{2}x_2^2-\beta x_1x_2\right]$$

where $\omega_1=\omega_2=\frac{1}{2}\left(\tilde{\omega}_++\tilde{\omega}_-\right)$, $\quad\beta\equiv\frac{1}{2}\left(\tilde{\omega}_+-\tilde{\omega}_-\right)<0$

- Natural reference state: factorized Gaussian

$$\psi_0(x_1,x_2) = \sqrt{\frac{\omega_0}{\pi}}\exp\left[-\frac{\omega_0}{2}\left(x_1^2+x_2^2\right)\right]$$

Sufficient set of gates to produce ψ_1 from ψ_0:

$$Q_{00} = e^{i\epsilon p_0 x_0}\ , \qquad Q_{i0} = e^{i\epsilon x_i p_0}\ , \qquad Q_{0i} = e^{i\epsilon x_0 p_i}\ ,$$
$$Q_{ij} = e^{i\epsilon x_i p_j}\ , \qquad Q_{ii} = e^{\frac{i\epsilon}{2}(x_i p_i + p_i x_i)} = e^{\epsilon/2} e^{i\epsilon x_i p_i}\ .$$

These act on an arbitrary state $\psi\left(x_1, x_2\right)$ as follows:

$$Q_{00}\,\psi(x_1,x_2) = e^{i\epsilon p_0 x_0}\psi(x_1,x_2) \qquad \text{global phase change}$$
$$Q_{10}\,\psi(x_1,x_2) = e^{i\epsilon p_0 x_1}\psi(x_1,x_2) \qquad \text{local phase change}$$
$$Q_{01}\,\psi(x_1,x_2) = \psi(x_1+\epsilon x_0, x_2) \qquad \text{shift } x_1 \text{ by } \epsilon x_0$$
$$Q_{21}\,\psi(x_1,x_2) = \psi(x_1+\epsilon x_2, x_2) \qquad \text{shift } x_1 \text{ by } \epsilon x_2 \text{ (entangling)}$$
$$Q_{11}\,\psi(x_1,x_2) = e^{\epsilon/2}\psi\left(e^{\epsilon}x_1, x_2\right) \qquad \text{rescale } x_1 \text{ to } e^{\epsilon}x_1 \text{ (scaling)}$$

Example

Consider the simple circuit U defined as

$$U\psi_0 = Q_{22}^{\alpha_3}\, Q_{21}^{\alpha_2}\, Q_{11}^{\alpha_1}\, \psi_0 = \psi_1$$

Length of circuit U given by *circuit depth*:

$$\mathcal{D}_1 = \sum_i |\alpha_i| = \frac{1}{\epsilon}\left[\frac{1}{2}\ln\left(\frac{\tilde{\omega}_+}{\omega_0}\right) + \frac{1}{2}\ln\left(\frac{\tilde{\omega}_-}{\omega_0}\right) + \frac{\tilde{\omega}_- - \tilde{\omega}_+}{\sqrt{\tilde{\omega}_+ + \tilde{\omega}_-}}\sqrt{\frac{\omega_0}{2\tilde{\omega}_+\tilde{\omega}_-}}\right]$$

where $\omega_1\omega_2 - \beta^2 > \omega_0^2,\;\; \tilde{\omega}_- > \tilde{\omega}_+$.

Cannot identify $\mathcal{D}_1(U)$ with $\mathcal{C}(\psi_1)$: no guarantee that this is the shortest circuit.

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Question: How do we find the minimal length circuit?

A geometric approach (quant-ph/0502070)

- Restrict to Gaussian states $\Longrightarrow$ space of 2×2 matrices

$$\left.\begin{aligned} \psi_0(x_1,x_2) &\simeq \exp\big[-\omega_0(x_1^2+x_2^2)\big] \\ \psi_1(x_1,x_2) &\simeq \exp\big[-\omega_1 x_1^2 - \omega_2 x_2^2 - 2\beta x_1 x_2\big] \end{aligned}\right\} \psi \simeq \exp[-x_i\, A_{ij}\, x_j]$$

- Set of unitary gates $Q_{ij} = \exp[\epsilon\, M_{ij}]$ act as $A' = Q_{ij}\, A\, Q_{ij}^T$
- Can think of circuits U as paths in $\mathrm{GL}(2,\mathbb{R})$
- Introduce a metric $\langle\cdot,\cdot\rangle$ on $\mathrm{GL}(2,\mathbb{R})$
- Circuit depth $\mathcal{D}_1\,(U)$ then becomes geometric length
- Allows one to apply variational calculus to quantum circuit design $\longrightarrow$ *optimum circuit is minimum geodesic*

Reference and target states given by

$$A_0 = \omega_0 \mathbb{1} \ , \qquad A_1 = \begin{pmatrix} \omega_1 & \beta \\ \beta & \omega_2 \end{pmatrix} \ , \ \text{with} \ \ \omega_1 \omega_2 - \beta^2 > 0$$

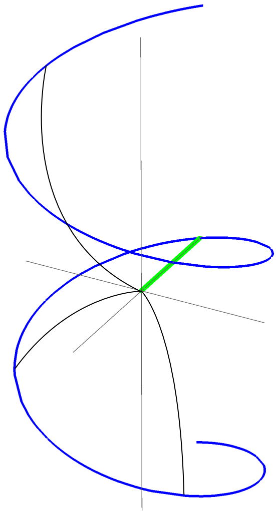
Circuit U a path in the space of (positive) quadratic forms

$$A_1 = U(1) A_0 U^T(1) \ , \qquad U(s) = \overleftarrow{\mathcal{P}} \exp \left[\int_0^s \mathrm{d}s' Y^I(s') M_I \right]$$

Parametrize $U \in \mathrm{GL}\,(2, \mathbb{R})$, components $Y^I = \mathrm{tr}(\mathrm{d}U U^{-1} M_I)$ allow construction of Euclidean geometry $\mathrm{d}s^2 = G_{IJ} Y^I Y^J$:

$$\begin{aligned} \mathrm{d}s^2 = \, & 2\mathrm{d}y^2 + 2\mathrm{d}\rho^2 + 2\cosh(2\rho)\cosh^2\rho \, \mathrm{d}\tau^2 \\ & + 2\cosh(2\rho)\sinh^2\rho \, \mathrm{d}\theta^2 - 2\sinh^2(2\rho) \, \mathrm{d}\tau \mathrm{d}\theta \end{aligned}$$

1-parameter family of solutions:



For our problem, minimal geodesic has

$$\tau(s) = 0\ , \quad \Delta\theta = 0 \implies \theta(s) = \theta_1 = \pi$$

Minimum path given by

$$U(s) = \exp\left[\begin{pmatrix} y_1 & -\rho_1 \\ -\rho_1 & y_1 \end{pmatrix} s\right]$$

Complexity $\mathcal{C} = \min\, \mathcal{D}$:

$$\mathcal{C} = \sum_I \left|Y^I(1)\right| = 2\left(\rho_1 + y_1\right)$$
$$= \frac{1}{2}\ln\left(\frac{\tilde{\omega}_+}{\omega_0}\right) + \frac{1}{2}\ln\left(\frac{\tilde{\omega}_-}{\omega_0}\right)$$

Normal-mode basis

Examine circuit in normal-mode basis, $\tilde{x}_\pm = (x_1 \pm x_2)/\sqrt{2}$ ($\tilde{x} = Rx$):

- Both reference and target states are *diagonal* in the normal-mode basis:

$$\tilde{A}_1 = R\,A_1\,R^T = \begin{pmatrix} \tilde{\omega}_+ & 0 \\ 0 & \tilde{\omega}_- \end{pmatrix}\,, \qquad \tilde{A}_0 = R\,A_0\,R^T = \begin{pmatrix} \omega_0 & 0 \\ 0 & \omega_0 \end{pmatrix}$$

- Minimal circuit simply scales up diagonal entries:

$$\tilde{U}(s) = \exp\left[\begin{pmatrix} y_1 - \rho_1 & 0 \\ 0 & y_1 + \rho_1 \end{pmatrix} s\right] = \exp\left[\begin{pmatrix} \frac{1}{2}\ln\frac{\tilde{\omega}_+}{\omega_0} & 0 \\ 0 & \frac{1}{2}\ln\frac{\tilde{\omega}_-}{\omega_0} \end{pmatrix} s\right]$$

- Normal-mode subspace is flat:

$$U(s) = \exp\left\{\,[M_{++}\,(y_1 - \rho_1) + M_{--}\,(y_1 + \rho_1)]\,s\right\},$$
$$[M_{++},\,M_{--}] = 0 \quad\Longrightarrow\quad \mathrm{d}s^2 = 2\mathrm{d}y^2 + 2\mathrm{d}\rho^2$$

Generalization to N oscillators

Reference and target states described by $N \times N$ matrices $\tilde{A}_0$, $\tilde{A}_1$

$$\psi_0(\tilde{x}_k) = \left(\frac{\omega_0}{\pi}\right)^{N/4} \exp\left[-\frac{1}{2}\tilde{x}^\dagger \tilde{A}_0 \tilde{x}\right], \qquad \tilde{A}_0 = \omega_0 \mathbb{1}$$

$$\psi_1(\tilde{x}_k) = \prod_{k=0}^{N-1} \left(\frac{\tilde{\omega}_k}{\pi}\right)^{1/2} \exp\left[-\frac{1}{2}\tilde{x}^\dagger \tilde{A}_1 \tilde{x}\right], \qquad \tilde{A}_1 = \mathrm{diag}(\tilde{\omega}_0, \ldots, \tilde{\omega}_{N-1})$$

Optimum circuit scales-up diagonal entries

$$\tilde{U}(s) = \exp\left[\tilde{Y}^{\tilde{I}} \tilde{M}_{\tilde{I}}\right], \quad \tilde{Y}^{\tilde{I}} \tilde{M}_{\tilde{I}} = \mathrm{diag}\left(\frac{1}{2}\ln\frac{\tilde{\omega}_0}{\omega_0}, \ldots, \frac{1}{2}\ln\frac{\tilde{\omega}_{N-1}}{\omega_0}\right)$$

Complexity for one-dimensional lattice of N oscillators:

$$\mathcal{C} = \sum_{\tilde{I}} \left|\tilde{Y}^{\tilde{I}}(1)\right| = \frac{1}{2}\sum_{k=0}^{N-1} \left|\ln\frac{\tilde{\omega}_k}{\omega_0}\right|$$

The continuum limit

Extend to $(d-1)$-dimensional lattice of N^{d-1} oscillators:

$$\mathcal{C} = \frac{1}{2} \sum_{\{k_i\}=0}^{N-1} \left| \ln \frac{\tilde{\omega}_{\vec{k}}}{\omega_0} \right|, \qquad \tilde{\omega}_{\vec{k}}^2 = \omega^2 + 4\Omega^2 \sum_{i=1}^{d-1} \sin^2 \frac{\pi k_i}{N}$$

Field theory parameters $\omega = m$, $\Omega = 1/\delta$.

Continuum limit: $N \to \infty$, $\delta \to 0$ with $N\delta$ fixed.

Leading order dominated by UV modes, $\tilde{\omega}_{\vec{k}} \sim 1/\delta \implies$

$$\mathcal{C} \approx \frac{N^{d-1}}{2} \left| \ln \frac{1}{\delta\omega_0} \right| \simeq \frac{V}{\delta^{d-1}} \left| \ln \frac{1}{\delta\omega_0} \right|, \qquad V = N^{d-1}\delta^{d-1}$$

$$\omega_0 = \begin{cases} \text{UV scale } e^{-\sigma}/\delta & \implies \mathcal{C} \approx \sigma \frac{V}{\delta^{d-1}} & (\text{C} = \text{A}) \\ \text{IR scale } \frac{\alpha}{\ell_{\text{AdS}}} \ll \frac{1}{\delta} & \implies \mathcal{C} \approx \frac{V}{\delta^{d-1}} \left| \ln \frac{\ell_{\text{AdS}}}{\alpha\delta} \right| & (\text{C} = \text{V}) \end{cases}$$

Summary & outlook

- Preliminary steps towards defining (circuit) complexity in quantum field theories
- Geometrical approach: optimum circuit a geodesic in the space of circuits
- Simple interpretation in normal-mode basis (Chapman et al.)
- Locality: penalty factors, more general metrics G_{IJ}?
- More general (non-Gaussian) states, interacting theories, fermions?
- Deeper relations to MERA, path-integral approach (Caputa et al., Czech)?