Algebraic properties of the monopole formula

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FWF Der Wissenschaftsfonds.

Based on arXiv:1605.00010 & 1611.07030 with A.Hanany

Outline

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Moduli space of supersymmetric vacua

Ex: $4d\ \mathcal{N}=1$ **theory**

- **gauge group** G: vector multiplets V_a
- **matter content**: representation $\mathcal{R}$ for chiral multiplets X_i
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$$\mathcal{M} \cong \{X \mid F_i = 0\ \forall i\} / \mathrm{G}^{\mathbb{C}}$$

$\implies$ $\mathcal{M}$ is Kähler

Chiral ring and Hilbert series

- **gauge-inv. scalar chiral operator** $\mathcal{O}$, i.e. $\bar{\mathcal{Q}}_{\dot{\alpha}}\mathcal{O}(x) = 0$

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R-charge: $[R, \bar{\mathcal{Q}}] = \bar{\mathcal{Q}}\,, \quad RW = 2W$

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$$H_{\mathfrak{R}}(t) := \sum_r \dim(\mathcal{H}_r) t^r = \frac{Q(t)}{\prod_i (1 - t^{d_i})}$$

Outline

bare monopole operators

for $3d$ $\mathcal{N} = 2$ theories: new class of chiral operators

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Dirac quantisation condition [Englert & Windey]

$$e^{2\pi i m} = \mathbb{1}_{\mathrm{G}} \quad \Leftrightarrow \quad m \in \Gamma_{\widehat{\mathrm{G}}}$$

$\Gamma_{\widehat{\mathrm{G}}}$ = weight lattice of GNO dual group [Goddard, Nuyts, Olive '77]

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- $\mathcal{N}=2$ v-plet $(A,\sigma) \longrightarrow$ chiral multiplet V_m for each m
- for each $m \in \Gamma_{\widehat{\mathrm{G}}}$ exists **a single** $N=2$ bare monopole V_m

Coulomb branch of $3d$ $\mathcal{N} = 4$

Specialise to $\mathcal{N} = 4$

$\mathrm{G}:$ $(\mathcal{N}{=}4 \text{ vector}) = (\mathcal{N}{=}2 \text{ vector}) \oplus (\mathcal{N}{=}2 \text{ chiral } \Phi \in \mathrm{adj})$

$\mathcal{R}:$ $(\mathcal{N}{=}4 \text{ hyper}) = (\mathcal{N}{=}2 \text{ chiral } X \in \mathcal{R}) \oplus (\mathcal{N}{=}2 \text{ chiral } X \in \bar{\mathcal{R}})$

with $W = \bar{X}\Phi X \rightarrow$ locally $\mathcal{M} = \mathcal{M}_C + \mathcal{M}_H$

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Coulomb branch $\boldsymbol{\mathcal{M}_C}$:

dressing of bare monopole V_m by residual gauge theory $\boldsymbol{T_m}$

- residual gauge group $\mathrm{H}_m = \mathrm{Stab}_{\mathrm{G}}(m)$
- residual matter fields Φ_α for $\alpha(m) = 0$

Coulomb branch of $3d$ $\mathcal{N} = 4$

Monopole formula counts dressed $\mathcal{N} = 2$ monopole operators

$$H(t) = \sum_{m \in \Gamma_{\widehat{G}}/\mathcal{W}} t^{\boldsymbol{\Delta}(\boldsymbol{m})} \boldsymbol{P}(\boldsymbol{t}, \boldsymbol{m})$$

[Cremonesi, Hanany, Zafferoni '13]

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$$\Delta(m) = \frac{1}{2} \sum_{\rho \in \mathcal{R}} |\rho(m)| - \sum_{\alpha \in \Phi_+} |\alpha(m)|$$

[Borokhov, Kapustin, Wu '02; Gaiotto, Witten '08; ...]

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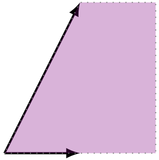
- Dressing by residual gauge theory T_m

$$P(t,m) = \frac{1}{\prod_{i=1}^{r}(1 - t^{d_i(m)})} = H_{T_m}(t)$$

Outline

Introduction of fan and Hilbert bases

Summation range $S = \Gamma_{\widehat{\mathrm{G}}}/\mathcal{W}$



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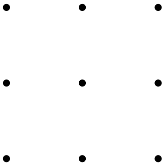

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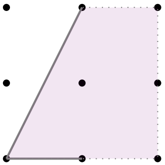

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weight lattice
$\Gamma_{\widehat{\mathrm{G}}}$

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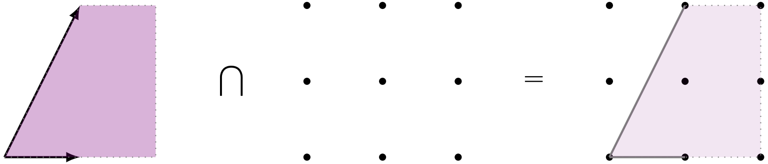

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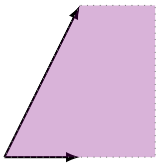

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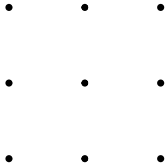

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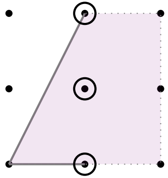

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Problem: absolute values $|\rho(m)|$ need be resolved

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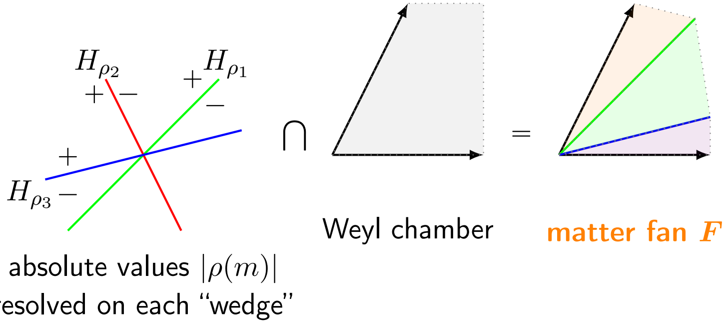

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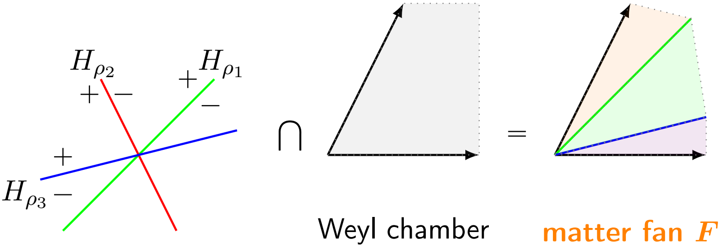

absolute values $|\rho(m)|$
resolved on each "wedge"

restrict to weight lattice $\longrightarrow$ **monoids** generated by Hilbert bases

Algebraic properties

$$H(t) = P(t,0) \quad \sum_{\tau \in F} \quad H_{M_\tau^{\textsf{Dress}}}(t) \quad H_{\omega(S_\tau)}(t)$$

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  - **dressed monopole operators** $\leftrightarrow$ $H_{M_\tau^{\mathsf{Dress}}}(t)$

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  - **dressed monopole operators** $\leftrightarrow H_{M_\tau^{\text{Dress}}}(t)$

    $\longrightarrow$ identify their finite number and all charges
- **pole structure**
  - order of pole of $H(t)$ at $t = 1$ is $\mathrm{rk}(\mathrm{G}) = \dim(\mathcal{M}_C)$
  - order of pole of $H(t)$ at $t \to \infty$ is $\mathrm{rk}(\mathrm{G}) = \dim(\mathcal{M}_C)$

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Geometric structure unveiled

- rewrite monopole formula
  - convenient for **implementation via** `Macaulay2`
  - **prove algebraic properties** of HS
  - appearing object mathematically well-studied
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Future directions

- Hilbert series with background charges $\longrightarrow$ resolution of $\mathcal{M}_C$
- implications for $\mathcal{N} = 2$ monopole formula

  [Cremonesi '15; Cremonesi, Mekareeya, Zafferoni '16]