Regge limit of scattering amplitudes from an anomalous dimension

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Massive scattering amplitudes

Modell

- Consider $\mathcal{N}=4$ SYM in the planar limit.
- Introduce masses via Higgs mechanism (all masses equal):

$$SU(N_c) \xrightarrow{\text{SSB}} SU(N_c-4)\times SU(4)\times U(1)$$

$\mathcal{N}=4$ SYM Lagrangian before SSB:

$$\mathcal{L}=\mathrm{tr}\Big\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}+2i\bar{\lambda}_{\dot{\alpha}A}\sigma_{\mu}^{\dot{\alpha}\beta}\mathcal{D}^{\mu}\lambda_{\beta}^{A}+(\mathcal{D}_{\mu}\phi^{I})(\mathcal{D}^{\mu}\phi^{I})$$

$$-\frac{1}{2}g^{2}[\phi^{I},\phi^{J}][\phi^{I},\phi^{J}]+\sqrt{2}g\lambda^{\alpha A}(T_{I})_{AB}[\phi^{I},\lambda_{\alpha}^{B}]-\sqrt{2}g\bar{\lambda}_{\dot{\alpha}A}(T_{I}^{\dagger})^{AB}[\phi^{I},\bar{\lambda}_{B}^{\dot{\alpha}}]\Big\}$$

with: $\mathcal{D}^{\mu}\Phi=\partial^{\mu}\Phi-ig[A^{\mu},\Phi]$, $\quad\Phi=\Phi^{a}T^{a}$, $\quad\Phi=\{\phi,\lambda,\bar{\lambda}\}$.

Setup

Modell

- Consider $\mathcal{N} = 4$ SYM in the planar limit.
- Introduce masses via Higgs mechanism (all masses equal):

$$SU(N_c) \xrightarrow{\text{SSB}} SU(N_c - 4) \times SU(4) \times U(1)$$

Scattering amplitudes

Consider color-ordered amplitude $Y\bar{Y} \to Y\bar{Y}$.

- Y: massless gauge boson, corresponding to an off-diagonal generator of the $SU(4)$.
- At loop level and in the planar limit, the interactions are mediated by massive W bosons.

These amplitudes have a dual conformal symmetry!
[Alday, Henn, Plefka, Schuster, '10]

Amplitude

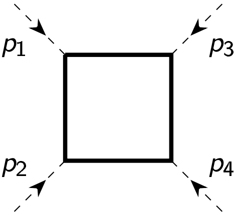
$$A = A_{\text{tree}}\, \mathcal{M}\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right)$$

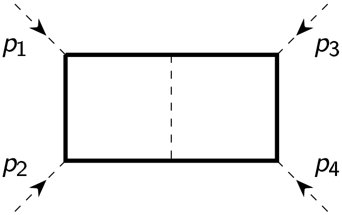
Variables:

$$s = (p_1 + p_2)^2\,, \qquad t = (p_2 + p_3)^2\,, \qquad m = \text{mass of } W \text{ bosons}$$

Perturbative expansion in $a = (g_{\text{YM}}^2 N_c)/(8\pi^2)$:

$$\mathcal{M} = 1 + a\,\mathcal{M}^{(1)} + a^2\,\mathcal{M}^{(2)} + a^3\,\mathcal{M}^{(3)} + \mathcal{O}\left(a^4\right)$$





Amplitude

Remarks on the amplitude / integrals:

- UV-finite
- IR-finite: mass of W bosons regulate infrared and collinear divergences.

$\Rightarrow$ Can calculate the integrals in $D = 4$ dimension.

Remarks on the calculation:

- Integrand derived using unitarity cuts. [Bern, Carrasco, Dennen, Huang, Ita, '10]
- Loop integrals evaluated up to three loops analytically in $D = 4$ dimensions using the differential equation method. [Caron-Huot, Henn, '14]

Kinematic limits

There are several interesting kinematic limits:

- High energy limit $s, t \to \infty$
  infrared and collinear divergences regulated by mass
- Threshold expansion $s \sim 4m^2$
  relation to hydrogen-like system
- Soft limit $s, t \to 0$
  effective action
- Forward limit $t = 0$
  total cross section, conjectured exact formula for $\lim_{s \to \infty} \sigma(s)$
- Regge limit $s \to \infty$
  Wilson line description, integrability

Calculation of the kinematic limits

Use the known differential equation for the integrals [Caron-Huot, Henn, '14]

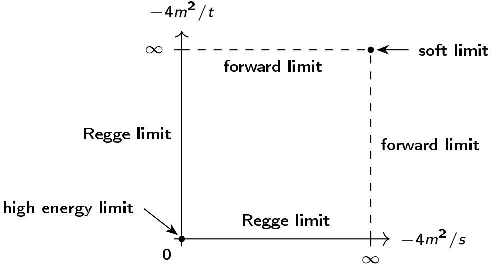
$$\mathrm{d}\vec{f} = \mathrm{d}A(s,t,m^2)\,\vec{f}\,.$$

Solve the it in an asymptotic expansion [Wasow, '65]

$$\text{DE: } x\,\frac{\partial \vec{f}}{\partial x} = \frac{\partial A}{\partial x}\vec{f}\,, \qquad \frac{\partial A}{\partial x} = \bar{A}_0 + \bar{A}_1 x + \mathcal{O}\left(x^2\right)$$

$$\text{Solution: } \vec{f}(x) = P(x)e^{\bar{A}_0 \log(x)}\,\vec{f}_0\,, \qquad P(x) = \mathbb{1} + P_1 x + \mathcal{O}\left(x^2\right)$$

Transportation of the (regularized) boundary value $\vec{f}_0$:



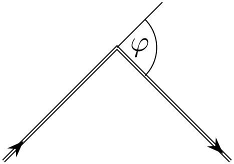
Regge limit

In the Regge limit $s \gg t, m^2$, the leading term is given by a simple power law [Henn, Naculich, Schnitzer, Spradlin, '10]

$$\lim_{s\to\infty} \mathcal{M}\left(\frac{4m^2}{-s}, \frac{4m^2}{-t}\right) = r_0(t) \left(\frac{4m^2}{-s}\right)^{\tilde{j}_0(t)} + \mathcal{O}(1/s)\,,$$

with

$$\tilde{j}_0(t) = -\Gamma_{\text{cusp}}(\varphi) \longrightarrow \text{anomalous dim. of a cusped Wilson loop}\,,$$
$$t = 4m^2 \sin^2(\varphi/2)\,.$$



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Single power law does not work for $\mathcal{O}(1/s)$ term.

Can improve situation by using better variables.

Regge limit

First improvement by a partial wave expansion.

$$A = \sum_j a_j P_j\big(\cos\theta_{SO(3)}\big)\,, \qquad \cos\theta_{SO(3)} = 1 + \frac{2t}{s}$$

This uses the rotational symmetry $SO(3)$.

In our setup we have a larger $SO(4)$ symmetry, due to dual conformal symmetry. The corresponding angle is given by

$$\cos\theta_{SO(4)} = 1 + \frac{2t}{s} - \frac{t}{2m^2}\,.$$

Using the $SO(4)$ variable $Y = e^{i\theta_{SO(4)}}$ we find in the Regge limit $Y \to 0$

$$\lim_{Y\to 0} \frac{1+Y}{1-Y}\,\mathcal{M} = \sum_{n=0}^{\infty} r_n(t)\, Y^{j_n(t)+1}\,.$$

Regge limit

Leading and subleading terms are described by single power laws:

$$\lim_{Y\to 0} \frac{1+Y}{1-Y} \mathcal{M} = r_0(t) Y^{\tilde{j}_0(t)} + r_1(t) Y^{\tilde{j}_1(t)+1} + \mathcal{O}\left(Y^2\right)$$

with $(4m^2)/(-t) = 4x/(1-x^2)$ we have

$$r_0 = 1 + \mathcal{O}\left(a\right) , \qquad r_1 = 2 + \mathcal{O}\left(a\right) ,$$

$$\tilde{j}_0 = a \frac{1-x}{1+x} \log(x) + \mathcal{O}\left(a^2\right) = -\Gamma_{\text{cusp}}(\varphi) ,$$

$$\tilde{j}_1 = -2a + a^2 \left[\frac{1}{4} \frac{1+x}{1-x} \left(\frac{4\log^3(x)}{3} + \frac{4}{3}\pi^2 \log(x) \right) \right.$$
$$\left. + 2\log^2(x) + 4\frac{1-x}{1+x}\log(x) + \frac{2\pi^2}{3} + 4 \right] + \mathcal{O}\left(a^3\right)$$

(calculated up to three loops)

Wilson loops

Cusp anomalous dimension in $\mathcal{N} = 4$ SYM

We consider the Maldacena Wilson loop operator

$$W[C] = P \exp \left[i g_{\text{YM}} \left(\int_C dx_\mu A^\mu + \int_C |dx| \sqrt{\dot{x}^2} \vec{n} \cdot \vec{\phi} \right) \right]$$

and choose $\vec{n} = (1, 0, 0, 0, 0, 0)$.

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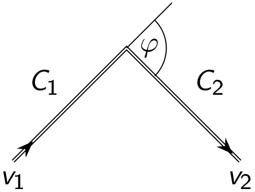
In particular interested in a Wilson loop with a cusp:

$$\boxed{W_{\text{cusp}} = \frac{1}{N_c} \mathrm{Tr}\big(W[C_1] W[C_2] \big)}$$



$$v_1^2 = v_2^2 = 1$$

$$\cos(\varphi) = v_1 \cdot v_2 = \frac{1}{2} \left(x + \frac{1}{x} \right)$$

$$x = e^{i\varphi}$$

Cusp anomalous dimension in $\mathcal{N} = 4$ SYM

- In the case of a straight line $\varphi = 0$, the operator W_{cusp} is protected.
- Known in planar $\mathcal{N} = 4$ SYM up to 4 loops. [Drukker, Forini, '06] [Correa, Henn, Maldacena, Sever '12] [Henn, Huber, '13]
- Planar case governed by integrability. [Drukker, '12] [Correa, Maldacena, Sever, '12]

Result for the anomalous dimension of W_{cusp}:

$$\Gamma_{\text{cusp}} = -a\,\frac{1-x}{1+x}\log(x) + \mathcal{O}\left(a^2\right)\,, \qquad x = e^{i\varphi}\,, \qquad a = \frac{g_{\text{YM}}^2 N_c}{8\pi^2}$$

checked up to three loops

$$\lim_{Y\to 0}\frac{1+Y}{1-Y}\,\mathcal{M} = r_0(t)\,Y^{\tilde{j}_0(t)} + r_1(t)\,Y^{\tilde{j}_1(t)+1} + \mathcal{O}\left(Y^2\right)$$

Wilson loop with scalar operator insertion

We want to reproduce $\tilde{j}_1$ with a Wilson loop calculation.

Let us consider a Wilson loop with a scalar inserted at the cusp

$$W_\phi = W[C_1]\,\phi_1(0)\,W[C_2]\,.$$

Even in the straight line case, this operator give rise to an anomalous dimension. [Alday, Maldacena, '07]

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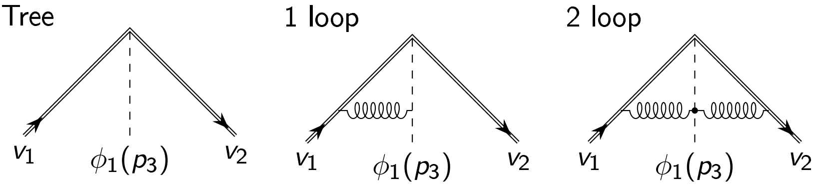
How to calculate the anomalous dimension Γ_ϕ?

- No tree level contribution to $\langle 0|W_\phi|0\rangle$.
- Therefore consider correlator with on-shell scalar $\phi_1(p_3)$ ($p_3^2 = 0$) in the initial state

$$\boxed{\langle 0|W_\phi|\phi_1(p_3)\rangle}$$

Wilson loop with scalar operator insertion

Typical Feynman diagrams for $\langle 0 | W_\phi | \phi_1(p_3) \rangle$:



We have three invariants:

$$\cos(\varphi) = v_1 \cdot v_2 = \frac{1}{2}\left(x + \frac{1}{x}\right), \quad s_1 = -2v_1 \cdot p_3\,, \quad s_2 = 2v_2 \cdot p_3$$

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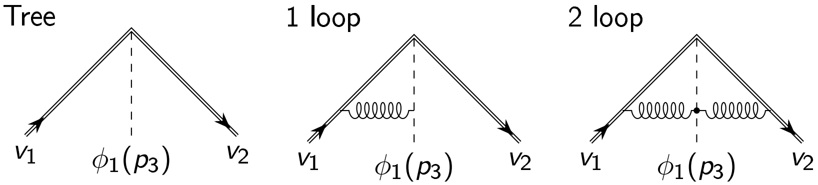

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$$\cos(\varphi) = v_1 \cdot v_2 = \frac{1}{2}\left(x + \frac{1}{x}\right), \quad s_1 = -2 v_1 \cdot p_3, \quad s_2 = 2 v_2 \cdot p_3$$

Divergences:

- UV divergences from the cusp: $\rightarrow Z_\phi(x)$
- IR and collinear divergences from on-shell scalar: $\rightarrow Z_{\mathrm{IR}}(s_1, s_2)$

Wilson loop with scalar operator insertion

The IR and collinear divergence are universal and have the form

$$\log\left((Z_{\mathrm{IR}})^{-1}\right) = \sum_{L\geq 1}\left(s_1^{-2\epsilon L} + s_2^{-2\epsilon L}\right) a^L \left(\frac{C_{L,1}}{\epsilon L} + \frac{C_{L,2}}{(\epsilon L)^2}\right) .$$

This allows us to use the "normal" Wilson cusp W_{cusp} to determine Z_{IR}.

Wilson loop with scalar operator insertion

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This allows us to use the "normal" Wilson cusp W_{cusp} to determine Z_{IR}.

Renormalization

$$\left.\begin{aligned} Z_{\mathrm{cusp}}^{-1}\, Z_{\mathrm{IR}}^{-1}\, \langle 0|W_{\mathrm{cusp}}|\phi_1(p_3)\rangle &= \text{finite} \\ Z_{\phi}^{-1}\, Z_{\mathrm{IR}}^{-1}\, \langle 0|W_{\phi}|\phi_1(p_3)\rangle &= \text{finite} \end{aligned}\right\} \implies Z_{\mathrm{IR}} \text{ and } Z_{\phi}$$

$$\tilde{j}_1 = -2a + a^2\left[\frac{1}{4}\frac{1+x}{1-x}\left(\frac{4\log^3(x)}{3} + \frac{4}{3}\pi^2\log(x)\right) + 2\log^2(x) + 4\frac{1-x}{1+x}\log(x) + \frac{2\pi^2}{3} + 4\right] + \mathcal{O}\left(a^3\right)$$

work in progress

verified

Summary and Outlook

Summary

- Massive scattering amplitude $Y\bar{Y} \to Y\bar{Y}$
  - calculated up to three loops
  - studied various kinematic limits
- Wilson loops
  - Regge limit described by anomalous dimension of Wilson loops:
    leading term $\to \Gamma_{\text{cusp}}$ (checked up to three loops)
    subleading term $\to \Gamma_{\phi}$ (checked up to two loops)

Outlook

- Γ_{ϕ} from integrability?
  [Gromov, Levkovich-Maslyuk, '15] $\to$ Calculation for the case where the inserted scalar does not couple to the Wilson line.
- Study subsubleading term in Regge limit.

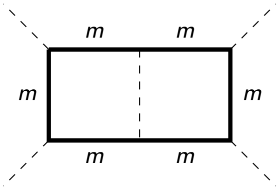
Backup slides

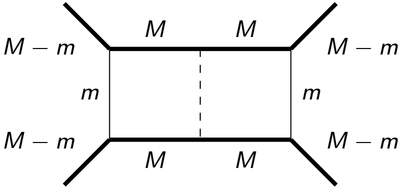
Three loop result for $\tilde{j}_1$

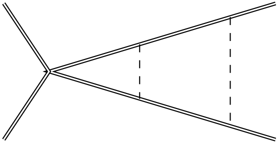
$$
\begin{aligned}
\tilde{j}_1 =& -4g^2 + g^4\left[\frac{-H_{1,1,1} - \frac{2}{3}\pi^2 H_1}{\xi} + 4H_{1,1} - 8H_1\xi + \frac{8\pi^2}{3} + 16\right] \\
&+ g^6\bigg[-16\xi^2 H_{1,1} + \xi\Big(-32H_{1,1} + 28H_{1,1,1} + 8\pi^2 H_1 + 96H_1 - 64H_2\Big) \\
&\quad + \Big(-8\pi^2 H_{1,1} - 24H_{1,1} + 32H_{1,2} + 16H_{1,1,1} - 20H_{1,1,1,1} \\
&\quad + 96\zeta_3 - \frac{8\pi^4}{3} - \frac{32\pi^2}{3} - 128\Big) \\
&\quad + \frac{1}{\xi}\Big(\frac{8}{3}\pi^2 H_{1,1,1} + 4H_{1,1,1} - 8H_{1,1,2} - 4H_{1,1,1,1} \\
&\quad + 6H_{1,1,1,1,1} - 24\zeta_3 H_1 + \frac{44\pi^4 H_1}{45} + \frac{8\pi^2 H_1}{3}\Big)\bigg]
\end{aligned}
$$

$H_{\vec{a}} = H_{\vec{a}}(1-x^2)$ [Harmonic Polylogarithm], $\xi = (1-x)/(1+x)$.

From the Regge limit to Wilson lines







[Henn, Naculich, Schnitzer, Spradlin, '10]

Amplitude:

$$u = \frac{4m^2}{-s}\,,\quad v = \frac{4m^2}{-t}$$

dual conformal symmetry

Amplitude:

$$u = \frac{4m^2}{-s}\,,\quad v = \frac{4M^2}{-t}$$

$m \ll M \ll s$

Wilson line:
cusp angle φ

Obtaining the $SO(4)$ angle $\theta_{SO(4)}$

We start with a change of variables to dual coordinate space

$$y_1 - y_4 = p_1\,, \qquad y_2 - y_1 = p_2\,, \qquad y_3 - y_2 = p_3\,, \qquad y_4 - y_3 = p_4\,.$$

Next we introduce vectors, which transform linear under the dual conformal symmetry $SO(4,2)$

$$Y = \left(\vec{y}, \frac{y^2 - m^2}{2\mu} + \frac{\mu}{2}, y^0, \frac{y^2 - m^2}{2\mu} - \frac{\mu}{2}\right)\,, \qquad Y^2 = m^2\,.$$

In the center of mass frame we have ($\mu = \sqrt{4m^2 - s}/2$)

$$Y_1 = \begin{pmatrix} \vec{p}_1 \\ -s/(4\mu) \\ 0 \\ -m^2/\mu \end{pmatrix}, Y_2 = \begin{pmatrix} \vec{0} \\ 0 \\ \sqrt{s}/2 \\ -\mu \end{pmatrix}, Y_3 = \begin{pmatrix} -\vec{p}_4 \\ -s/(4\mu) \\ 0 \\ -m^2/\mu \end{pmatrix}, Y_4 = \begin{pmatrix} \vec{0} \\ 0 \\ -\sqrt{s}/2 \\ -\mu \end{pmatrix}.$$

$\theta_{SO(4)}$ is the angle between the spacelike components of Y_1 and Y_3