Cluster algebras, Steinmann relations and scattering amplitudes

Georgios Papathanasiou

Deutsches Elektronen-Synchrotron

DESY String Theory Seminar
May 4, 2017

`1412.3763 [hep-th] with Drummond & Spradlin`
`1612.08976 [hep-th] + work in progress with`
`Dixon, Drummond, Harrington, McLeod & Spradlin`
`1606.08807 [hep-th] with Del Duca,Druc,`
`Drummond,Duhr,Dulat,Marzucca,Verbeek`

Outline

$\mathcal{N} = 4$ Super Yang Mills Theory & Why Should We Care

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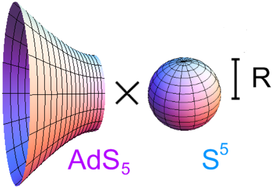

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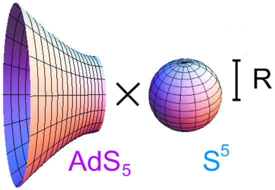

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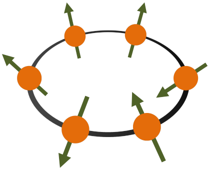

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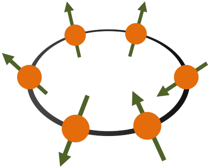

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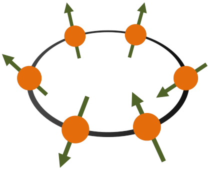

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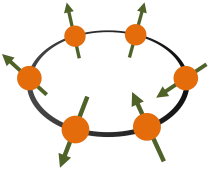

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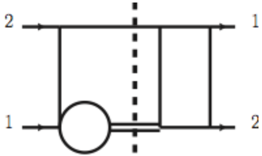

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Then apply to QCD, e.g. $|gg \to Hg|^2$ for N^{3}LO Higgs cross-section!

[Anastasiou,Duhr,Dulat,Herzog,Mistlberger]

Scattering Amplitudes: $d\sigma \propto |\mathcal{A}|^2$

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Can thus use helicity $h = \vec{S} \cdot \hat{p}$ to classify on-shell particle content,

$$\begin{array}{ccccc} h: -1 & -1/2 & 0 & 1/2 & 1 \\ G^- \xrightarrow{Q^1} & \bar{\Gamma}^A \xrightarrow{Q^2} & \Phi_{AB} \xrightarrow{Q^3} & \Gamma_A \xrightarrow{Q^4} & G^+ \end{array}$$

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For the gluons $G^\pm$, the gluinos $\Gamma, \bar{\Gamma}$, and the scalars Φ. For n gluons,

$$\begin{aligned} &\mathcal{A}_n^{L\text{-loop}}(\{k_i, h_i, a_i\}) \\ &\quad = \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}})\ A_n^{(L)}(\sigma(1^{h_1}),\ldots,\sigma(n^{h_n})) \\ &\qquad\qquad + \text{multitrace terms, subleading by powers of } 1/N^2\,. \end{aligned}$$

$A_n^{(L)}$: color-ordered amplitude, all color factors removed.

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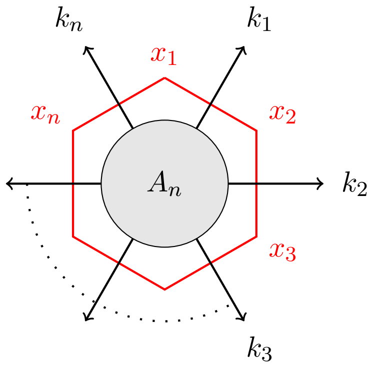
They also have remarkable properties, namely they

- are dual to null polygonal Wilson loops.
  [Alday,Maldacena] [Drummond,Korchemsky,Sokatchev] [Brandhuber,Heslop,Travaglini]



$$k_i \equiv x_{i+1} - x_i \equiv x_{i+1,i}\,,$$

$$k_i^2 = x_{i+1,i}^2 = 0$$

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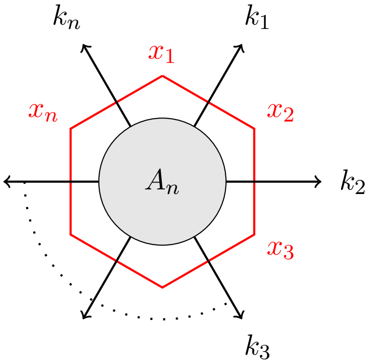

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- exhibit (formally) dual conformal invariance (DCI) under $x_i^\mu \to \frac{x_i^\mu}{x_i^2}$

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$$\boxed{W_n = W_n^{BDS} e^{R_n(u_1,\ldots,u_m)}}$$

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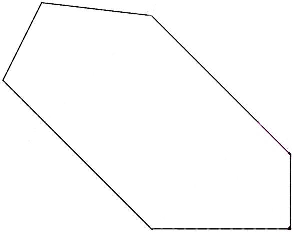
- # of independent u_i: $m = 4n - n - 15 = 3n - 15$

Nonperturbative Definition via the Collinear Limit



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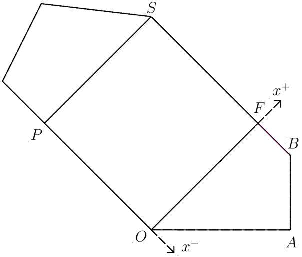

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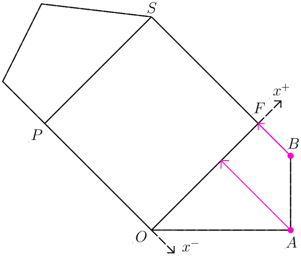

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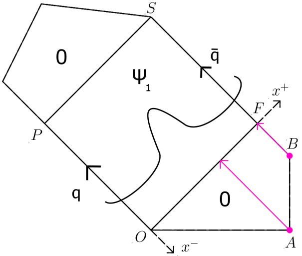

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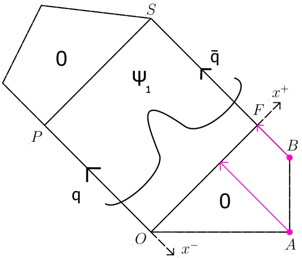

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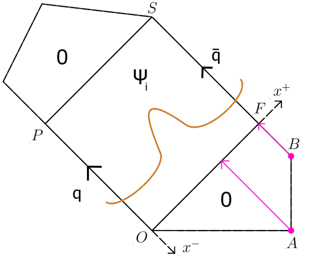

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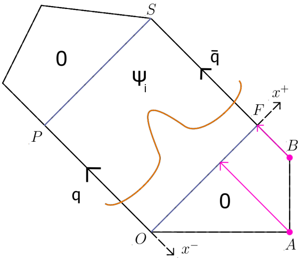

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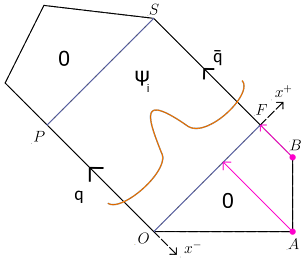

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⇒ WL 'Operator Product Expansion' (OPE)

[Alday,Gaiotto,Maldacena,Sever,Vieira]

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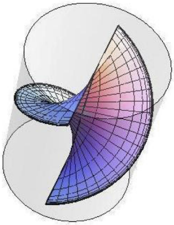

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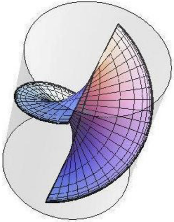

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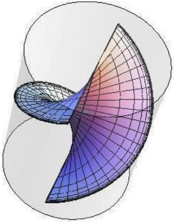

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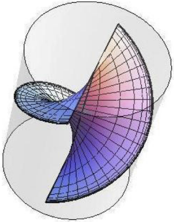
If exact S-matrix within reach, look at many "data points" at weak/strong coupling to extract its general pattern.



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Motivated by this progress, we upgraded this procedure for $n = 7$, with information from the cluster algebra structure of the kinematical space.

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2. What the function *arguments* (encoding the kinematics) are

B. Fix the coefficients of the ansatz by imposing consistency conditions (e.g. collinear data we described in previous part of talk)

Motivated by this progress, we upgraded this procedure for $n = 7$, with information from the cluster algebra structure of the kinematical space.

> Surprisingly, we found that heptagon bootstrap is more powerful than the hexagon one! Obtained the symbol of $R_7^{(3)}$ from very little input. [Drummond,GP,Spradlin]

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Very convenient tool for describing them: The **symbol** $\mathcal{S}(f_k)$, encapsulating recursive application of above definition (on $f_{k-1}^{(\alpha)}$ etc)

$$\mathcal{S}(f_k) = \sum_{\alpha_1,\ldots,\alpha_k} f_0^{(\alpha_1,\alpha_2,\ldots,\alpha_k)}\left(\phi_{\alpha_1}\otimes\cdots\otimes\phi_{\alpha_k}\right).$$

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Empeirical evidence: L-loop amplitudes=GPLs of weight $k = 2L$
[Duhr,Del Duca,Smirnov] [Arkani-Hamed,Bourjaily,Cachazo,Goncharov,Postnikov,Trnka] [GP]

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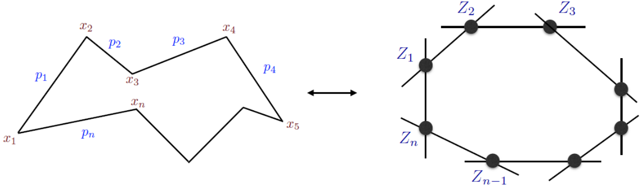
The latter is a collection of n ordered *momentum twistors* Z_i on $\mathbb{P}^3$, (an equivalent way to parametrise massless kinematics), modulo dual conformal transformations.



$$x_i \sim Z_{i-1} \wedge Z_i$$

$$(x_i - x_j)^2 \sim \epsilon_{IJKL} Z^I_{i-1} Z^J_i Z^K_{j-1} Z^L_j = \det(Z_{i-1} Z_i Z_{j-1} Z_j) \equiv \langle i-1 i j-1 j \rangle$$

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- $(x_{i+i} - x_i)^2 = 0 \quad \Rightarrow X_i = Z_{i-1}\wedge Z_i$

$\mathrm{Conf}_n(\mathbb{P}^3)$ and Graßmannians

Can realize $\mathrm{Conf}_n(\mathbb{P}^3)$ as $4 \times n$ matrix $(Z_1|Z_2|\ldots|Z_n)$ modulo rescalings of the n columns and $SL(4)$ transformations, which resembles a Graßmannian $Gr(4, n)$.

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Comparing the two matrices,

$$\boxed{\mathrm{Conf}_n(\mathbb{P}^3) = Gr(4,n)/(C^*)^{n-1}}$$

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Here, finite number of cluster variables:

$$a_3 = \frac{1+a_2}{a_1}\,, \quad a_4 = \frac{1+a_1+a_2}{a_1 a_2}\,, \quad a_5 = \frac{1+a_1}{a_2}\,, \quad a_6 = a_1\,, \quad a_7 = a_2$$

Cluster algebras (cont'd)

For our purposes, can be described by quivers, where each variable a_k of a cluster corresponds to node k.

Example: A_3 Cluster algebra

- Initial cluster: $a_1 \longrightarrow a_2 \longrightarrow a_3$

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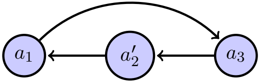
- Mutate at 2: 

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- In this manner, obtain new quiver/cluster where

$$a_k \to a'_k = \frac{1}{a_k}\left(\prod_{\text{arrows } i\to k} a_i + \prod_{\text{arrows } k\to j} a_j\right)$$

Example: A_3 Cluster algebra

- Initial cluster: $$\begin{tikzcd} a_1 \arrow[r] & a_2 \arrow[r] & a_3 \end{tikzcd}$$
- Mutate at 2: $$\begin{tikzcd} a_1 \arrow[rr, bend left] & a'_2 \arrow[l] & a_3 \arrow[l] \end{tikzcd}$$
- Leads to new cluster $\{a_1, a'_2, a_3\}$ with $a'_2 = (a_1 + a_3)/a_2$ and so on

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- Crucial observation: For all known cases, symbol alphabet of n-point amplitudes for $n = 6, 7$ are $Gr(4,n)$ cluster variables (also known as $\mathcal{A}$-coordinates) [Golden,Goncharov,Spradlin,Vergu,Volovich]

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Fundamental assumption of "cluster bootstrap"

Symbol alphabet is made of cluster $\mathcal{A}$-coordinates on $\mathrm{Conf}_n(\mathbb{P}^3)$. For the heptagon, 42 of them.

Heptagon Symbol Letters

Multiply $\mathcal{A}$-coordinates with suitable powers of $\langle i\, i+1\, i+2\, i+3\rangle$ to form conformally invariant cross-ratios,

$$a_{11} = \frac{\langle 1234\rangle\langle 1567\rangle\langle 2367\rangle}{\langle 1237\rangle\langle 1267\rangle\langle 3456\rangle}\,, \qquad a_{41} = \frac{\langle 2457\rangle\langle 3456\rangle}{\langle 2345\rangle\langle 4567\rangle}\,,$$

$$a_{21} = \frac{\langle 1234\rangle\langle 2567\rangle}{\langle 1267\rangle\langle 2345\rangle}\,, \qquad a_{51} = \frac{\langle 1(23)(45)(67)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

$$a_{31} = \frac{\langle 1567\rangle\langle 2347\rangle}{\langle 1237\rangle\langle 4567\rangle}\,, \qquad a_{61} = \frac{\langle 1(34)(56)(72)\rangle}{\langle 1234\rangle\langle 1567\rangle}\,,$$

where

$$\langle ijkl\rangle \equiv \langle Z_i Z_j Z_k Z_l\rangle = \det(Z_i Z_j Z_k Z_l)$$

$$\langle a(bc)(de)(fg)\rangle \equiv \langle abde\rangle\langle acfg\rangle - \langle abfg\rangle\langle acde\rangle\,,$$

together with a_{ij} obtained from a_{i1} by cyclically relabeling $Z_m \to Z_{m+j-1}$.

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Define **heptagon symbol**: A symbol of the aforementioned 42-letter alphabet, obeing 1 & 2.

Results

Weight k =	1	2	3	4	5	6
Number of heptagon symbols	7	42	237	1288	6763	?
well-defined in the $7 \parallel 6$ limit	3	15	98	646	?	?
which vanish in the $7 \parallel 6$ limit	0	6	72	572	?	?
well-defined for all $i{+}1 \parallel i$	0	0	0	1	?	?
with MHV last entries	0	1	0	2	1	4
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The symbol of the two-loop seven-particle MHV remainder function $R_7^{(2)}$ is the only weight-4 heptagon symbol which is well-defined in all $i{+}1 \parallel i$ collinear limits.

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The symbol of the three-loop seven-particle MHV remainder function $R_7^{(3)}$ is the only weight-6 heptagon symbol which satisfies the last-entry condition and which is finite in the $7 \parallel 6$ collinear limit.

Comparison with the hexagon case

Weight k =	1	2	3	4	5	6
Number of hexagon symbols	3	9	26	75	218	643
well-defined (vanish) in the $6 \parallel 5$ limit	0	2	11	44	155	516
well-defined (vanish) for all $i{+}1 \parallel i$	0	0	2	12	68	307
with MHV last entries	0	3	7	21	62	188
with both of the previous two	0	0	1	4	14	59

Table: Hexagon symbols and their properties.

Surprisingly, heptagon bootstrap more powerful than hexagon one! Fact that $\lim_{7\parallel 6} R_7^{(3)} = R_6^{(3)}$, as well as discrete symmetries such as cyclic $Z_i \to Z_{i+1}$, flip $Z_i \to Z_{n+1-i}$ or parity symmetry **follow for free**, not imposed a priori.

Steinmann Relations Upgrade

[Steinmann] [Cahill,Stapp] [Bartels,Lipatov,Sabio Vera]

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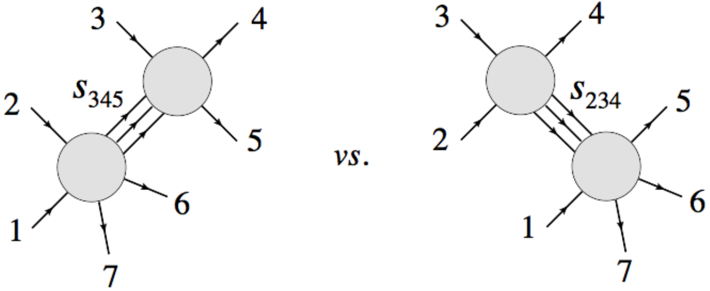
  $\Rightarrow$ 5-loop Hexagon [Caron-Huot,Dixon,McLeod,von Hippel]
- More powerful at 7 points $\Rightarrow$ 3-loop NMHV/4-loop MHV Heptagon [Dixon,Drummond,Harrington,McLeod,GP,Spradlin]

Steinmann Relations

Double discontinuities vanish for any set of overlapping channels



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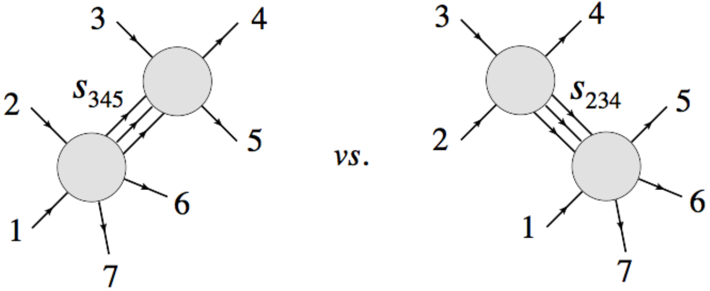

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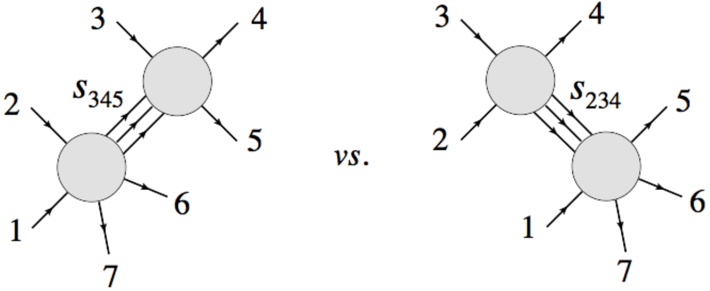

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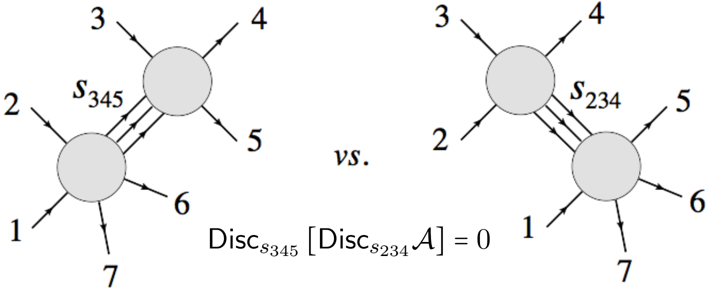

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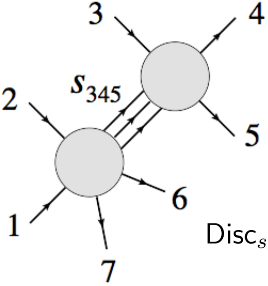

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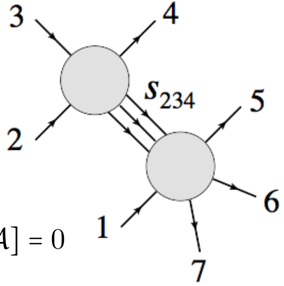

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$$\boxed{\text{Disc}_{a_{1i}}\left[\text{Disc}_{a_{1j}}\mathcal{A}\right] = 0 \quad \text{if } j \neq i, i+3, i+4\,.}$$

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BDS-like normalized amplitudes obey Steinmann relations, BDS normalized ones do not!

Results: Steinmann Heptagon symbols

Weight k =	1	2	3	4	5	6	7	$7''$
parity +, flip +	4	16	48	154	467	1413	4163	3026
parity +, flip −	3	12	43	140	443	1359	4063	2946
parity −, flip +	0	0	3	14	60	210	672	668
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Total	7	28	97	322	1030	3192	9570	7309

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Results: 4-loop MHV Heptagon

Loop order $L =$	1	2	3	4
Steinmann symbols	28	322	3192	?
MHV final entry	1	1	2	4
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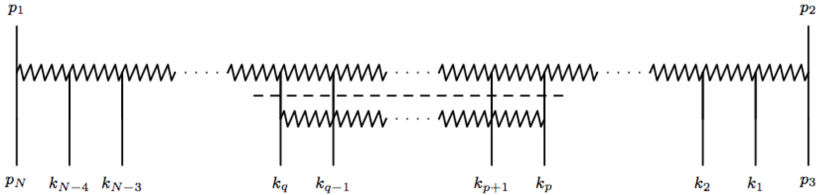
> Strong tension between collinear properties and Steinmann relations.

Further checks: Multi-Regge limit

Phenomenologically relevant high-energy gluon scattering



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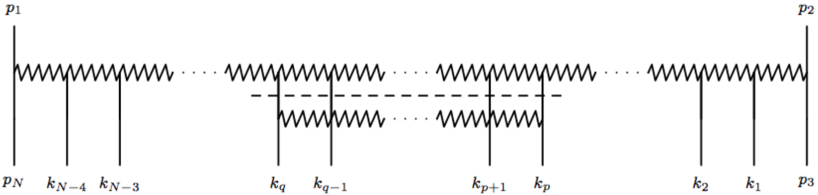

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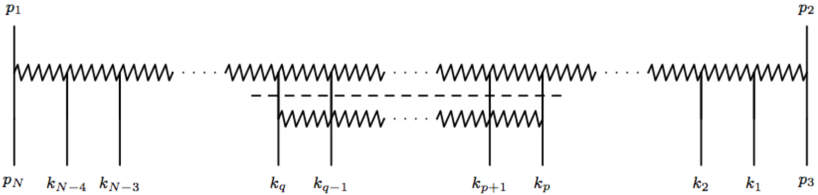

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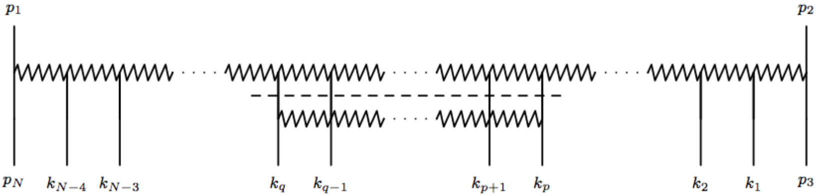

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Beyond seven particles

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For $N \geq 8$, $Gr(N,8)$ cluster algebra becomes infinite

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\frac{\langle 1234\rangle\langle 1256\rangle}{\langle 1236\rangle\langle 1245\rangle} \arrow[r] \arrow[d] & \frac{\langle 1235\rangle\langle 1267\rangle}{\langle 1237\rangle\langle 1256\rangle} \arrow[r] \arrow[d] & \cdots\cdots \arrow[r] & \frac{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle}{\langle 123\,N\rangle\langle 12\,N-2\,N-1\rangle} \arrow[d] \\
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- However, in multi-Regge limit, $Gr(N,8) \to A_{N-5} \times A_{N-5}$: finite!
  [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]

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\xcancel{\frac{\langle 1235\rangle\langle 1456\rangle}{\langle 1256\rangle\langle 1345\rangle}} \arrow[r] \arrow[d] & \xcancel{\frac{\langle 1236\rangle\langle 1245\rangle\langle 1567\rangle}{\langle 1235\rangle\langle 1267\rangle\langle 1456\rangle}} \arrow[r] \arrow[d] \arrow[ul] & \cdots\cdots \arrow[r] \arrow[ul] & \xcancel{\frac{\langle 123\,N-1\rangle\langle 12\,N-3\,N-2\rangle\langle 1\,N-2\,N-1\,N\rangle}{\langle 123\,N-2\rangle\langle 12\,N-1\,N\rangle\langle 1\,N-3\,N-2\,N-1\rangle}} \arrow[d] \arrow[ul] \\
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- However, in multi-Regge limit, $Gr(N,8) \to A_{N-5} \times A_{N-5}$: finite!
  [Del Duca,Druc,Drummond,Duhr,Dulat,Marzucca,GP,Verbeek]
- The two A_{N-5} factors not independent: Related by single-valuedness

Beyond seven particles

For $N \geq 8$, $Gr(N,8)$ cluster algebra becomes infinite

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Exploiting this analytic structure, and generalizing the BFKL-type dispersion formula to N-pts, obtained LLA contributions of MHV amplitudes to 5 loops for any N, and NMHV amplitudes up to 4 loops and $N = 8$.

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- Steinmann relations on analytic structure of amplitude massively reduce the size of this space $\Rightarrow$ much simpler to single it out
- Surprizingly, 7-particle bootstrap more powerful than 6-particle one! Minimal input $\Rightarrow$ obtained symbols of 3-loop NMHV and 4-loop MHV

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- Similar story with the multi-Regge kinematics and BFKL approach.

> Ultimately, can the integrability of planar SYM theory, together with a thorough knowledge of the analytic structure of its amplitudes, lead us to the theory's exact S-matrix?

Imposing Constraints: Integrable Words

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Given a random symbol $\mathcal{S}$ of weight $k > 1$, there does not in general exist any function whose symbol is $\mathcal{S}$. A symbol is said to be **integrable**, (or, to be an **integrable word**) if it satisfies

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Define a **heptagon symbol**: An integrable symbol with alphabet a_{ij} that obeys first-entry condition.

MHV Constraints: Yangian anomaly equations

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Particularly here: Only the 14 letters a_{2j} and a_{3j} may appear in the last symbol entry of R_7.

Imposing Constraints: The Collinear Limit

It is baked into the definition of the BDS normalized n-particle L-loop MHV remainder function that it should smoothly approach the corresponding $(n-1)$-particle function in any simple collinear limit:

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> A function has a well-defined $i+1 \parallel i$ limit only if its symbol is independent of all nine of these letters.

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"Just" linear algebra, however for e.g. 4-loop MHV hexagon A boils down to a size of 941498×60182. Tackled with fraction-free variants of Gaussian elimination that bound the size of intermediate expressions, implemented in `Integer Matrix Library` and `Sage`. [Storjohann]

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- $(67) = (76) \equiv [12345]$ *Dual superconformal* R-invariants, with

$$[abcde] = \frac{\delta^{0|4}\big(\chi_a \langle bcde\rangle + \text{cyclic}\big)}{\langle abcd\rangle\langle bcde\rangle\langle cdea\rangle\langle deab\rangle\langle eabc\rangle}, \quad \chi_i^A = \sum_{j=1}^{i-1} \langle ji\rangle \eta_j^A .$$

Results: 3-loop NMHV Heptagon

Loop order L =	1	2	3
Steinmann symbols	15×28	$15{\times}322$	$15{\times}3192$
NMHV final entry	42	85	226
Dihedral symmetry	5	11	31
Well-defined collinear	0	0	0

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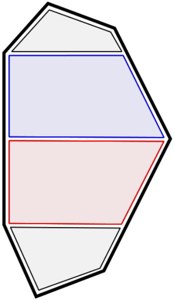
4. We also need collinear limit of R-invariants

Further check: Heptagon Wilson loop OPE

This is an expansion in two variables $e^{-\tau_1}, e^{-\tau_2}$ near the double collinear limit $\tau_1 \to \infty, \tau_2 \to \infty$.



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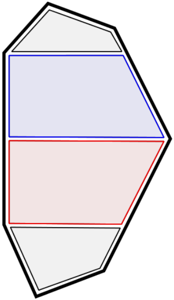
This is an expansion in two variables $e^{-\tau_1}, e^{-\tau_2}$ near the double collinear limit $\tau_1 \to \infty, \tau_2 \to \infty$.



Integrability predicts linear terms in $e^{-\tau_i}$ to all loops in integral form, e.g.[Basso,Sever,Vieira 2]

$$h = e^{i(\phi_1+\phi_2)}\, e^{-\tau_1-\tau_2} \int \frac{du dv}{(2\pi)^2}\, \mu(u)\, P_{FF}(-u|v)\mu(v) \times$$
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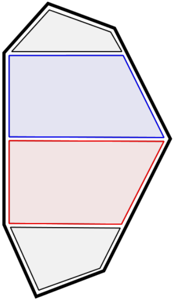

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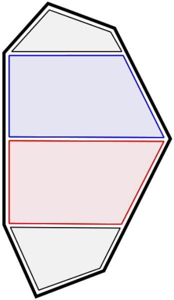

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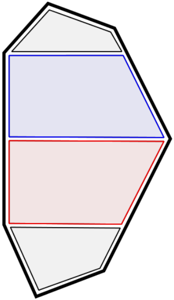

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Perfect match, currently computing 4 loops

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NMHV final entry conditions

[Caron-Huot]

$$(34)\log a_{21},\quad (14)\log a_{21},\quad (15)\log a_{21},\quad (16)\log a_{21},\quad (13)\log a_{21},\quad (12)\log a_{21},$$

$$(45)\log a_{37},\quad (47)\log a_{37},\quad (37)\log a_{37},\quad (27)\log a_{37},\quad (57)\log a_{37},\quad (67)\log a_{37},$$

$$(45)\log\frac{a_{34}}{a_{11}},\quad (14)\log\frac{a_{34}}{a_{11}},\quad (14)\log\frac{a_{11}a_{24}}{a_{46}},\quad (14)\log\frac{a_{14}a_{31}}{a_{34}},$$

$$(24)\log\frac{a_{44}}{a_{42}},\qquad (56)\log a_{57},\qquad (12)\log a_{57},\qquad (16)\log\frac{a_{67}}{a_{26}},$$

$$(13)\log\frac{a_{41}}{a_{26}a_{33}} + ((14)-(15))\log a_{26} - (17)\log a_{26}a_{37} + (45)\log\frac{a_{22}}{a_{34}a_{35}} - (34)\log a_{33}\,,$$