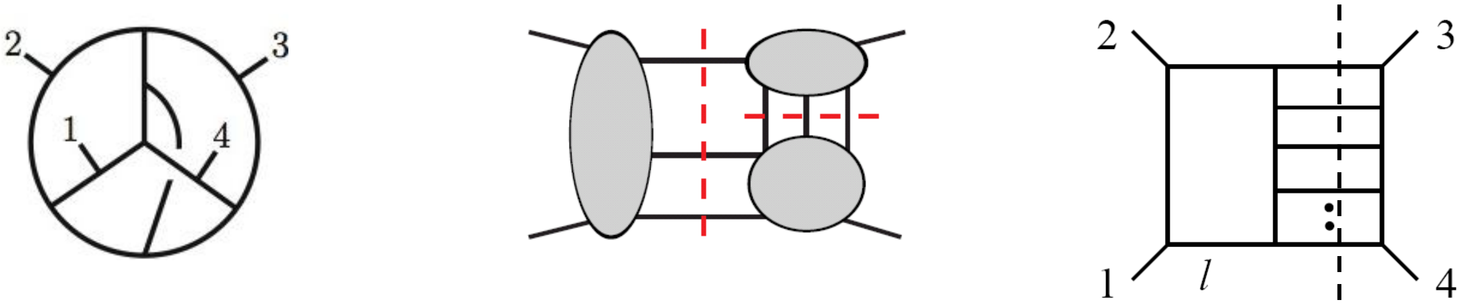
# Harmony of Scattering Amplitudes: From QCD to Gravity

Berlin, October 8, 2009

QCD: The Modern View of the Strong Interactions

Zvi Bern, UCLA

Will present results from papers with:
J.J. Carrasco, L. Dixon, D. Forde, H. Ita, H. Johansson,
D. Kosower, V. Smirnov, M. Spradlin, R. Roiban and A. Volovich.



# Outline

**This talk will present some recent results in gauge and gravity theories using on-shell methods and will expose a remarkable harmony amongst scattering amplitudes.**

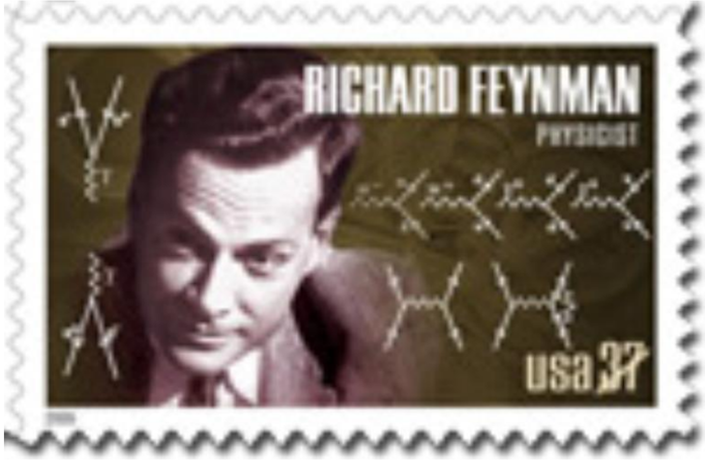
- **QCD: Applications to LHC physics**

  **talks from Badger, Dixon and Glover**

- **Supersymmetric gauge theory: resummation of certain planar *N* = 4 super-Yang-Mills scattering amplitudes to *all* loop orders.**

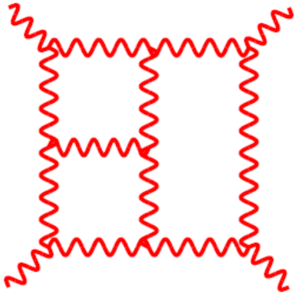
- **Quantum gravity: reexamination of standard wisdom on ultraviolet properties of quantum gravity. Four-loop demonstration of novel UV cancellations.**

# Why are Feynman diagrams so difficult for high-loop or high-multiplicity processes?



- Vertices and propagators involve gauge-dependent off-shell states. An important origin of the complexity.

$$\int \frac{d^4p}{(2\pi)^4}$$



$p^2 \neq m^2$

- To get at root cause of the trouble we must rewrite perturbative quantum field theory.

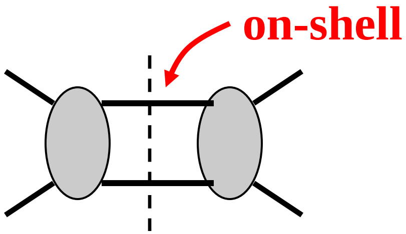
- **All steps should be in terms of gauge invariant on-shell states. On-shell formalism.** $p^2 = m^2$

ZB, Dixon, Dunbar, Kosower

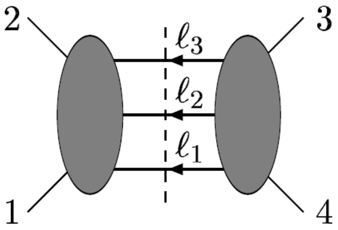
Bern, Dixon, Dunbar and Kosower

# Modern Unitarity Method

**Two-particle cut:**



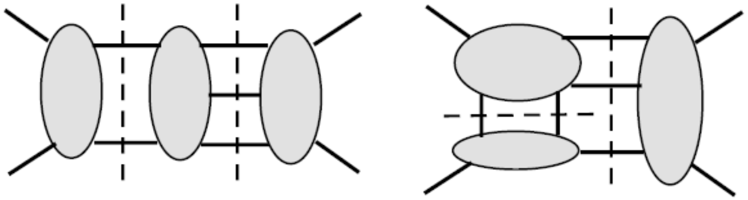
**Three-particle cut:**



**Systematic assembly of complete amplitudes from cuts for any number of particles or loops.**

**Generalized unitarity as a practical tool:**

Bern, Dixon and Kosower
Britto, Cachazo and Feng



**Different cuts merged to give an expression with correct cuts in all channels.**

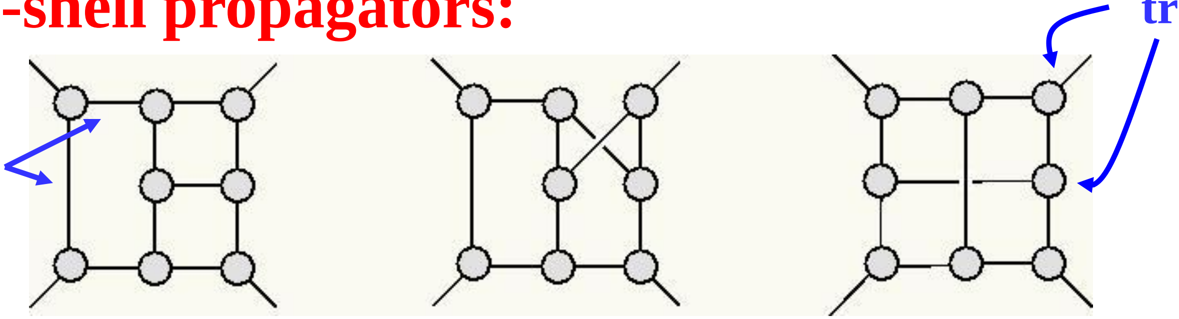
**Generalized cut interpreted as cut propagators not canceling.**

# Method of Maximal Cuts

ZB, Carrasco, Johansson, Kosower

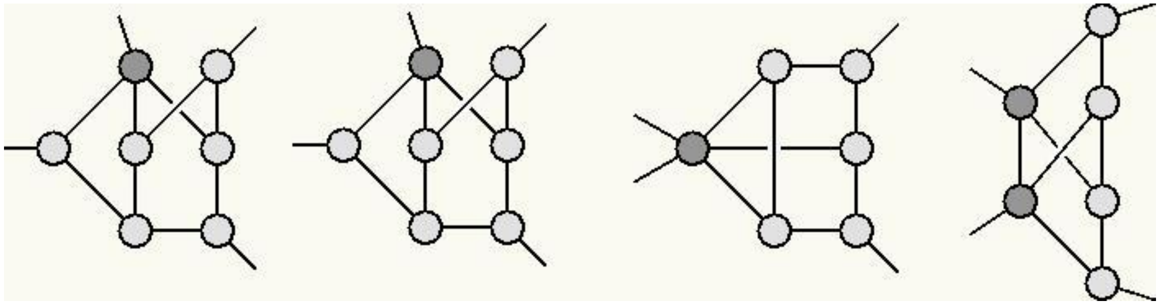
**A refinement of unitarity method for constructing complete higher-loop amplitudes is "Method of Maximal Cuts". Systematic construction in *any* massless theory.**

**To construct the amplitude we use cuts with maximum number of on-shell propagators:**



**Maximum number of propagator placed on-shell.**

**Then systematically release cut conditions to obtain contact terms:**

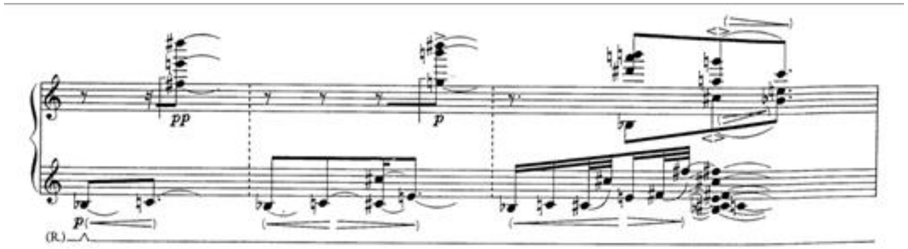


**Fewer propagators placed on-shell.**

**Related to subsequent leading singularity method for maximal susy**

Cachazo and Skinner; Cachazo; Cachazo, Spradlin, Volovich; Spradlin, Volovich, Wen

# Examples of Harmony



# Gravity vs Gauge Theory

**Consider the gravity Lagrangian**

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

flat metric

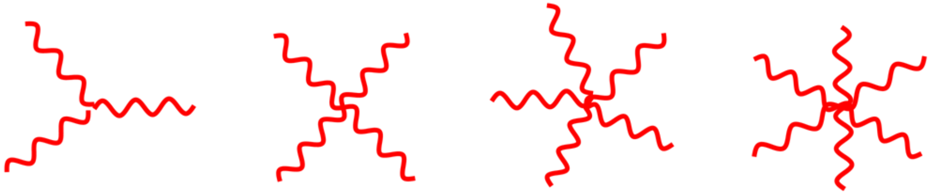
graviton field

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

metric

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

**Infinite number of complicated interactions**



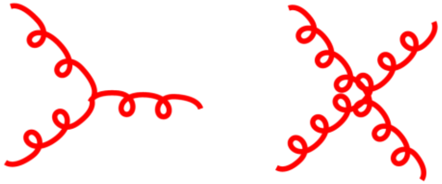
$+ \cdots$

**a mess**

Non-renormalizable

**Compare to Yang-Mills Lagrangian on which QCD is based**

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$

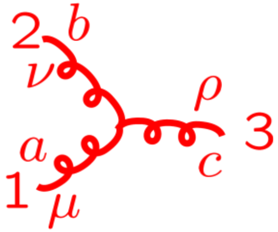


**Only three and four point interactions**

**Gravity seems so much more complicated than gauge theory.**

**Does not look harmonious!**

# Three Vertices



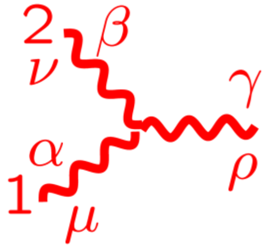
**Three-gluon vertex:**

$$V_{3\,\mu\nu\sigma}^{abc} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

**Three-graviton vertex:**

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1, k_2, k_3) = \\
&\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$



**About 100 terms in three vertex**

**Naïve conclusion: Gravity is a nasty mess.**

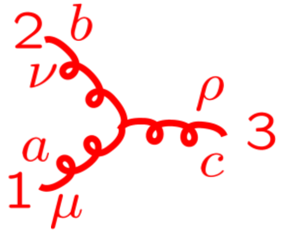
**Not very harmonious!**

# Simplicity of Gravity Amplitudes

*On-shell* three vertices contains all information:

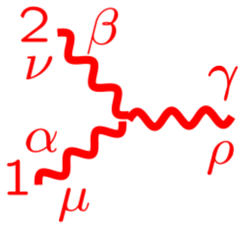
$k_i^2 = 0$

**gauge theory:**



$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

**gravity:**



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

**Any gravity scattering amplitude constructible solely from *on-shell* 3 vertex.**

- **BCFW on-shell recursion for tree amplitudes.**

Britto, Cachazo, Feng and Witten; Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

**See talks from Badger and Glover**

- **Unitarity method for loops.**

ZB, Dixon, Dunbar and Kosower; ZB, Dixon, Kosower; Britto, Cachazo, Feng; ZB, Morgan; Buchbinder and Cachazo; ZB, Carrasco, Johansson, Kosower; Cachzo and Skinner.

# Gravity vs Gauge Theory

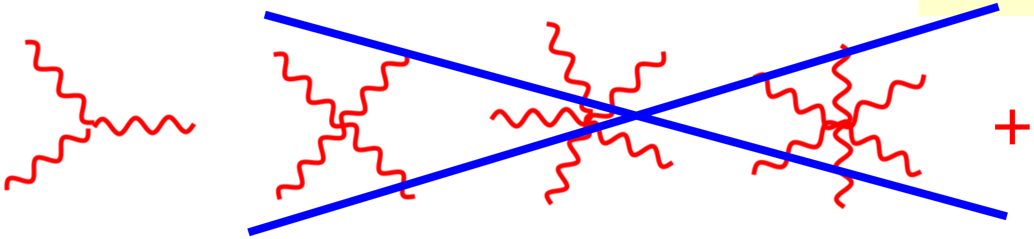
## Consider the gravity Lagrangian

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R$$

$$\underset{\text{metric}}{g_{\mu\nu}} = \overset{\text{flat metric}}{\eta_{\mu\nu}} + \kappa \overset{\text{graviton field}}{h_{\mu\nu}}$$

$$\kappa^2 = 32\pi\, G_{\text{Newton}}$$

**Infinite number of irrelevant interactions!**

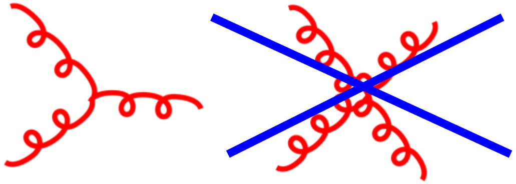


$+ \cdots$

**Simple relation to gauge theory**

## Compare to Yang-Mills Lagrangian

$$L_{\text{YM}} = \frac{1}{g^2}F^2$$



**Only three-point interactions**

**Gravity seems ~~so much~~ no more complicated than gauge theory.**

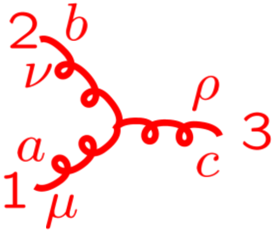
**Does ~~not~~ look harmonious!**

# Harmony of Color and Kinematics

ZB, Carrasco, Johansson

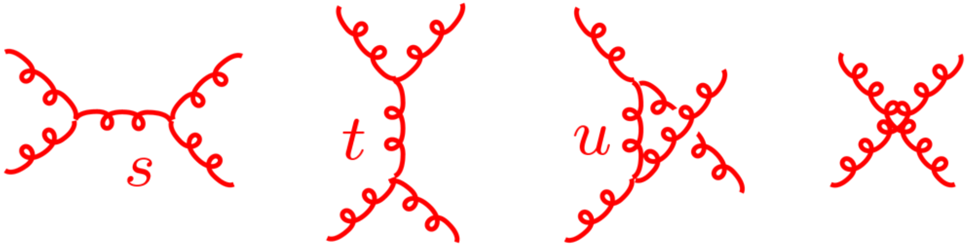
**coupling constant** **color factor** **momentum dependent kinematic factor**

$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



**Color factors based on a Lie algebra:** $[T^a, T^b] = if^{abc}T^c$

**Jacobi identity** $[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$



Use 1 = ***s/s* = *t/t* = *u/u***
to assign 4-point diagram
to others.

$$A_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right)$$

$$s = (k_1 + k_2)^2 \qquad u = (k_1 + k_3)^2$$
$$t = (k_1 + k_4)^2$$

**Color factors satisfy Jacobi identity:** $c_u = c_s - c_t$

**Numerator factors satisfy similar identity:** $n_u = n_s - n_t$

**Color and kinematics are singing same tune!**

# Harmony of Color and Kinematics

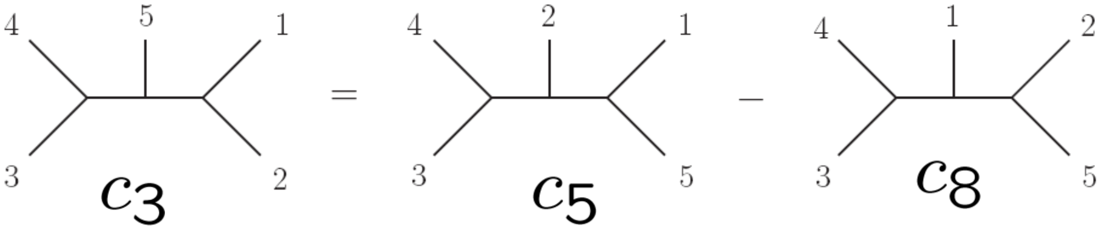
**At higher points similar structure:**

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i\, n_i}{D_i}$$

**color factor**

**kinematic numerator factor**

**Feynman propagators**



$$c_3 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2}\,, \qquad c_5 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5}\,, \qquad c_8 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$\boxed{c_3 - c_5 + c_8 = 0 \quad \Leftrightarrow \quad n_3 - n_5 + n_8 = 0}$$

**Claim: We can always find a rearrangement so color and kinematics satisfy the *same* Jacobi constraint equations.**

- **Color and kinematics sing same tune!**
- **Nontrivial constraints on amplitudes.**

Recent string theory understanding. Bjerrum-Bohr, Damgaard, Vanhove; Stieberger; Mofra

# Higher-Point Gravity and Gauge Theory

ZB, Carrasco, Johansson

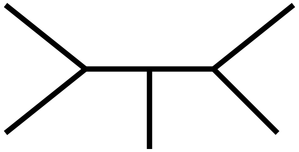
**QCD:**

$$\mathcal{A}_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i \, n_i}{D_i}$$

**sum over diagrams with only 3 vertices**

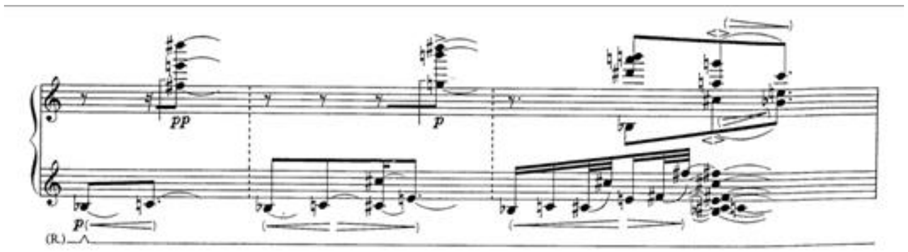
**Einstein Gravity:**

$$\mathcal{M}_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i}$$



**This relation extremely useful in high-loop gravity calculations.**

**Gravity and QCD kinematic numerators sing same tune!**



**Cries out for a unified description of the sort given by string theory.**

# Applications to AdS/CFT

# *N* = 4 Super-Yang-Mills to All Loops

**Since ‘t Hooft’s paper thirty years ago on the planar limit of QCD we have dreamed of solving QCD in this limit. This is too hard. *N* = 4 sYM is much more promising.**

- **Special theory because of AdS/CFT correspondence.**

**talks from Haack and Staudacher**

- **Maximally supersymmetric – 1 gluon, 4 gluinos, 6 scalars.**
- **Simplicity both at strong and weak coupling.**

**Remarkable relation**

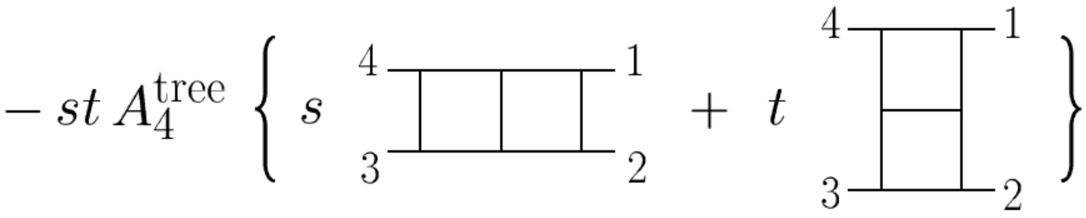
Alday and Maldacena

**scattering at strong coupling in *N* = 4 sYM $\longleftrightarrow$ classical string theory in AdS space**

**To make this link need to evaluate *N* = 4 super-Yang-Mills amplitudes to *all* loop orders. Seems impossible even with modern methods.**

# Loop Iteration of the $N = 4$ Amplitude

**The planar four-point two-loop amplitude undergoes fantastic simplification. Computed via unitarity method.**



**ZB, Rozowsky, Yan**

$$M_4^{\text{2-loop}}(s,t) = \frac{1}{2}\Big(M_4^{\text{1-loop}}(s,t)\Big)^2 + f(\epsilon)\,M_4^{\text{1-loop}}(s,t)\big|_{\epsilon\to 2\epsilon} - \frac{1}{2}\zeta_2^2$$

$$M_4^{\rm loop} = A_4^{\rm loop}/A_4^{\rm tree} \qquad f(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2$$

**Anastasiou, ZB, Dixon, Kosower**

$f(\epsilon)$ is universal function related to IR singularities $\qquad D = 4 - 2\epsilon$

**This gives two-loop four-point planar amplitude as iteration of one-loop amplitude.**

**Three loop satisfies similar iteration relation. Rather nontrivial.**

**ZB, Dixon, Smirnov**

# All-Loop Generalization

**Why not be bold and guess scattering amplitudes for all loop and all legs, at least for simple helicity configurations?**

$$\mathcal{A}_n = A_n^{\text{tree}} A_n^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_n^{\text{1-loop}} + C\right]$$

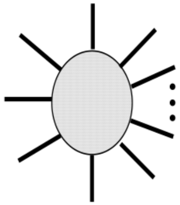
**constant independent of kinematics.**

**all-loop resummed amplitude**

**IR divergences**

**cusp anomalous dimension**

**finite part of one-loop amplitude**



**"BDS conjecture"**

Anastasiou, ZB, Dixon, Kosower
ZB, Dixon and Smirnov

- **IR singularities agree with Magnea and Sterman formula.**
- **Limit of collinear momenta gives us key analytic information, at least for MHV amplitudes. Warning: This argument has a loophole. Multiloop collinear limits rather subtle.**

**Gives a definite prediction for *all* values of coupling given BES integral equation for the cusp anomalous dimension.**

Beisert, Eden, Staudacher

**See Staudacher's talk**

# Alday and Maldacena Strong Coupling

**For MHV amplitudes:** $F_4^{\text{1-loop}} = \frac{1}{2}\ln^2(s/t) + \frac{2\pi^2}{3}$ ZB, Dixon, Smirnov

$$\mathcal{A}_4 = A_4^{\text{tree}} A_4^{\text{divergent}} \exp\left[\frac{1}{4}\gamma_K F_4^{\text{1-loop}} + C\right]$$

**constant independent of kinematics.**

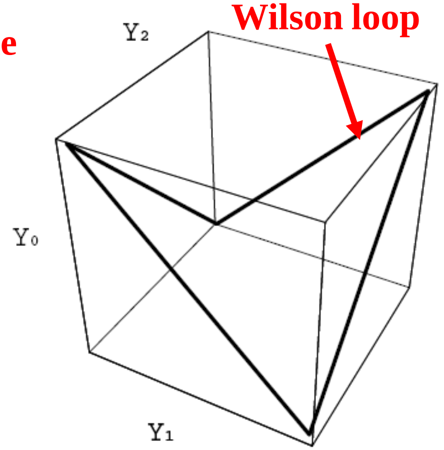
**all-loop resummed amplitude**

**IR divergences**

**cusp anomalous dimension**

**finite part of one-loop amplitude**

**In a beautiful paper Alday and Maldacena confirmed the conjecture for 4 gluons at *strong coupling* from an AdS string theory computation. Minimal surface calculation.**



**Very suggestive link to Wilson loops even at weak coupling.**

Drummond, Korchemsky, Sokatchev ; Brandhuber, Heslop, and Travaglini
ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich

- **Identication of new symmetry: "dual conformal symmetry"**
- **Link to integrability**
- **Yangian structure!**

Drummond, Henn, Korchemsky, Sokatchev ;Berkovits and Maldacena;
Beisert, Ricci, Tseytlin, Wolf Brandhuber, Heslop, Travaglini;
Drummond, Henn, Plefka; Bargheer,
Beisert, Galleas, Loebbert, McLoughlin.

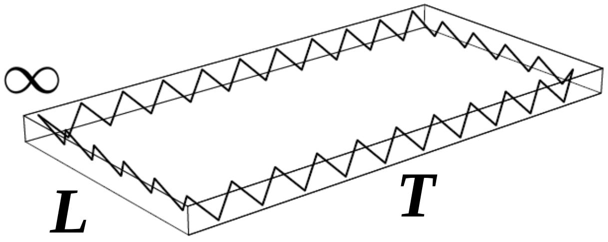
# Trouble at Higher Points

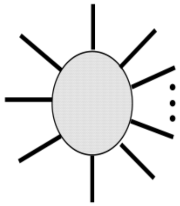
**For various technical reasons it is hard to solve for minimal surface for large numbers of gluons.**

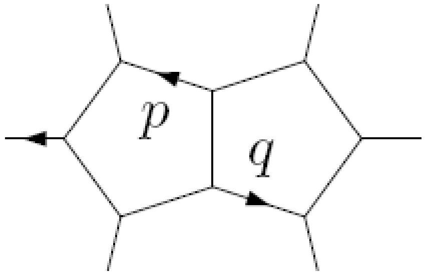
**Alday and Maldacena realized certain terms can be calculated at strong coupling for an infinite number of gluons.**

**Disagrees with BDS conjecture**

$T/L \to \infty$







**May also be trouble also in the Regge limit.**

Bartels, Lipatov, Sabio Vera; Del Duca, Duhr and Glover; Brower, Nastase , Schnitzer, Tan

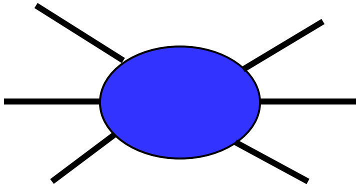
**Explicit computation at 2-loop 6 points.**

**Need to modify conjecture!**

ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich
Drummond, Henn, Korchemsky, Sokatchev

**Can the BDS conjecture be repaired for six and higher points?**

# In Search of the Holy Grail



log of the amplitude

discrepancy

$$A^{\mathrm{truth}} = A^{\mathrm{div}} + A^{\mathrm{BDS}} + R$$

**Can we figure out the discrepancy?**

**Important new information from regular polygons should serve as a guide.**

**Explicit solution at eight points**

$$k_i = x_{i+1} - x_i$$

$$A_{BDS} = -\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1, j\neq i,i-1}^{n} \log\frac{x_j^+ - x_i^+}{x_{j+1}^+ - x_i^+}\log\frac{x_j^- - x_{i-1}^-}{x_j^- - x_i^-}$$

Alday and Maldacena (2009)

$$A = A_{div} + A_{BDS} + R$$

$$R = -\frac{1}{2}\log(1+\chi^-)\log(1+\frac{1}{\chi^+}) + \frac{7\pi}{6} + \int_{-\infty}^{\infty} dt \frac{|m|\sinh t}{\tanh(2t+2i\phi)}\log\left(1+e^{-2\pi|m|\cosh t}\right)$$

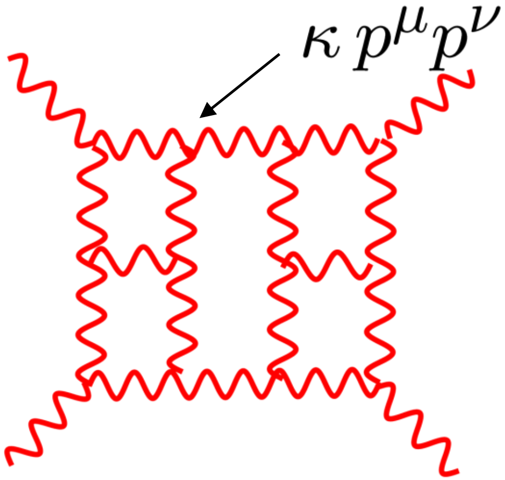
**Solution only valid for strong coupling and special kinematics**

Together with the conformal and dual conformal symmetry, this should help guide us.

# UV Properties of $\mathcal{N} = 8$ Supergravity

# Is a UV finite theory of gravity possible?

$\kappa = \sqrt{32\pi G_N}$ ← Dimensionful coupling



**Gravity:** $$\int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa\, p_j^\mu p_j^\nu)\cdots}{\text{propagators}}$$

**Gauge theory:** $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g\, p_j^\nu)\cdots}{\text{propagators}}$$

**Extra powers of loop momenta in numerator means integrals are badly behaved in the UV**

**Much more sophisticated power counting in supersymmetric theories but this is the basic idea.**

# *N* = 8 Supergravity

Consider the *N* = 8 supergravity theory of Cremmer and Julia.

256 massless states

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $N = 8$ : | 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 |
| helicity : | $-2$ | $-\frac{3}{2}$ | $-1$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 |
| | $h^-$ | $\psi_i^-$ | $v_{ij}^-$ | $\chi_{ijk}^-$ | $s_{ijkl}$ | $\chi_{ijk}^+$ | $v_{ij}^+$ | $\psi_i^+$ | $h^+$ |

**Reasons to focus on *N* = 8 maximal supergravity:**

- **With more susy suspect better UV properties.**
- **High symmetry implies simplicity. Much simpler than expected.**

# Opinions from the 80's

Unfortunately, in the absence of further mechanisms for cancellation, the analogous $N = 8$ $D = 4$ supergravity theory would seem set to diverge at the three-loop order.

Howe, Stelle (1984)

It is therefore very likely that all supergravity theories will diverge at three loops in four dimensions. … The final word on these issues may have to await further explicit calculations.

Marcus, Sagnotti (1985)

**The idea that *all* supergravity theories diverge has been widely accepted for over 25 years**

# Reasons to Reexamine UV Behavior

1) Discovery of remarkable cancellations at 1 loop – the "no-triangle property". Key evidence for all-loop cancellations!

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr, Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Bjerrum-Bohr, Vanhove; Arkani-Hamed, Cachazo, Kaplan; ZB, Dixon, Roiban

2) *Every* explicit loop calculation to date finds $N = 8$ supergravity has identical power counting as $N = 4$ super-Yang-Mills theory, which is UV finite. Green, Schwarz and Brink; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; ZB, Carrasco, Dixon, Johansson, Kosower, Roiban.

3) Interesting hint from string dualities. Chalmers; Green, Vanhove, Russo

– Dualities restrict form of effective action. May prevent divergences from appearing in $D = 4$ supergravity, athough indirect and nontrivial issues with decoupling of towers of massive states.

4) Interesting string non-renormalization theorem from Berkovits. Suggests divergence delayed to 9 loops. Would like to redo arguments in field theory not string theory. Green, Vanhove, Russo

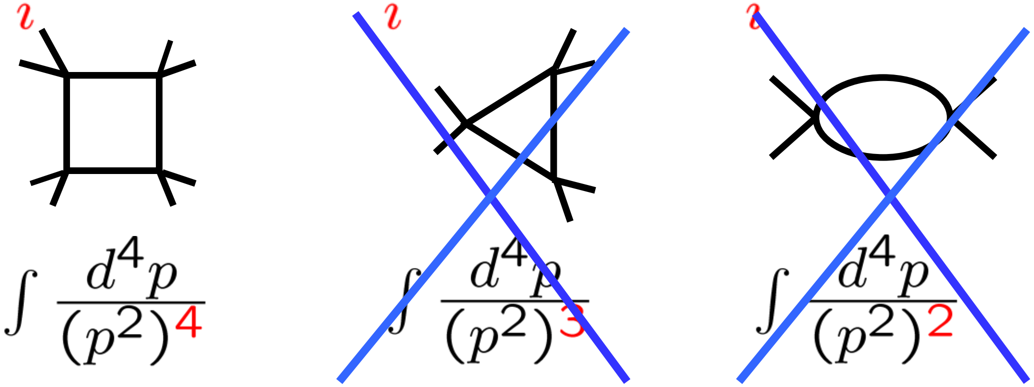
# *N* = 8 Supergravity No-Triangle Property

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr and Dunbar; Bjerrum-Bohr, Dunbar, Ita, Perkins, Risager; Proofs by Bjerrum-Bohr and Vanhove; Arkani-Hamed, Cachazo and Kaplan.

One-loop *D* = 4 theorem: Any one loop amplitude is a linear combination of scalar box, triangle and bubble integrals with rational coefficients:

Brown, Feynman; Passarino and Veltman, etc

$$A_n^{\text{1-loop}} = \sum_i d_i I_4^{(i)} + \sum_i c_i I_3^{(i)} + \sum_i b_i I_2^{(i)}$$



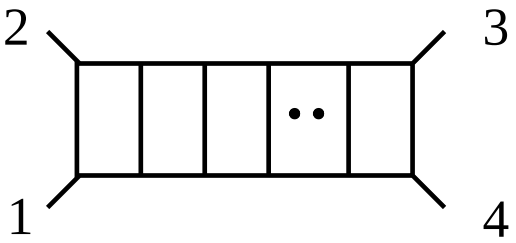
$$\int \frac{d^4p}{(p^2)^4} \qquad \cancel{\int \frac{d^4p}{(p^2)^3}} \qquad \cancel{\int \frac{d^4p}{(p^2)^2}}$$

- In *N* = 4 Yang-Mills *only box* integrals appear. No triangle integrals and no bubble integrals.
- The “no-triangle property” is the statement that same holds in *N* = 8 supergravity. Non-trivial constraint on analytic form of amplitudes.
- Unordered nature of gravity is important for this property

Bjerrum-Bohr and Vanhove

# *N* = 8 *L*-Loop UV Cancellations

ZB, Dixon, Roiban (2006)

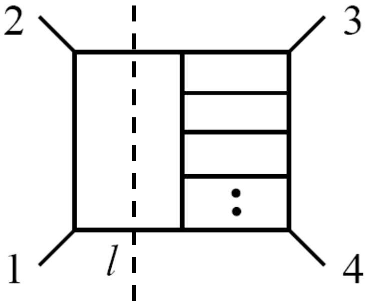


$$[(k_1 + k_2)^2]^{2(L-2)}$$

numerator factor

$$D < \frac{6}{L} + 4$$

From 2 particle cut:

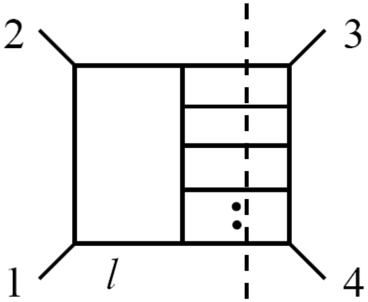


1 in *N* = 4 YM

$$[(l + k_4)^2]^{2(L-2)}$$

numerator factor

*L*-particle cut



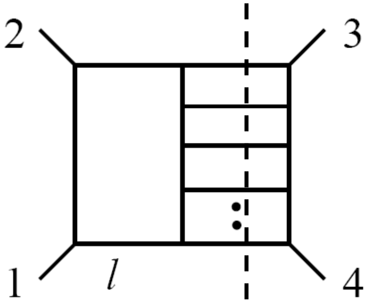
- Numerator violates one-loop “no-triangle” property.
- Too many powers of loop momentum in one-loop subamplitude.
- After cancellations behavior is same as in *N* = 4 Yang-Mills!

- UV cancellation exist to *all* loop orders! (not a proof of finiteness)
- Existence of these cancellations drive our calculations!
- These all-loop cancellations *not* explained by any known supersymmetry argument.

# Origin of Cancellations?

There does not appear to be a supersymmetry explanation for *observed* all-loop cancellations.

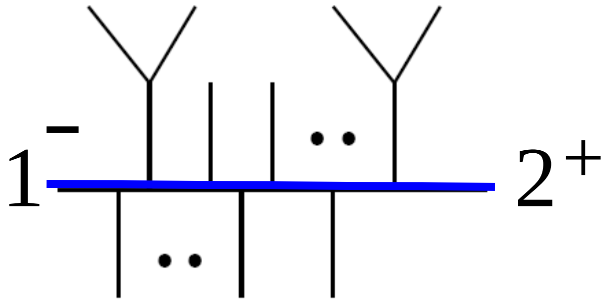
If it is *not* supersymmetrywhat might it be?



# Origin of Cancellations?

First consider tree level

ZB, Carrasco, Forde, Ita, Johansson



$$k_1^\mu \to k_1^\mu + zq \qquad k_2^\mu \to k_2^\mu - zq$$

$$q^2 = 0 \qquad k_1 \cdot q = k_2 \cdot q = 0$$

*m* propagators and *m*+1 vertices between legs 1 and 2

Yang-Mills scaling: $z^{m+1} \times \frac{1}{z^m} \times \frac{1}{z^2} \sim \frac{1}{z}$ well behaved

vertices propagators polarizations

$z \longrightarrow \infty$

gravity scaling: $z^{2(m+1)} \times \frac{1}{z^m} \times \frac{1}{z^4} \sim z^{m-2}$ poorly behaved

Summing over all Feynman diagrams, correct gravity scaling is:

$$M_n^{\text{tree}}(z) \sim \frac{1}{z^2}$$

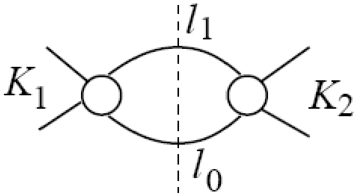
Remarkable tree-level cancellations.
Better than gauge theory!

$z^{n-5}$ cancels to $\frac{1}{z^2}$

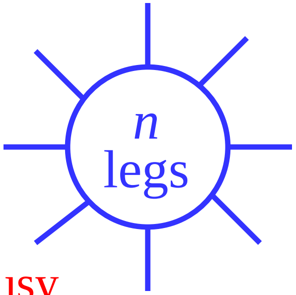
Bedford, Brandhuber, Spence, Travaglini;
Cachazo and Svrcek;
Benincasa, Boucher-Veronneau, Cachazo
Arkani-Hamed, Kaplan; Hall

# Loop Cancellations in Pure Gravity

Powerful new one-loop integration method due to Forde.
Allows us to link one-loop cancellations to tree-level cancellations.



Observation: Most of the one-loop cancellations observed in $N$ = 8 supergravity leading to “no-triangle property” are already present in non-susy gravity.



$$(l^\mu)^{2n} \rightarrow (l^\mu)^{n+4} \times (l^\mu)^{-8}$$

maximum powers of loop momenta

Cancellation generic to Einstein gravity

Cancellation from $N$ = 8 susy

ZB, Carrasco, Forde, Ita, Johansson

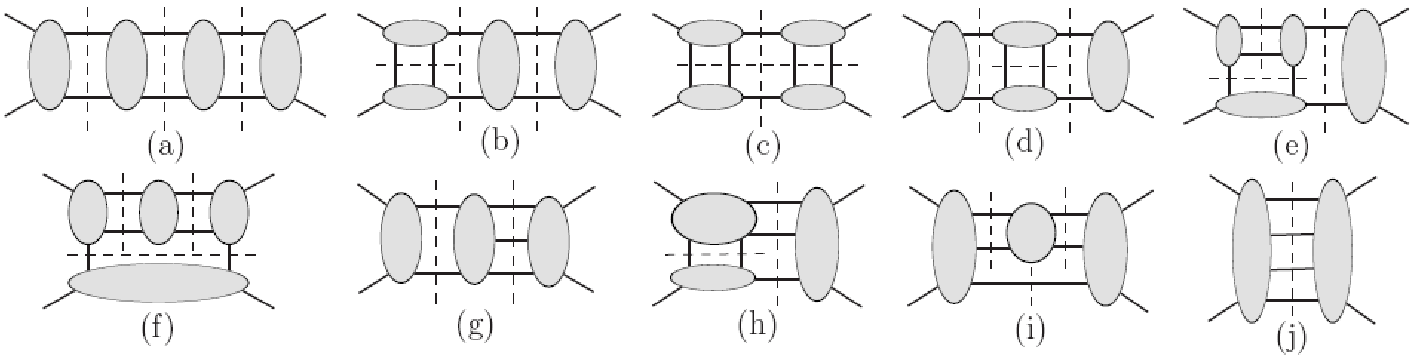
Proposal: This continues to higher loops, so that most of the observed $N$ = 8 multi-loop cancellations are *not* due to susy but in fact are generic to gravity theories!

All-loop finiteness of $N$ = 8 supergravity would follow from a combination of susy cancellations on top of novel but generic cancellations present even in pure Einstein gravity.

# Full Three-Loop Calculation

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban

Need following cuts:



For cut (g) have:

reduces everything to product of tree amplitudes

$$\sum_{N=8\ \text{states}} M_4^{\text{tree}}(1,2,l_3,l_1)\times M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\times M_5^{\text{tree}}(3,4,-q_1,-q_2,-q_3)$$

Use Kawai-Lewellen-Tye tree relations

ZB, Dixon, Dunbar, Perelstein and Rozowsky.

$$M_4^{\text{tree}}(1,2,l_3,l_1) = -is_{12}A_4^{\text{tree}}(1,2,l_3,l_1)A_4^{\text{tree}}(2,1,l_3,l_1)$$

$$M_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1) = i\,s_{l_1q_1}s_{l_3q_3}A_5^{\text{tree}}(-l_1,-l_3,q_3,q_2,q_1)\,A_5^{\text{tree}}(-l_1,q_1,q_3,-l_3,q_2) + \{l_1\leftrightarrow l_3\}\,,$$

supergravity

super-Yang-Mills

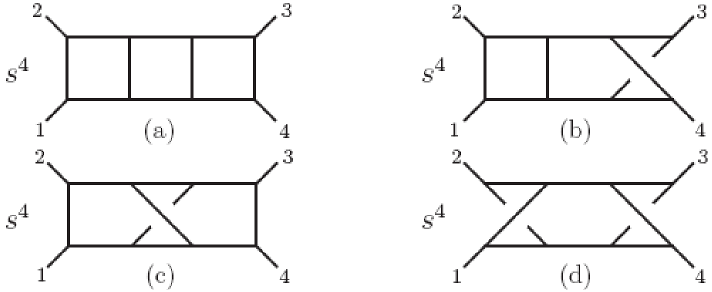
**$N$ = 8 supergravity cuts are sums of products of $N$ = 4 super-Yang-Mills cuts. Harmony!**

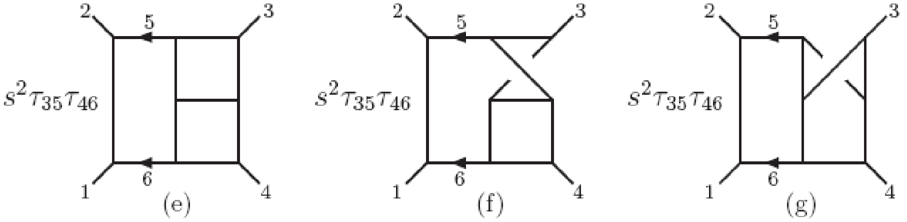
# Complete Three-Loop $N$ = 8 Supergravity Result

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112
ZB, Carrasco, Dixon, Johansson, Roiban arXiv:0808.4112 [hep-th]

$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 stu M_4^{\rm tree} \sum_{S_3} \left[ I^{(a)} + I^{(b)} + \frac{1}{2} I^{(c)} + \frac{1}{4} I^{(d)} + 2 I^{(e)} + 2 I^{(f)} + 4 I^{(g)} + \frac{1}{2} I^{(h)} + 2 I^{(i)} \right]$$

$$\tau_{ij} = 2 k_i \cdot k_j$$

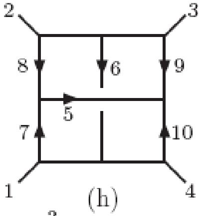




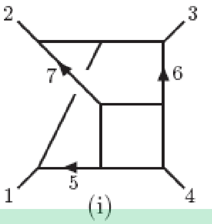
Cancellations beyond those needed for finiteness in $D$ = 4.

Finite for $D$ < 6

$$(s(\tau_{26}+\tau_{36}) + t(\tau_{15}+\tau_{25}) + st)^2$$
$$+(s^2(\tau_{26}+\tau_{36}) - t^2(\tau_{15}+\tau_{25}))(\tau_{17}+\tau_{28}+\tau_{39}+\tau_{4,10})$$
$$+s^2(\tau_{17}\tau_{28}+\tau_{39}\tau_{4,10}) + t^2(\tau_{28}\tau_{39}+\tau_{17}\tau_{4,10})$$
$$+u^2(\tau_{17}\tau_{39}+\tau_{28}\tau_{4,10})$$



$$(s\tau_{45} - t\tau_{46})^2 - \tau_{27}(s^2\tau_{45} + t^2\tau_{46})$$
$$-\tau_{15}(s^2\tau_{47} + u^2\tau_{46}) - \tau_{36}(t^2\tau_{47} + u^2\tau_{45})$$
$$+l_5^2 s^2 t + l_6^2 s t^2 - \tfrac{1}{3} l_7^2 stu$$



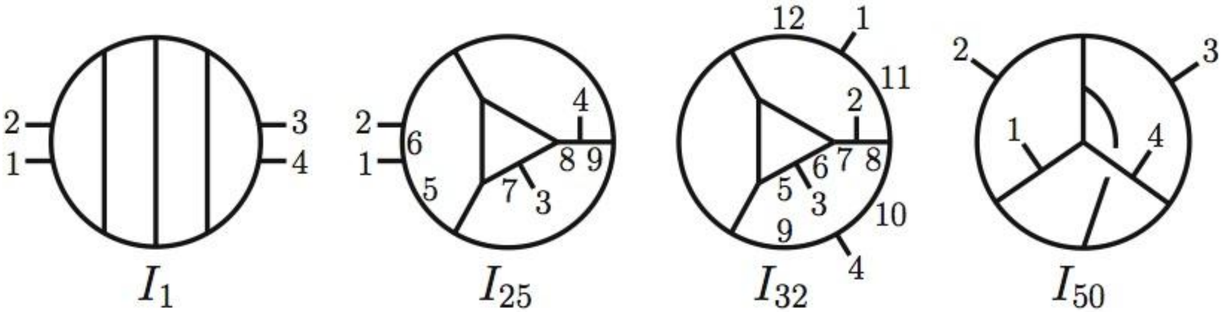
$$\text{UVpole}_{D=6-2\epsilon} = \frac{1}{\epsilon} \frac{5\zeta_3}{(4\pi)^9} \left(\frac{\kappa}{2}\right)^8 (stu)^2 M_4^{\rm tree}$$

**Identical power count as $N$ = 4 super-Yang-Mills**

# Four-Loop Amplitude Construction

ZB, Carrasco, Dixon, Johansson, Roiban

**Get 50 distinct diagrams or integrals (ones with two- or three-point subdiagrams not needed).**



**arXiv submission has mathematica files with all 50 diagrams**

$$M_4^{\text{4-loop}} = \left(\frac{\kappa}{2}\right)^{10} stu M_4^{\text{tree}} \sum_{S_4} \sum_{i=1}^{50} c_i I_i$$

**Integral**

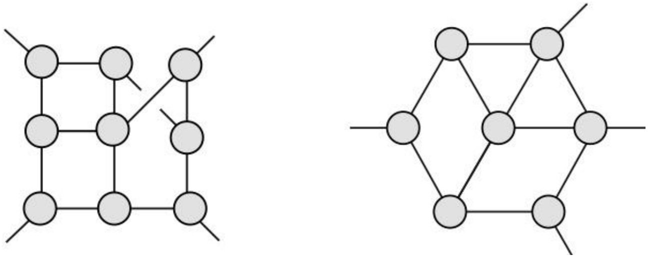
**leg perms**

**symmetry factor**

# Four-Loop Construction

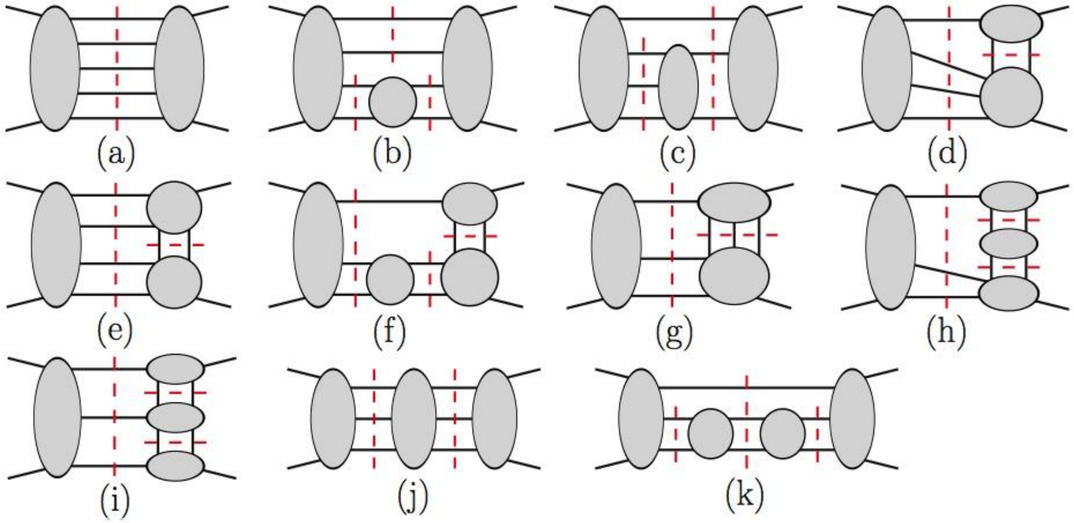
numerator

$$I_i = \int d^D l_1 d^D l_2 d^D l_3 d^D l_4 \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$



**Determine numerators from 2906 maximal and near maximal cuts**

**Completeness of expression confirmed using 26 generalized cuts sufficient for obtaining the complete expression**



**11 most complicated cuts shown**

# UV Finiteness at Four Loops

$$I_i = \int d^Dl_1 d^Dl_2 d^Dl_3 d^Dl_4 \, \frac{N_i(l_j, k_j)}{l_1^2 l_2^2 l_3^2 l_4^2 l_5^2 l_6^2 l_7^2 l_8^2 l_9^2 l_{10}^2 l_{11}^2 l_{12}^2 l_{13}^2}$$

$$N_i \sim O(k^4 l^8)$$

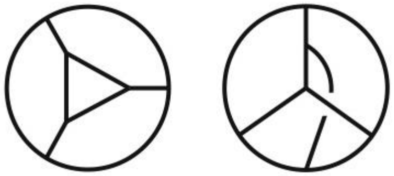
$k_i$: external momenta

$l_i$ : loop momenta

**The $N_i$ are rather complicated objects, but it is straightforward to analyze UV divergences.**

**Manfestly finite for $D = 4$, but no surprise here.**

**Leading terms can be represented by two vacuum diagrams which cancel in the sum over all contributions.**



**coefficients vanish**

$$O(k^4 l^8)$$

- **If no further cancellation corresponds to $D = 5$ divergence.**

# UV Finiteness in *D* = 5 at Four Loops

ZB, Carrasco, Dixon, Johansson, Roiban

$N \sim O(k^6 l^6)$ **corresponds to *D* = 5 divergence.**

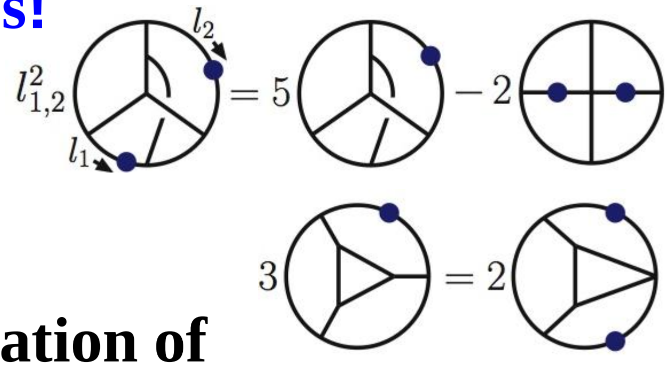
Expand numerator and propagators in small *k*

$$\frac{1}{(l_j + K_n)^2}$$

$$N^{(6)} + N^{(7)} \frac{K_i \cdot l_j}{l_j^2} + N^{(8)} \left( \frac{K_i^2}{l_j^2} + \frac{K_i \cdot l_j \; K_m \cdot l_n}{l_j^2 l_n^2} \right)$$

**Marcus & Sagnotti**
**UV extraction method**

**Cancels after using *D* = 5 integral identities!**



**UV finite for *D* = 4 and 5**

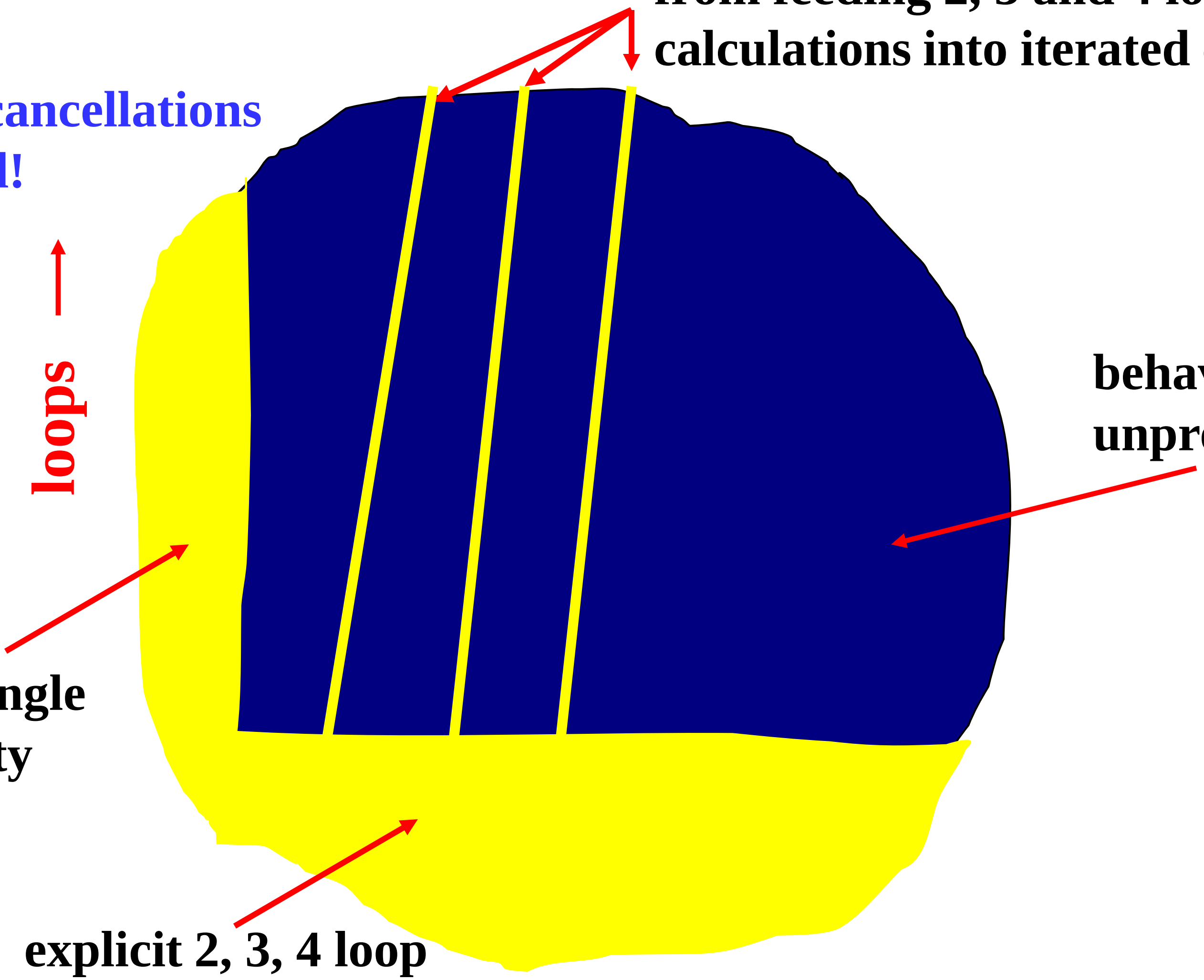
**actually finite for *D* < 5.5**

1. **Shows potential supersymmetry explanation of three loop result by Bossard, Howe, Stelle does *not* work!**
2. **The cancellations are stronger at 4 loops than at 3 loops, which is in turn stronger than at 2 loops. Rather surprising from traditional susy viewpoint.**

# Schematic Illustration of Status

🟨 **Same power count as *N*=4 super-Yang-Mills**

🟦 **UV behavior unknown**



**All-loop UV cancellations demonstrated!**

**from feeding 2, 3 and 4 loop calculations into iterated cuts.**

**behavior unproven**

**No triangle property**

**explicit 2, 3, 4 loop computations**

**loops →**

**# terms →**

# Summary

**Scattering amplitudes have a surprising simplicity and harmony.**

- ***N*** **= 4 supersymmetric gauge theory:**
  - **Scattering amplitudes open an exciting new venue for studying Maldacena's AdS/CFT conjecture.**
  - **Examples valid to *all* loop orders, matching strong coupling!**
  - **Can we repair BDS conjecture at 6 points and beyond?**
  - **New symmetries and explicit results.**
- **Quantum gravity:  Surprisingly simple structures emerge.**
  - **Gravity as the "square" of gauge theory.**
  - **Is a point-like perturbatively UV finite quantum gravity theory possible?  Multiloop arguments and explicit four-loop evidence.**

**On-shell methods have led to a variety of nontrivial new results in a broad range of topics.  We can expect many more such results in the years to come.**

# Extra

# Comments on Consequences of Finiteness

- Suppose $N = 8$ SUGRA is finite to all loop orders. Would this prove that it is a nonperturbatively consistent theory of quantum gravity? Of course not!
- At least two reasons to think it needs a nonperturbative completion:

  — Likely $L!$ or worse growth of the order $L$ coefficients,

  $$\sim L! \ (s/M_{\mathrm{Pl}}^2)^L$$

  — Different $E_{7(7)}$ behavior of the perturbative series (invariant!), compared with the $E_{7(7)}$ behavior of the mass spectrum of black holes (non-invariant!)
- Note QED is renormalizable, but its perturbation series has zero radius of convergence in $\alpha$: $\sim L! \ \alpha^L$ . But it has many point-like nonperturbative UV completions —asymptotically free GUTS.

# Where is First *D*= 4 UV Divergence in *N* = 8 Sugra?

Various opinions over the years:

| | | |
|---|---|---|
| **3 loops** | Conventional superspace power counting | Green, Schwarz, Brink (1982)<br>Howe and Stelle (1989)<br>Marcus and Sagnotti (1985) |
| **5 loops** | Partial analysis of unitarity cuts; *If* $\mathcal{N}$ = 6 harmonic superspace exists; algebraic renormalisation argument | Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998)<br>Howe and Stelle (2003,2009) |
| **6 loops** | *If* $\mathcal{N}$ = 7 harmonic superspace exists | Howe and Stelle (2003) |
| **7 loops** | *If* $\mathcal{N}$ = 8 harmonic superspace exists;<br>lightcone gauge locality arguments;<br>Algebraic renormalization arguments | Grisaru and Siegel (1982);<br>Kallosh (2009)<br>Bossard,Howe, Stelle (yesterday) |
| **8 loops** | Explicit identification of potential susy invariant counterterm with full non-linear susy | Kallosh; Howe and Lindström (1981) |
| **9 loops** | *Assume* Berkovits' superstring non-renormalization theorems can be carried over to *D*=4 $\mathcal{N}$ = 8 supergravity and extrapolate to 9 loops. Speculations on *D*=11 gauge invariance. 7-loop calculation needed. | Green, Russo, Vanhove (2006)<br>Stelle (2006)<br>Berkovits, Green, Russo, Vanhove (2009) |

No divergence demonstrated above. Arguments based on lack of susy protection!

To end debate, we need solid results!

# Divergences in Gravity

**Vanish on shell**

**One loop:** $R^2, \; R^2_{\mu\nu}, \; R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ **vanishes by Gauss-Bonnet theorem**

Pure gravity 1-loop finite (but not with matter)

‘t Hooft, Veltman (1974)

**Two loop:** Pure gravity counterterm has non-zero coefficient:

$$R^3 \equiv R^{\lambda\rho}{}_{\mu\nu} R^{\mu\nu}{}_{\sigma\tau} R^{\sigma\tau}{}_{\lambda\rho}$$

Goroff, Sagnotti (1986); van de Ven (1992)

**Any supergravity:**

$R^3$ is ***not*** a valid supersymmetric counterterm.

Produces a helicity amplitude $(-,+,+,+)$ forbidden by susy.

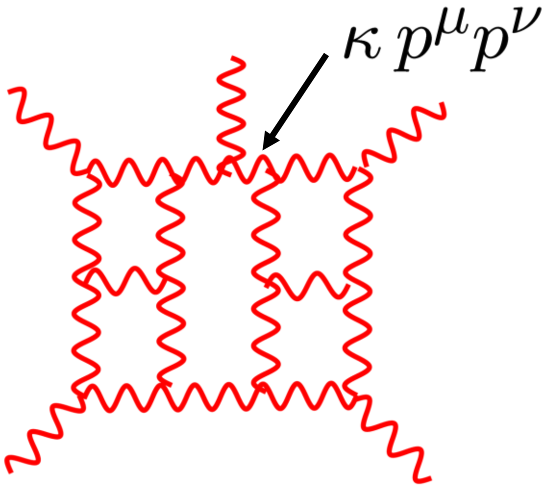
Grisaru (1977); Tomboulis (1977)

**The first divergence in *any* supergravity theory can be no earlier than three loops.**

$R^4$ **squared Bel-Robinson tensor expected counterterm**

Deser, Kay, Stelle (1977); Kaku, Townsend, van Nieuwenhuizen (1977), Ferrara, Zumino (1978)

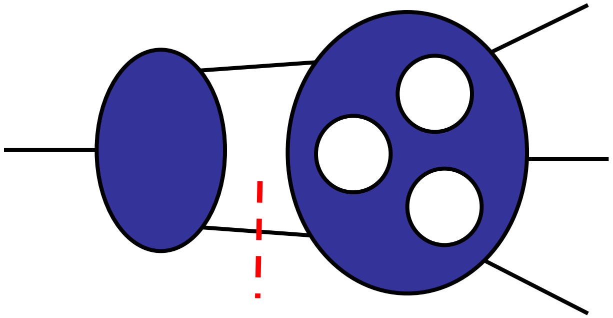
# Higher Point Divergences?



**Add an extra leg:**

1. extra $\kappa\, p^\mu p^\nu$ in vertex
2. extra $1/p^2$ from propagator

**Adding legs generically does not worsen power count.**



**Cutting propagators exposes lower loop higher-point amplitudes.**

- Higher-point divergences should be visible in high-loop four-point amplitudes.
- A proof of UV finiteness would need to systematically rule out higher-point divegences.

# KLT Relations Between Gravity and Gauge Theory

At *tree level* Kawai, Lewellen and Tye derived a relationship between closed and open string amplitudes.
In field theory limit, relationship is between gravity and gauge theory.

$$M_4^{\text{tree}}(1,2,3,4) = s_{12} A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)\,,$$

$$\begin{aligned} M_5^{\text{tree}}(1,2,3,4,5) &= s_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)A_5^{\text{tree}}(2,1,4,3,5) \\ &\quad + s_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)\,A_5^{\text{tree}}(3,1,4,2,5) \end{aligned}$$

**Gravity amplitude**

**Color stripped gauge theory amplitude**

where we have stripped all coupling constants

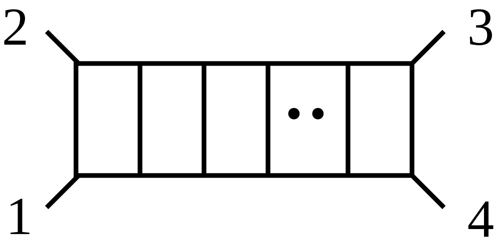
$$\mathcal{A}_4^{\text{tree}} = g^2 \sum\nolimits_{\text{non-cyclic}} \text{Tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\, A_4^{\text{tree}}(1,2,3,4)$$

**Full gauge theory amplitude**

**Holds for any external states.**
**See review: gr-qc/0206071**

**Progress in gauge theory can be imported into gravity theories**

# *L*-Loops *N* = 4 Super-Yang-Mills Warmup

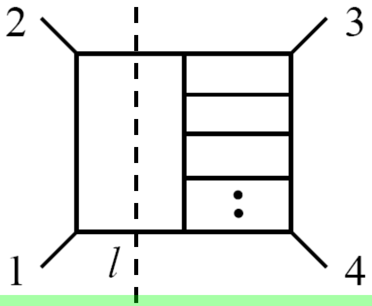


$$[(k_1 + k_2)^2]^{(L-2)}$$

numerator factor

ZB, Dixon, Dunbar, Perelstein, Rozowsky (1998)

From 2 particle cut:



$$[(l + k_4)^2]^{(L-2)}$$

numerator factor

Power counting this gives UV finiteness for :

$$D < \frac{6}{L} + 4$$

bound saturated for $L \leq 4$

**Power count of UV behavior follows from supersymmetry alone.**

A bit better than more conventional superspace power counts of *N* = 4 sYM

- Confirmed by explicit calculation through *L* = 5.
- Confirmed by Howe and Stelle using *N* = 4 harmonic superspace.
- Through *L* = 6 agrees with Berkovits, Green, Russo and Vanhove, who use low-energy limit for open strings with Berkovits' pure spinor formalism.
- Though *L* = 4, *all* cancellations exposed by unitarity method! (full calculations agree with partial '98 partial analysis).