Mass Effects at Higher Orders of QCD

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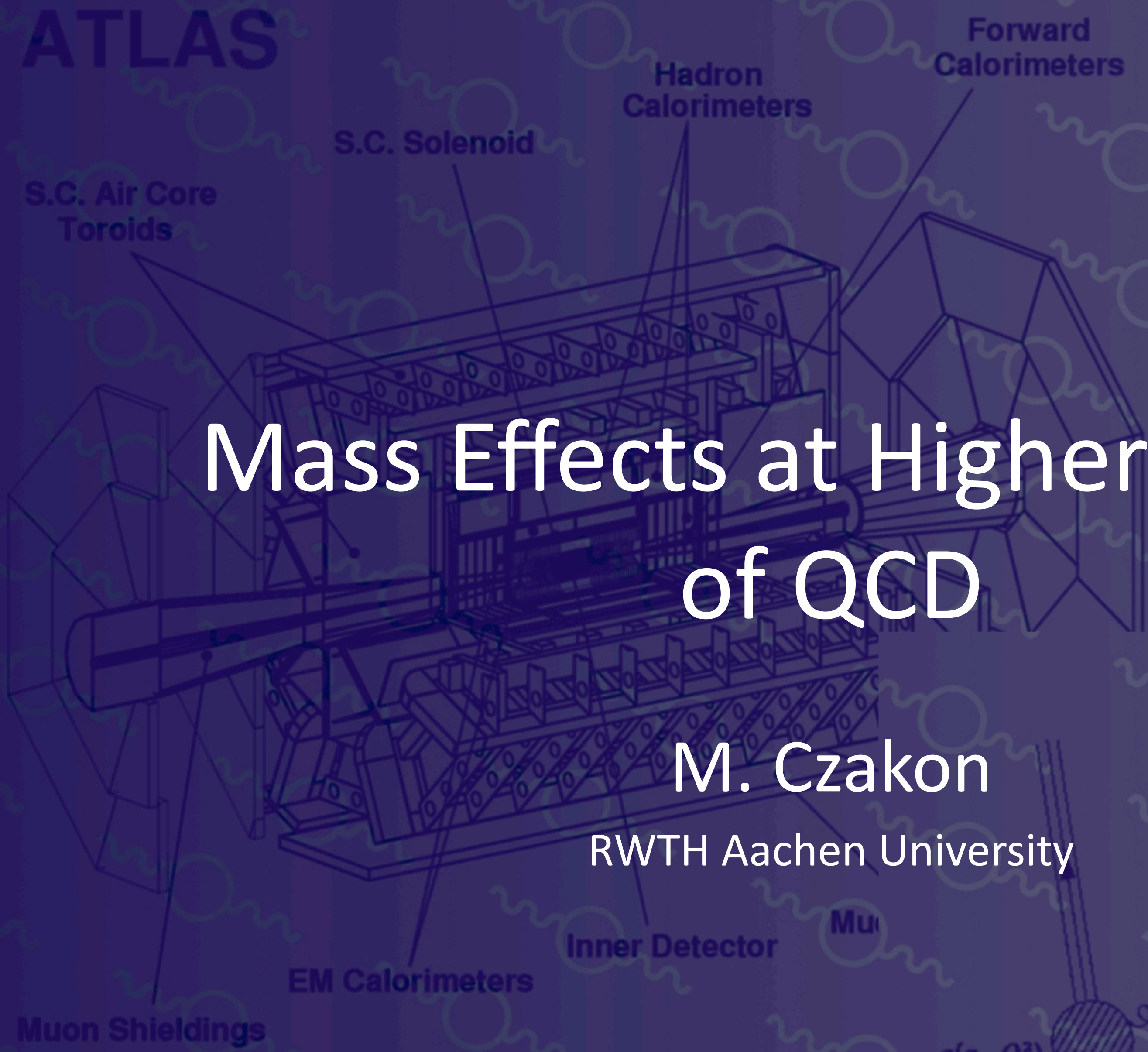

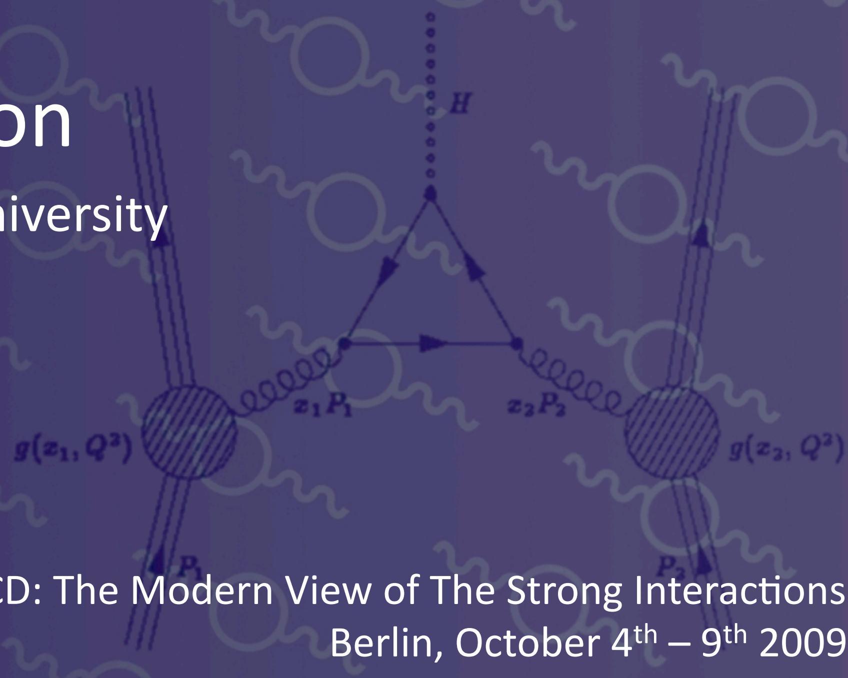

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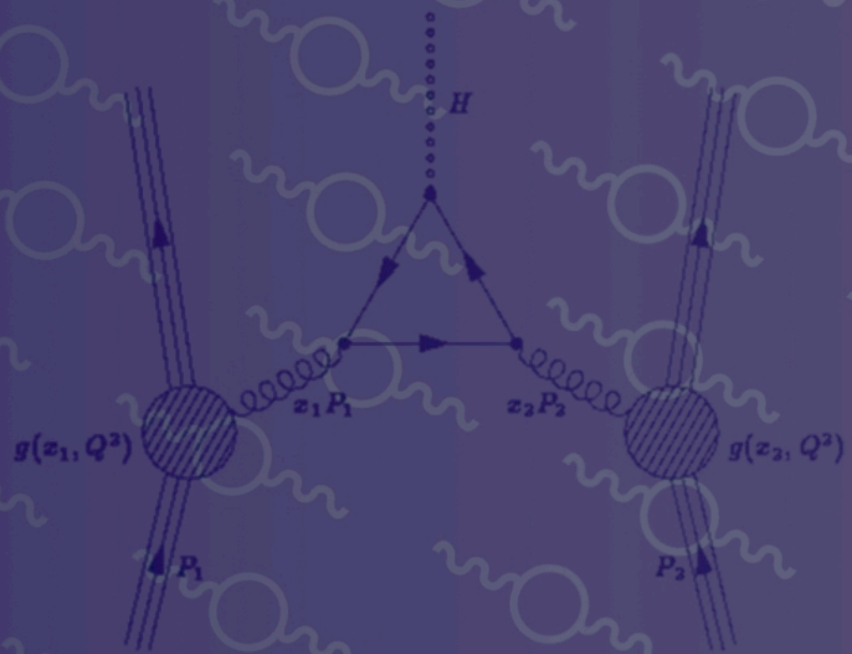
QCD: The Modern View of The Strong Interactions
Berlin, October 4th – 9th 2009

0. Top Motivation

A. Factorization and Divergences

B. Factorization and Resummation

C. Fixed Orders

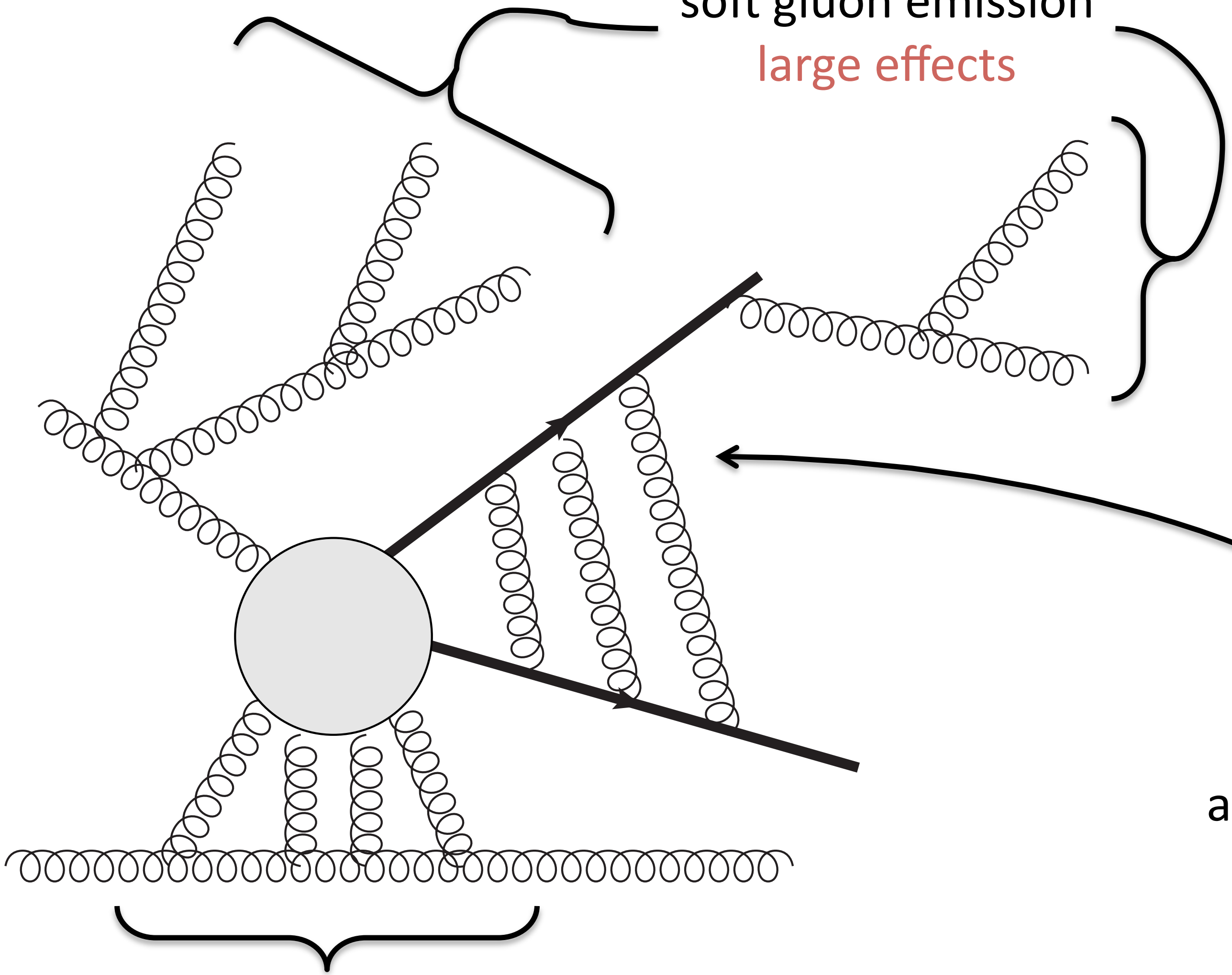


Top Quark Pairs at the LHC

- Total cross section numbers published by three groups
  - Moch, Uwer '08
  - Cacciari, Frixione, Mangano, Nason, Ridolfi '08
  - Kidonakis, Vogt '08
- No consensus on the error estimate !!!
- We expect a common understanding within the next few months
- New contributions all at or below 1% level !!!

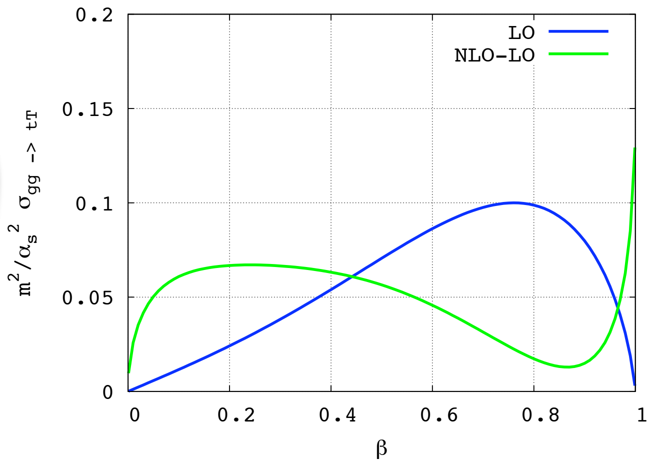
The Physics

soft gluon emission
large effects



Coulomb
attraction/repulsion
small effects

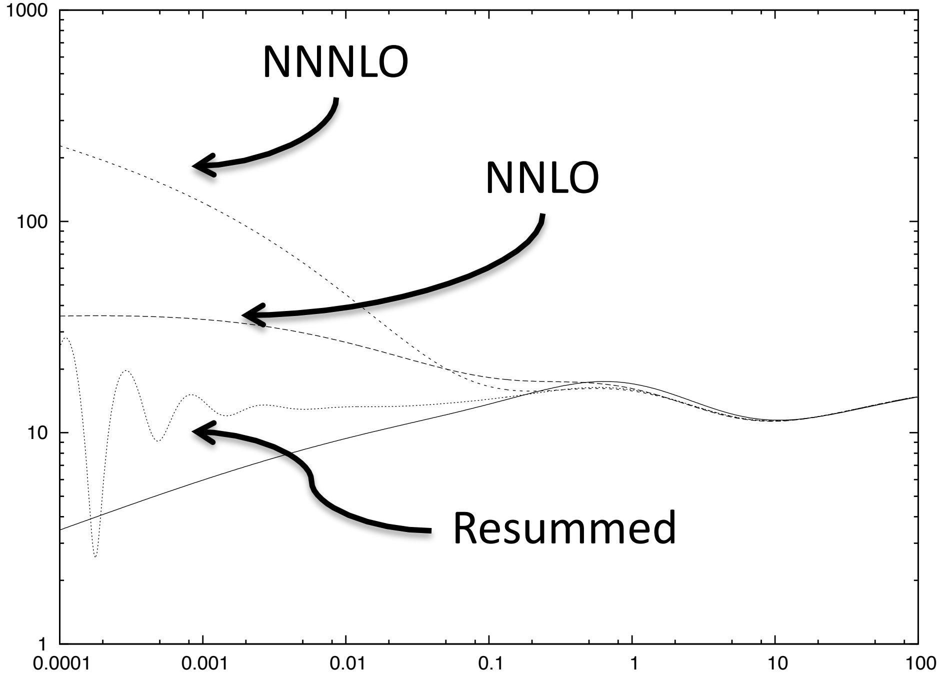
t-channel gluon exchange
negligible effects

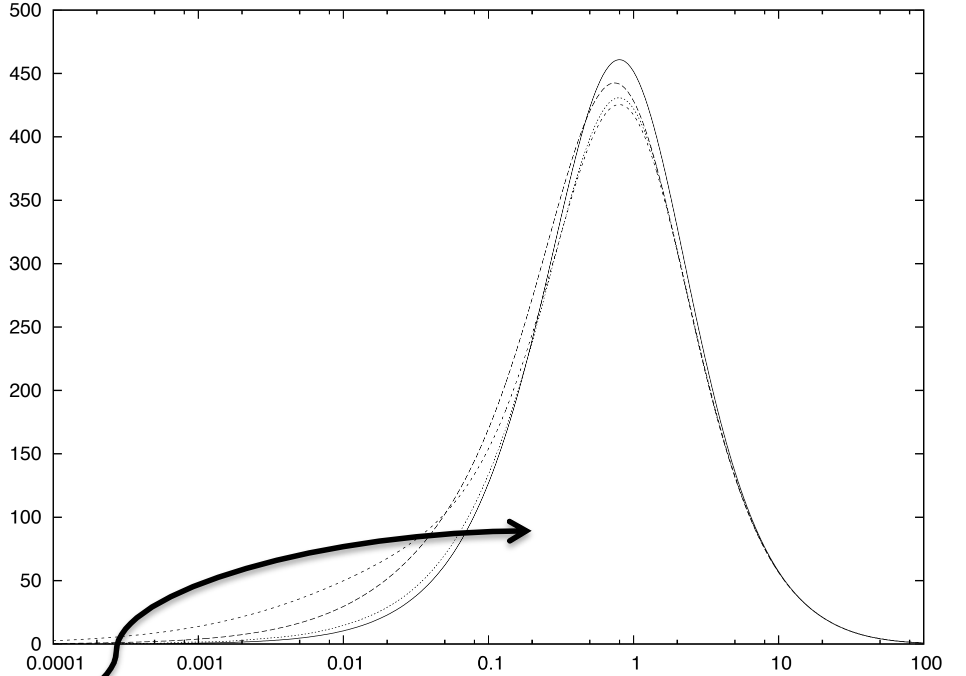


All effects can be resummed !!!

Do We Need Resummation ?

- Partonic gg cross section





- After multiplication
  with the flux

- The third order of the expansion contributes less than 1%

NO, WE DON'T !!!

Fixed Order NNLO Effects

$$\hat{\sigma}^{(1)}_{gg\to t\bar{t}} = \hat{\sigma}^{(0)}_{gg\to t\bar{t}}\left\{96\ln^2\beta - 9.5165\ln\beta + 35.322 + 5.1698\frac{1}{\beta}\right\},$$

$$\hat{\sigma}^{(2)}_{gg\to t\bar{t}} = \hat{\sigma}^{(0)}_{gg\to t\bar{t}}\left\{4608\ln^4\beta - 1894.9\ln^3\beta + \left(-3.4811 + 496.30\frac{1}{\beta}\right)\ln^2\beta + \left(3144.4 + 321.17\frac{1}{\beta}\right)\ln\beta + 68.547\frac{1}{\beta^2} - 196.93\frac{1}{\beta} + C^{(2)}_{gg}\right\}$$

Moch, Uwer `08

- Pick a guess at $C_{gg}^{(2)}$… Why not ≈ 1000 ?

- The correction is relative to the LO value (no need to integrate)

$$\left(\frac{\alpha_s}{4\pi}\right)^2 \hat{\sigma}^{(0)}_{gg\to t\bar{t}}\, C^{(2)}_{gg} \approx \left(\frac{10\%}{12}\right)^2 \hat{\sigma}^{(0)}_{gg\to t\bar{t}}\, 1000 \approx 10\%\, \hat{\sigma}^{(0)}_{gg\to t\bar{t}}$$

- Since the NLO is a further 50% away and there is another channel, the total is about 5%

- In conclusion: NNLO could be anything around 5% !!!

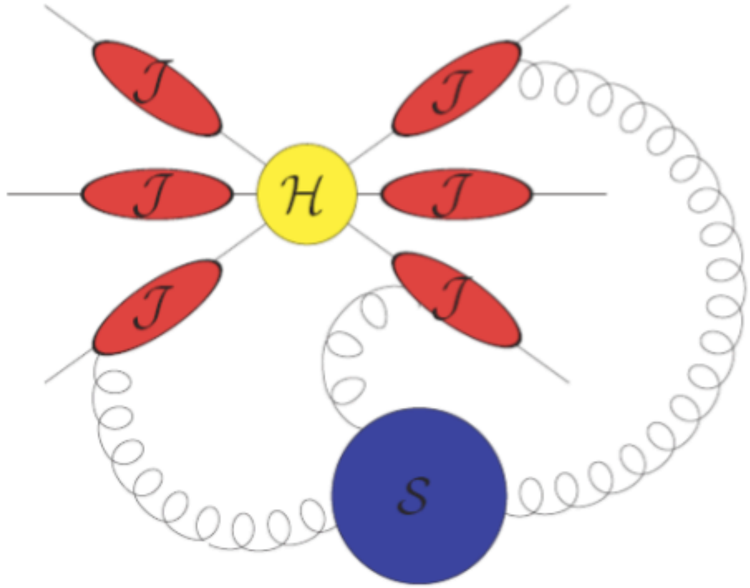
Factorization for Amplitudes

- By the factorization theorem every amplitude can be decomposed into three functions

  - jet

  - soft

  - hard



jet function
collinear/mass singularities

$$|\mathcal{M}_{\mathrm{p}}\rangle = \mathcal{J}_0^{[\mathrm{p}]}\left(\frac{Q^2}{\mu^2}, \alpha_{\mathrm{s}}(\mu^2), \varepsilon\right) \mathcal{S}_0^{[\mathrm{p}]}\left(\{k_i\}, \frac{Q^2}{\mu^2}, \alpha_{\mathrm{s}}(\mu^2), \varepsilon\right) |\mathcal{H}_{\mathrm{p}}\rangle$$

soft function
coherent color flow

hard part
IR finite

Divergences at Two Loops

- Ambiguities are fixed by requiring the jet functions to be square roots of form factors for example

- The soft function can be obtained from renormalization group arguments

$$\mathbf{s}(\epsilon, \ldots) = \mathcal{P} \exp \left\{ - \int_0^1 \frac{dx}{1-x} \, \mathbf{\Gamma}_S \left(\bar{\alpha}_s \left[(1-x)^2 Q^2 \right] \right) \right\}$$

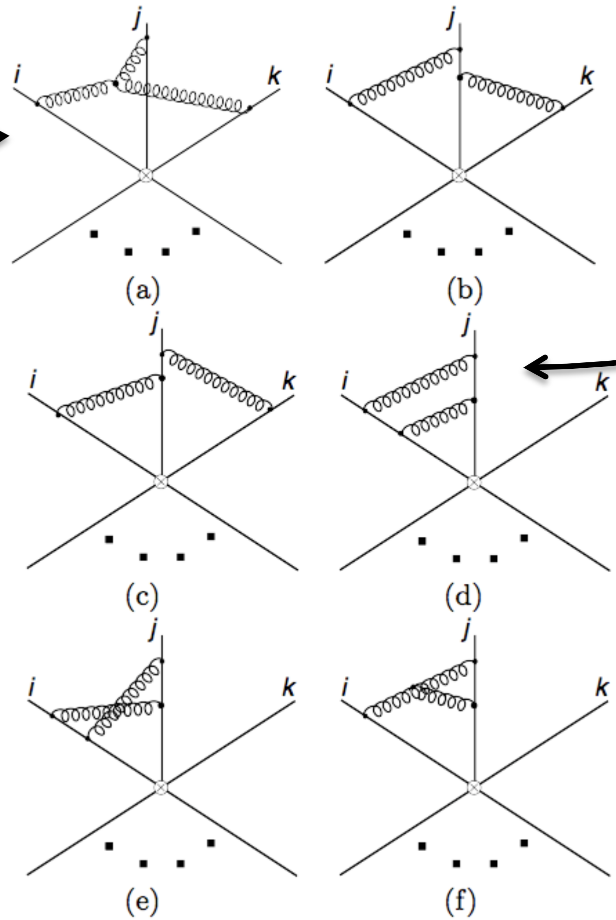
- The divergences are then

$$\begin{aligned} M^{(1)}(\epsilon) &= \left\{ \frac{1}{\epsilon} \Gamma_1 + J^{(1)} \right\} M^{(0)} + \mathcal{O}(\epsilon^0) \,, \\ M^{(2)}(\epsilon) &= \left\{ J^{(2)} - \left(J^{(1)} \right)^2 + \frac{1}{\epsilon} \left(-J^{(1)} \Gamma_1 + \Gamma_2 \right) + \frac{1}{\epsilon^2} \left(-\frac{1}{2} (\Gamma_1)^2 - \frac{\beta_0}{4} \Gamma_1 \right) \right\} M^{(0)} \\ &+ \left\{ \frac{1}{\epsilon} \Gamma_1 + J^{(1)} \right\} M^{(1)} + \mathcal{O}(\epsilon^0) \,. \end{aligned}$$

How to Get the Anomalous Dims?

- The soft anomalous dimensions are obtained from calculations in the eikonal approximation (scattering of Wilson lines)

3-particle correlators



2-particle correlators

How to Get the Anomalous Dims?

- As long as only two particle correlations are needed it is sufficient to use form factors and similar results at two loops

$$\mathbf{\Gamma}_S^{(2)} = \frac{1}{2}\sum_{(i\neq j)=1}^{n} T_i\cdot T_j\,\frac{K}{2}\,\ln\left(-\frac{\mu^2}{\sigma_{ij}}\right) + \frac{1}{2}\sum_{(i\neq j)\in\mathcal{N}_m} T_i\cdot T_j\,P_{ij}^{(2)} + 3E\text{ terms}$$

$$x_{ij} = \frac{\sqrt{1-\frac{4m^2}{s_{ij}}}-1}{\sqrt{1-\frac{4m^2}{s_{ij}}}+1}$$

$$P^{(2)} = \frac{K}{2}P^{(1)} + P^{(2),\mathrm{m}}$$

MC, Mitov, Sterman `09

$$\begin{aligned} P^{(2),\mathrm{m}}(x) \;=\; & \frac{C_A}{(1-x^2)^2}\Bigg\{-\frac{(1+x^2)^2}{2}\mathrm{Li}_3(x^2) + \left(\frac{(1+x^2)^2}{2}\ln(x) - \frac{1-x^4}{2}\right)\mathrm{Li}_2(x^2) \\ & + \frac{x^2(1+x^2)}{3}\ln^3(x) + x^2(1-x^2)\ln^2(x) \\ & + \left(-(1-x^4)\ln\left(1-x^2\right) + x^2(1+x^2)\zeta_2\right)\ln(x) + x^2(1-x^2)\zeta_2 + 2x^2\zeta_3\Bigg\}, \end{aligned}$$

- Others obtained this result with different methods

Kidonakis `09 (trivial color case)
Becher, Neubert `09 (general, based on old results by Korchemsky and Radyushkin)

Checks and Predictions

- Checked the qq -> tt divergences against explicit calculation

  MC, Mitov, Sterman `09

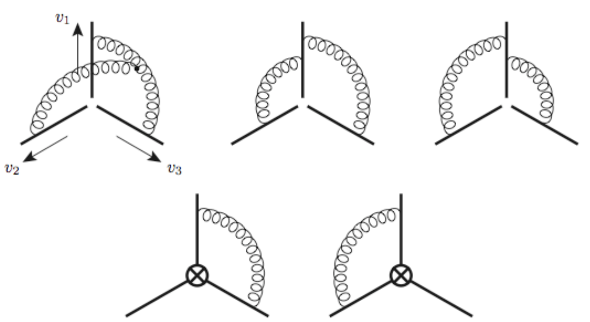
- Given explicit formulae for divergences in both channels

  Ferroglia, Neubert, Pecjak, Yang `09

- Confirmed the gg -> tt poles

  Baernreuther, MC, in preparation

- The divergences are in priciple known for any massive 2-loop amplitude, thanks to the evaluation of triple correlators



Ferroglia, Neubert, Pecjak, Yang `09

Total Cross Section Resummation

- Until recently resummation in Mellin space was written as

$$\frac{\hat{\sigma}^{N}_{ij,I}(m_t{}^2,\mu_f^2,\mu_r^2)}{\hat{\sigma}^{(0),N}_{ij,I}(m_t{}^2,\mu_f^2,\mu_r^2)} = g^{0}_{ij,I}(m_t{}^2,\mu_f^2,\mu_r^2)\cdot\exp\left(G^{N+1}_{ij,I}(m_t{}^2,\mu_f^2,\mu_r^2)\right) + \mathcal{O}\left(N^{-1}\ln^n N\right)$$

with

$$G^{N}_{q\bar{q}/gg,I} = G^{N}_{\mathrm{DY/Higgs}} + \delta_{I,8}\, G^{N}_{Q\bar{Q}}$$

$$G^{N}_{\mathrm{DY/Higgs}} = \int_0^1 dz\, \frac{z^{N-1}-1}{1-z} \int_{\mu_f^2}^{4m_t{}^2(1-z)^2} \frac{dq^2}{q^2}\, 2A_i(\alpha_s(q^2)) + D_i(\alpha_s(4m_t{}^2[1-z]^2))$$

$$G^{N}_{Q\bar{Q}} = \int_0^1 dz\, \frac{z^{N-1}-1}{1-z}\, D_{Q\bar{Q}}(\alpha_s(4m_t{}^2[1-z]^2))$$

final state radiation in the octet state

required at the two-loop level

- The requirements stand for NNLL resummation

The Matching Story

- For years matching coefficients taken from a numerical evaluation of the cross sections, color dependence ignored

- First determined correctly in the context of Coulomb corrections in

  Hagiwara, Sumino, Yokoya `08

- Determination based on quarkonium production results

  Kühn, Mirkes `93 (singlet)

  Petrelli et al. `98 (octet, incorrect, missing decoupling constant)

- Importance for soft gluon resummation first realised in

  MC, Mitov `08

- Values confirmed by direct evaluation, the largest numerical effect discovered until now

The Soft Function

- Same factorization as for amplitudes but in Mellin space

$$J_1(N,\alpha_s(\mu^2))\dots J_k(N,M/\mu,m/\mu,\alpha_s(\mu^2))$$
$$\times\ \mathrm{Tr}\left[\mathbf{H}^P\left(\frac{M^2}{\mu^2},\frac{m^2}{\mu^2},\alpha_s(\mu^2)\right)\mathbf{S}^P\left(\frac{N^2\mu^2}{M^2},\frac{M^2}{m^2},\alpha_s(\mu^2)\right)\right]+\mathcal{O}(1/N)$$

- The soft function satisfies the same renormalization group equation for cross sections and amplitudes, but the solution is different

$$\overline{\mathcal{P}}\exp\left\{\int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\,\mathbf{\Gamma}_S^\dagger\left(\beta_i\cdot\beta_j,\alpha_s\left((1-x)^2M^2\right)\right)\right\}$$
$$\times\mathbf{S}\left(1,\beta_i\cdot\beta_j,\alpha_s\left(M^2/N^2\right)\right)$$
$$\times\mathcal{P}\exp\left\{\int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\,\mathbf{\Gamma}_S\left(\beta_i\cdot\beta_j,\alpha_s\left((1-x)^2M^2\right)\right)\right\}$$

same anomalous dimension to be expanded at threshold

- The boundary is crucial for NNLL

MC, Mitov, Sterman `09

$$\mathbf{S}\left(1,\alpha_s\left(Q^2/N^2\right)\right)=\mathbf{S}^{(0)}\left[1+\frac{\alpha_s\left(Q^2/N^2\right)}{\pi}C_A\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}+\dots\right]$$
$$=\mathbf{S}^{(0)}\left[1+C_A\frac{\alpha_s\left(\mu^2\right)}{\pi}\left\{1+\frac{\alpha_s\left(\mu^2\right)}{\pi}\frac{\beta_0}{4}\ln\left(\frac{N^2\mu^2}{Q^2}\right)\right\}\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}+\dots\right]$$

NNLL Resummation

- The resummation looks now rather like

$$\frac{\sigma^P(N,m^2,\mu^2)}{\sigma^P_{\rm Born}(N)} = {\rm Tr}\left[\hat{\mathbf{H}}^P(m^2,\mu^2)\mathbf{S}_P^{(0)}\left[1 + \frac{\alpha_s\left(Q^2/N^2\right)}{\pi} C_A\, \mathbf{\Pi}_8\right] \exp\left\{\int_0^1 dx \frac{x^{N-1}-1}{1-x}\right.\right.$$
$$\left.\left. \times \left(\int_{\mu_F^2}^{4m^2(1-x)^2} \frac{dq^2}{q^2}\, 2\, A_P\left(\alpha_s\left[q^2\right]\right) \mathbf{1} + \hat{D}^P_{Q\overline{Q}}\left(\alpha_s\left[4m^2(1-x)^2\right]\right)\right)\right\}\right] + \mathcal{O}(1/N, N^3LL)$$

- Can be cast into the traditional formula with a modified D_{QQ}

MC, Mitov, Sterman \`09

$$D^P_{Q\overline{Q}} = \frac{\alpha_s(\mu^2)}{\pi}(-C_A)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2 \left\{D_{\rm P}^{(2)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(-C_A\frac{K}{2} - \frac{\zeta_3 - 1}{2}C_A^2 - C_A\frac{\beta_0}{2}\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

- The same result obtained with SCET

Beneke, Falgari, Schwinn \`09 (separation of Coulomb effects made transparent)

- A fixed order expansion of the cross section soon available

Beneke, Falgari, MC, Mitov, Schwinn, in preparation

Fixed Order Cross Section

Difficult 2-loop amplitudes
1-loop amplitude squared not-easy
trivial phase-space

Similar to tT+jet, but
difficulties at the phase space
boundaries

A window to new methods
for 1-loop integrals

$$d\hat{\sigma}^{NNLO}_{t\bar{t}} = d\hat{\sigma}^{VV}_{2} + d\hat{\sigma}^{VR}_{2+1} + d\hat{\sigma}^{RR}_{2+2}$$

Trivial amplitude
difficult phase space

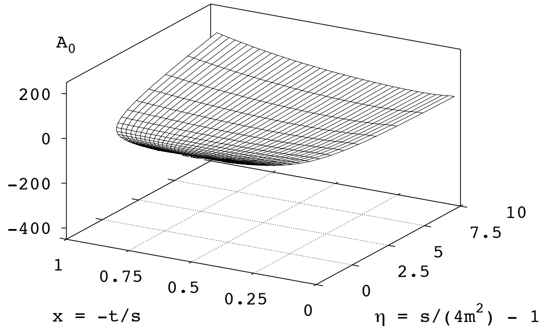
$$d\hat{\sigma}_n = d\Phi_n |M_n|^2$$

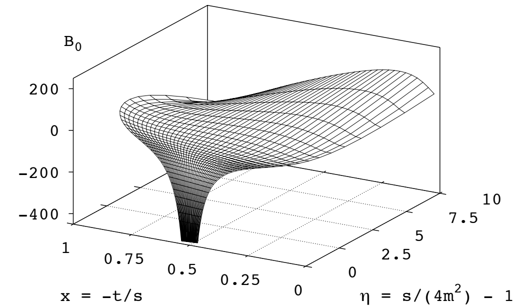
Amplitude

Phase space

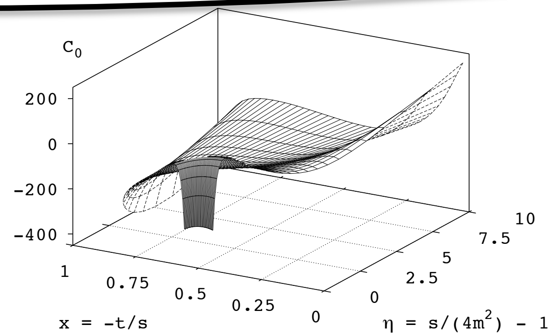
Amplitudes for Quark Annihilation

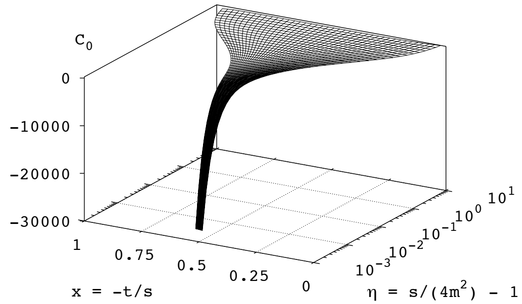
- complexity
  - 190 diagrams
  - 2812 integrals
  - 145 masters





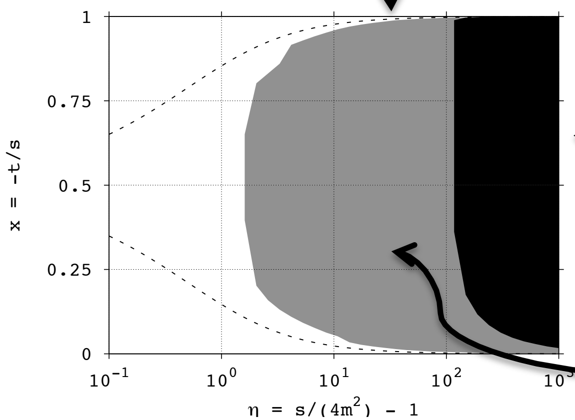
High precision numerics





Convergence region for
A small mass expansion

MC ‘08



1% accuracy
MC, Mitov, Moch ‘07

0.1% from the
expansion

MC ‘08

$$m^2 = 0.2s,\ t = -0.45s$$

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
A	0.22625	1.391733154	-2.298174307	-4.145752449	17.37136599
B	-0.4525	-1.323646320	8.507455541	6.035611156	-35.12861106
C	0.22625	-0.06808683395	-18.00716652	6.302454931	3.524044913
D_l		-0.22625	0.2605057339	-0.7250180282	-1.935417247
D_h			0.5623350684	0.1045606449	-1.704747998
E_l		0.22625	-0.3323207300	7.904121951	2.848697837
E_h			-0.5623350684	4.528240788	12.73232424
F_l					-1.984228442
F_{lh}					-2.442562819
F_h					-0.07924540546

Numerical Techniques

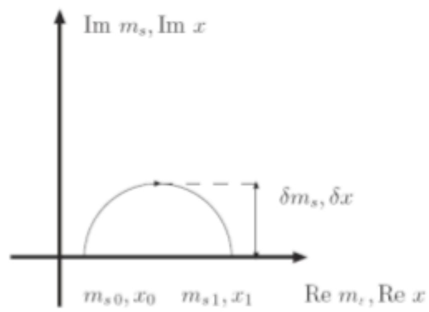
The new invention   based on some earlier ideas by Czyz, Caffo, Remiddi '02

- Determine the coefficients of the mass expansions using differential equations in m_s obtaining the power corrections

$$m_s \frac{d}{dm_s} M_i(m_s, x, \epsilon) = \sum_j C_{ij}(m_s, x, \epsilon) M_j(m_s, x, \epsilon)$$

- Evaluate the expansions for $m_s \ll 1$ to obtain the desired numerical precision of the boundaries

- Evolve the functions from the boundary point with differential equations first in m_s and then in x (**ZVODE**)



The hardest part   invented for Bhabha scattering by MC, Gluza, Riemann '06

- Compute the high energy asymptotics of the master integrals using Mellin-Barnes representations in order to obtain the leading behaviour of the amplitude

Beyond Differential Equations

- Problems at the singular points of the differential equations

Jacobian singularity	branching	allowed	interpretation
$m_s = 0$	yes		collinear singularity
$m_s = 1/4$	yes		s-channel threshold
$m_s = -1/4$			
$x = 0$	yes		t-channel threshold
$x = 1$	yes		u-channel threshold
$x = 1/2$		yes	perpendicular scattering
$m_s = x(1-x)$			forward/backward scattering
$m_s = x$			
$m_s = 1 - x$			
$m_s = -x$			
$m_s = x - 1$			
$m_s = 1/2x(1-x)$		yes	
$m_s = 1/2x$		yes	
$m_s = 1/2(1-x)$		yes	
$m_s = 1/2(1-x^2)$			
$m_s = -1/2(1-x)^2$			

finite part of the bosonic + heavy lepton contribution

$m_s = 0.2, \qquad d = 1/10(x - 0.45)$

- Taylor expansions around arbitrary points as a possible solution

$$469.555 - 14.5383\,d + 17.4298\,d^2 - 0.448744\,d^3 - 0.0600637\,d^4 + 0.0325809\,d^5 - 0.00888086\,d^6 + 0.00212687\,d^7 - 0.000528458\,d^8 + 0.000118944\,d^9 - 0.0000286279\,d^{10} + 6.39261\times 10^{-6}\,d^{11} - 1.51017\times 10^{-6}\,d^{12} + 3.37839\times 10^{-7}\,d^{13} - 7.88119\times 10^{-8}\,d^{14} + 1.76849\times 10^{-8}\,d^{15} - 4.09387\times 10^{-9}\,d^{16} + 9.10803\times 10^{-10}\,d^{17} - 2.2892\times 10^{-10}\,d^{18} + 9.93296\times 10^{-12}\,d^{19} - 8.09343\times 10^{-11}\,d^{20}$$

- Extremely efficient implement with sparse matrix multiplication and multiple precision (e.g. 128 digits needs 10 seconds for 21 terms)

High Precision Total Cross Section

- Contribution to the total cross section at $m_s = 0.2$, obtained with 2 Taylor expansions around x = 0.45 and x = 0.55 (unrenormalized)

$$\int_{-1}^{+1} d\cos\theta \,(1-\cos^2\theta)^{-\epsilon}\, 2\,\Re\langle M^{(0)}|M^{(2)}\rangle \approx$$

with 17 terms

$$\frac{53.0963}{\epsilon^4}-\frac{665.629}{\epsilon^3}+\frac{2524.60}{\epsilon^2}-\frac{21.6349}{\epsilon}+16206.6$$

with 18 terms

$$\frac{53.0963}{\epsilon^4}-\frac{665.630}{\epsilon^3}+\frac{2524.60}{\epsilon^2}-\frac{21.6359}{\epsilon}+16206.6$$

with 19 terms

$$\frac{53.0963}{\epsilon^4}-\frac{665.630}{\epsilon^3}+\frac{2524.60}{\epsilon^2}-\frac{21.6353}{\epsilon}+16206.6$$

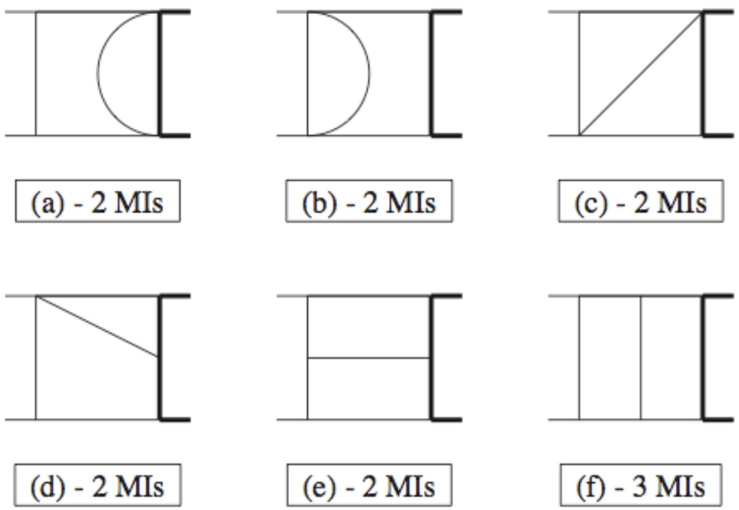
Analytic Results

- Fermionic and leading color results in the same channel have been obtained recently through analytic integration of DEQs

  Bonciani, Ferroglia, Gehrmann, Maitre, Studerus `08

  Bonciani, Ferroglia, Gehrmann, Studerus `09

- Only a handful of new integrals to determine



$$A_{-4} = \frac{1}{24(1+y)^2},$$

$$A_{-3} = \frac{1}{96(1+y)^2}\Big[-10G(-1;y) + 3G(0;x) - 6G(1;x)\Big],$$

$$A_{-2} = \frac{1}{192(1+y)^2}\Big[-47\zeta(2) - 24G(-1;y)G(0;x) + 48G(-1;y)G(1;x) + 32G(-1,-1;y) -6G(0,-1;y)\Big],$$

$$\begin{aligned}A_{-1} = \frac{1}{192(1+y)^2}\Big[&-85\zeta(3) + 188\zeta(2)G(-1;y) + 96\zeta(2)G(1;x) - 96\zeta(2)\,G(-1/y;x)\\
&+96G(0;x)G(-1,-1;y) - 96G(1;x)G(-1,-1;y) - 48G(-1/y;x)G(-1,-1;y)\\
&-48G(-y;x)G(-1,-1;y) - 24G(0;x)G(0,-1;y) + 24G(-1/y;x)G(0,-1;y)\\
&+24G(-y;x)G(0,-1;y) + 48G(-1;y)G(1,0;x) - 96G(-1;y)G(1,1;x)\\
&-24G(-1;y)G(-1/y,0;x) + 48G(-1;y)G(-1/y,1;x) - 24G(-1;y)G(-y,0;x)\\
&+48G(-1;y)G(-y,1;x) + 64G(-1,-1,-1;y) - 24G(-1,0,-1;y)\\
&-12G(0,-1,-1;y) + 6G(0,0,-1;y) + 24G(1,0,0;x) - 48G(1,0,1;x)\\
&-48G(1,1,0;x) + 96G(1,1,1;x) - 24G(-1/y,0,0;x) + 48G(-1/y,0,1;x)\\
&+24G(-1/y,1,0;x) - 48G(-1/y,1,1;x) + 24G(-y,1,0;x) - 48G(-y,1,1;x)\Big].\end{aligned} \quad (C.34)$$

Obvious Advantages

- Fast evaluation almost out of the box, because functions known (and the authors provide a numerical code)

- Expansions can be obtained by third party

$$
\begin{aligned}
A(\beta,\xi) &= \frac{A^{(-4)}(\beta,\xi)}{\varepsilon^4} + \frac{A^{(-3)}(\beta,\xi)}{\varepsilon^3} + \frac{A^{(-2)}(\beta,\xi)}{\varepsilon^2} + \frac{A^{(-1)}(\beta,\xi)}{\varepsilon} + A^{(0)}(\beta,\xi) + \mathcal{O}(\varepsilon)\ , \\
A^{(-4)} &= 0.25 - 0.5\beta^2(1-\xi)\xi + \mathcal{O}(\beta^3)\ , \\
A^{(-3)} &= 1.68185 + 0.5L_\mu + \beta(1-2\xi) - \beta^2\Big[0.5 + \xi(1-\xi)(5.86371 + L_\mu)\Big] + \mathcal{O}(\beta^3)\ , \\
A^{(-2)} &= -2.67119 - 0.302961L_\mu + 0.5L_\mu^2 + \beta\Big[1.84475 + 2L_\mu - \xi(3.68951 + 4L_\mu)\Big] \\
&\quad + \beta^2\Big[0.777936 - L_\mu - \xi(1-\xi)(7.03784 + 4.39408L_\mu + L_\mu^2)\Big] + \mathcal{O}(\beta^3)\ , \\
A^{(-1)} &= -8.15701 - 5.7593L_\mu - 2.13629L_\mu^2 + 0.333333L_\mu^3 + \beta\Big[-9.83935 - 3.64382L_\mu \\
&\quad + 2L_\mu^2 + \xi(19.6787 + 7.28765L_\mu - 4L_\mu^2)\Big] + \beta^2\Big[2.78693 + 5.22254L_\mu \\
&\quad - L_\mu^2 + \xi(1-\xi)(46.7006 + 8.75816L_\mu - 0.727411L_\mu^2 - 0.666667L_\mu^3)\Big] + \mathcal{O}(\beta^3)\ , \\
A^{(0)} &= 23.5701 + 7.82592L_\mu + 0.754463L_\mu^2 - 2.03531L_\mu^3 + 0.166667L_\mu^4 + \\
&\quad \beta\Big[0.505501 - 11.5953L_\mu - 7.31049L_\mu^2 + 1.33333L_\mu^3 + \xi(-1.011 + 23.1906L_\mu \\
&\quad + 14.621L_\mu^2 - 2.66667L_\mu^3)\Big] + \beta^2\Big[-4.351 - 3.60348L_\mu + 7.05587L_\mu^2 \\
&\quad - 0.666667L_\mu^3 + \xi(1-\xi)(-5.36823 + 43.1864L_\mu + 6.73063L_\mu^2 + 0.737281L_\mu^3 \\
&\quad - 0.333333L_\mu^4)\Big] + \mathcal{O}(\beta^3)\ . \qquad (5.4)
\end{aligned}
$$

and by the way… we love analytic results

Conclusions

- Required theoretical precision for top quark pair production not met yet
- Divergences at 2-loops fully understood in massive QCD
- NNLL resummation of total cross sections fully understood
- New insights with effective theories, in particular Coulomb enhancements vs. soft gluon enhancements
- Fixed order amplitudes almost finished with various methods
- Integrated real radiation is the next challenge