Integrability and Exact Anomalous Dimensions in Yang-Mills Theory

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Prologue

More than 35 years ago 't Hooft proposed a simplifying limit for Yang-Mills theories with gauge symmetry SU(N): The large N or planar limit.

Even though planar QCD has not yet been solved exactly, the idea has had very strong impact on mathematical physics. Among the reasons are:

- Many models with matrix-valued fields simpler than QCD turn out to be exactly solvable in the planar limit. There is an ill-understood, mysterious connection between the $N \to \infty$ limit and integrability.

- And indeed many models with matrix-valued fields turn out to possess a, likewise often mysterious, dual description by a string theory.

Gauge/String Duality

To date, many, seemingly quite distinct, examples for gauge/string dualities, have appeared. These are, at least partially, exactly solvable.

- Two-dim. quantum gravity: Matrix models/Liouville string theory.
- Two-dimensional QCD and its string description.
- Yang-Mills matrix models and M-theory.
- $\mathcal{N} = 4$ Gauge theory and String theory on $AdS_5 \times S^5$ (AdS/CFT correspondence).

The last one is of course the most interesting!

$\mathcal{N} = 4$ Supersymmetric Gauge Theory, I

[Brink, Schwarz, Scherk '77; Gliozzi, Scherk, Olive '77]

Fields: All fields are in the adjoint representation, they are $N \times N$ matrices.

- gauge field $\mathcal{A}_\mu$ with $\mu = 0, 1, 2, 3$ of dimension $\Delta = 1$
- field strength $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i\,[\mathcal{A}_\mu, \mathcal{A}_\nu]$, $\Delta = 2$
- $\mathbf{6}$ real scalars Φ_m, with $m = 1, \ldots, 6$, $\Delta = 1$
- $4 \times \mathbf{4}$ real fermions $\Psi_{\alpha a}, \dot{\Psi}^a_{\dot{\alpha}}$ mit $\alpha, \dot{\alpha} = 1, 2$, $a = 1, 2, 3, 4$, $\Delta = \frac{3}{2}$
- covariant derivatives: $\mathcal{D}_\mu = \partial_\mu - i\,\mathcal{A}_\mu$, $\Delta = 1$

$\mathcal{N} = 4$ Supersymmetric Gauge Theory, II

Action:

$$S = \frac{N}{\lambda} \int d^4x \, 2 \, \mathrm{Tr} \Bigg(\tfrac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} + \tfrac{1}{2} \mathcal{D}^\mu \Phi^m \, \mathcal{D}_\mu \Phi_m - \tfrac{1}{4} [\Phi^m, \Phi^n][\Phi_m, \Phi_n]$$

$$+ \dot{\Psi}^a_{\dot{\alpha}} \sigma_\mu^{\dot{\alpha}\beta} \mathcal{D}^\mu \Psi_{\beta a} - \tfrac{i}{2} \, \Psi_{\alpha a} \sigma_m^{ab} \epsilon^{\alpha\beta} [\Phi^m, \Psi_{\beta b}] - \tfrac{i}{2} \, \dot{\Psi}^a_{\dot{\alpha}} \sigma^m_{ab} \epsilon^{\dot{\alpha}\dot{\beta}} [\Phi_m, \dot{\Psi}^b_{\dot{\beta}}] \Bigg)$$

$$+ \frac{\theta_{\mathrm{YM}}}{16 \, \pi^2} \int d^4x \, 2 \, \mathrm{Tr} \, \mathcal{F}^{\mu\nu} \tilde{\mathcal{F}}_{\mu\nu}$$

Free parameters: $\lambda = N g^2_{\mathrm{YM}}$ and N and θ_{YM}.

Unique Model completely fixed by supersymmetry. λ ist dimensionless.

$\mathcal{N} = 4$ supersymmetries $+ \ \beta = 0 \Longrightarrow$ Superconformal Yang-Mills theory.

Why should we discuss this model at this conference?

Because of the emerging fact that this four-dimensional Yang-Mills theory is exactly solvable, at least in the planar limit.

The model might become the "harmonic oscillator" for gauge field theory!

Symmetries of $\mathcal{N} = 4$ Supersymmetric Gauge Theory

The "most beautiful" four-dimensional gauge theory. Many symmetries:

- Nonabelian local gauge symmetry: $\mathsf{SU}(N)$.
- Global symmetry: $\mathsf{PSU}(2, 2|4)$. Contains:
  Conformal group $\mathsf{SU}(2, 2) \simeq SO(2, 4)$, R-symmetry $\mathsf{SU}(4) \simeq SO(6)$.
  The latter two groups are connected by $\mathcal{N} = 4$ supersymmetries.
- Olive-Montonen symmetry: $\mathsf{SL}(2, \mathbb{Z})$
- At $N \to \infty$ new "hidden" symmetries $\mathrm{U}(1)^\infty$ appear: integrability

The AdS/CFT Correspondence

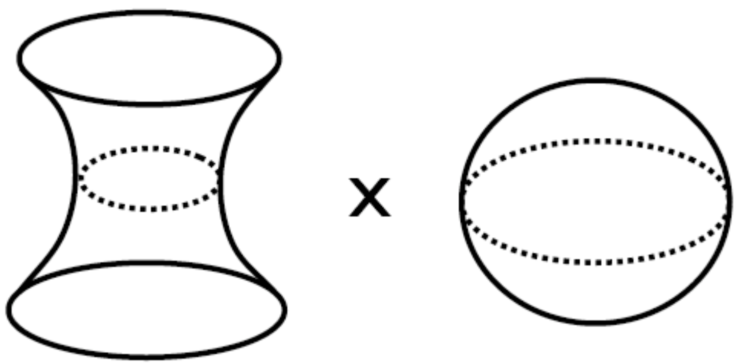
[Maldacena '97]

$\mathcal{N} = 4\ SU(N)$ supersymmetric gauge theory

't Hooft coupling: $\lambda = N g_{\mathrm{YM}}^2$ 1/color number: $\frac{1}{N}$ theta angle: θ_{YM}

$\mathcal{N} = 4$ SYM was conjectured to be dual to a string theory:

IIB Superstrings on $AdS_5 \times S^5$



tension: $\frac{R^2}{\alpha'} = \sqrt{\lambda}$ coupling: $g_s = \frac{\lambda}{4\pi N}$ axion: $\langle C \rangle = \theta_{\mathrm{YM}}$

The Spectral Problem of $\mathcal{N} = 4$ SYM

Scale-invariance leads to scale-covariance of composite operators $\mathcal{O}$:

$$\boxed{\left[\,\mathfrak{D}, \mathcal{O}(0)\,\right] = i\,\Delta_n\,\mathcal{O}(0)}$$

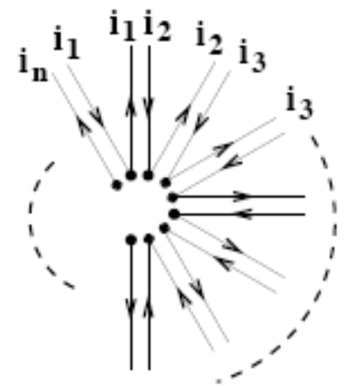
$\mathfrak{D}$ is the Lie-generator of dilatations and Δ the scaling dimension of $\mathcal{O}$.

Leads to integrable spin chain picture of gauge theory: [Lipatov '98, Minahan, Zarembo '02]

$$\boxed{\mathcal{O} = \mathrm{Tr}\left(\mathcal{X}\mathcal{Y}\mathcal{Z}\mathcal{F}_{\mu\nu}\Psi_\alpha^A(\mathcal{D}_\mu\mathcal{Z})\ldots\right) + \ldots}$$

Recall

$$\mathrm{Tr}\,(M^n) = \sum_{i_1,i_2,\ldots,i_n} M_{i_1,i_2}\,M_{i_2,i_3}\ldots M_{i_n,i_1} \quad \leftrightarrow$$



The partons carry additive, protected Lorentz and R-symmetry charges S_1, S_2, J_1, J_2, J_3. The sixth charge generically depends on λ: $\Delta = \Delta(\lambda)$.

Mixing Problem in $\mathcal{N} = 4$ SYM and Spin Chains

Consider twist operators, in close analogy to QCD:

$$\boxed{\mathcal{O} = \mathrm{Tr}\left(\mathcal{D}^{S_1}\mathcal{Z}^{J_3}\right) + \ldots .}$$

$\mathcal{D} = \mathcal{D}_1 + i\,\mathcal{D}_2$ mit $\mathcal{D}_\mu = \partial_\mu + i\,A_\mu$ is a covariant lightcone derivative.

The dilatation operator is regarded as the Hamiltonian of a spin chain.

The spin chain is

$$\boxed{\mathrm{Tr}\Big((\mathcal{D}^{s_1}\mathcal{Z})(\mathcal{D}^{s_2}\mathcal{Z})\ldots(\mathcal{D}^{s_{J_3-1}}\mathcal{Z})(\mathcal{D}^{s_{J_3}}\mathcal{Z})\Big)\,,}$$

where $S_1 = s_1 + s_2 + \ldots + s_{J_3-1} + s_{J_3} := M =$ Magnon number.

Apparently, the Hamiltonian is all-loop integrable, and diagonalizable by Bethe ansatz.

The Asymptotic All-Loop AdS/CFT Bethe Equations

[Beisert, MS '05,'06]

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}},$$

$$\left(\frac{x^+_{4,k}}{x^-_{4,k}}\right)^L = \prod_{\substack{j=1\\j\neq k}}^{K_4} \left(\frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i}\,\sigma^2(x_{4,k},x_{4,j})\right) \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}},$$

$$1 = \prod_{\substack{j=1\\j\neq k}}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}},$$

--- -- ------------------------------------

$$E(g) = 2\sum_{j=1}^{K_4}\left(\frac{i}{x^+_{4,j}}-\frac{i}{x^-_{4,j}}\right) = \frac{1}{g^2}\sum_{j=1}^{K_4}\left(\sqrt{1+16\,g^2\,\sin\frac{p_j}{2}}-1\right), \qquad \Delta = \Delta_0 + g^2\,E(g)\,, \qquad K_4 = M\,.$$

$$1 = \prod_{j=1}^{K_4}\left(\frac{x^+_{4,j}}{x^-_{4,j}}\right) = \prod_{j=1}^{K_4} e^{ip_j}\,, \qquad u_k = x_k + \frac{g^2}{x_k}\,, \qquad u_k \pm \tfrac{i}{2} = x^\pm_k + \frac{g^2}{x^\pm_k}\,.$$

The Interpolating Scaling Function

The scaling dimension of operators of low twist J_3 behaves in a very interesting logarithmic way at large spin $M = S_1 \to \infty$:

$$\boxed{\Delta - S_1 - J_3 \;=\; f(g)\;\log S_1 \;+\; O(S_1^0)\,.}$$

$f(g)$ is the universal scaling function, where $g^2 = \lambda/16\,\pi^2$.

Also appears in the structure of MHV-amplitudes und in lightcone segmented Wilson loops $\mathcal{W}$! Gluon 4-point function in $4 - 2\,\epsilon$ dimensions:

[Bern, Dixon, Smirnov]

$$\boxed{\mathcal{M}_4^{\mathrm{All-Loop}} \simeq \exp\left[f(g)\,\mathcal{M}_4^{\mathrm{One-Loop}}\right]\,, \qquad \mathcal{M}_4^{\mathrm{All-Loop}} \simeq \langle \mathcal{W} \rangle\,.}$$

Gauge Theory Meets String Theory

The asymptotic Bethe ansatz allows to derive an integral equation for a certain scaling function $f(g)$ at arbitrary values of g which interpolates between gauge theory and string theory. [Eden, MS '06; Beisert, Eden, MS '06]

At weak coupling this "BES" equation was (numerically) tested up to four loop order in gauge theory: [Bern, Czakon, Dixon, Kosower, Smirnov, '06]

$$\boxed{f(g) = 8\,g^2 - \tfrac{8}{3}\,\pi^2 g^4 + \tfrac{88}{45}\,\pi^4 g^6 - 16\left(\tfrac{73}{630}\,\pi^6 + 4\,\zeta(3)^2\right) g^8 \pm \ldots\,.}$$

At strong coupling the scaling function agrees with string theory to the three known perturbative orders [Gubser, Klebanov, Polyakov '02], [Frolov, Tseytlin '02], [Roiban, Tirziu, Tseytlin '07; Roiban, Tseytlin '07] as was derived analytically from the integral equation (G=Catalan's constant): [Basso, Korchemsky, Kotański '07]:

$$\boxed{f(g) = 4\,g\; - \frac{3\log 2}{\pi} - \frac{G}{4\pi^2}\,\frac{1}{g} - \ldots}$$

$\rightarrow$ The AdS/CFT correspondence is exactly true !

AdS/CFT is Exactly True

- Therefore, independently of all attempts to use string theory as a "theory of everything", it has thus been established that string theory can be a "theory of something": A $4D$ Yang-Mills theory.

- Turning this around, it has therefore also been established that theories with no apparent trace of gravity (i.e. Yang-Mills theories) can in a hidden way contain quantum gravity.

- Planar Feynman diagrams of a $4D$ Yang-Mills theory can really be summed to all orders, and analytically continued to strong coupling.

Transcendentality and Bethe Ansatz

The asymptotic Bethe ansatz is fraught with a wrapping problem. In the case of twist-two operators, this is not a problem up to three loops: [MS '04]

$$E_0 = 8\,S_1\,,$$

$$E_2 = -8\left(S_3 + S_{-3} - 2\,S_{-2,1} + 2\,S_1\left(S_2 + S_{-2}\right)\right),$$

$$\begin{aligned} E_4 = -16\Big(&2\,S_{-3}\,S_2 - S_5 - 2\,S_{-2}\,S_3 - 3\,S_{-5} + 24\,S_{-2,1,1,1} + 6\left(S_{-4,1} + S_{-3,2} + S_{-2,3}\right) \\ &-12\left(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\right) - \left(S_2 + 2\,S_1^2\right)\left(3\,S_{-3} + S_3 - 2\,S_{-2,1}\right) \\ &-S_1\left(8\,S_{-4} + S_{-2}^2 + 4\,S_2\,S_{-2} + 2\,S_2^2 + 3\,S_4 - 12\,S_{-3,1} - 10\,S_{-2,2} + 16\,S_{-2,1,1}\right)\Big), \end{aligned}$$

expressed in terms of recursively defined harmonic sums $(a, b, c > 0)$

$$S_{\pm a}(M) = \sum_{m=1}^{M} \frac{(\pm 1)^m}{m^a}, \qquad S_{\pm a,b,c,\ldots}(M) = \sum_{m=1}^{M} \frac{(\pm 1)^m}{m^a}\, S_{b,c,\ldots}(m)\,.$$

The 4-loop ABA result was obtained in [Kotikov, Lipatov, Rej, MS, Velizhanin, '07], it reads ...

$$
\begin{aligned}
&\boldsymbol{4\,S_{-7} + 6\,S_{7}} + 2\,(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1}) + 3\,(-S_{-2,5} \\
&+ S_{-2,3,-2}) + 4\,(S_{-2,1,4} + -S_{-2,-2,-2,1} - S_{-2,1,2,-2} - S_{-2,2,1,-2} - S_{1,-2,1,3} \\
&- S_{1,-2,2,2} - S_{1,-2,3,1}) + 5\,(-S_{-3,4} + S_{-2,-2,-3}) + 6\,(-S_{5,-2} \\
&+ S_{1,-2,4} - S_{-2,-2,1,-2} - S_{1,-2,-2,-2}) + 7\,(-S_{-2,-5} + S_{-3,-2,-2} \\
&+ S_{-2,-3,-2} + S_{-2,-2,3}) + 8\,(S_{-4,1,2} + S_{-4,2,1} - S_{-5,-2} - S_{-4,3} \\
&- S_{-2,1,-2,-2} + S_{1,-2,1,1,-2}) + 9\,S_{3,-2,-2} - 10\,S_{1,-2,2,-2} + 11\,S_{-3,2,-2} \\
&+ 12\,(-S_{-6,1} + S_{-2,2,-3} + S_{1,4,-2} + S_{4,-2,1} + S_{4,1,-2} - S_{-3,1,1,-2} - S_{-2,2,-2,1} \\
&- S_{1,1,2,3} - S_{1,1,3,-2} - S_{1,1,3,2} - S_{1,2,1,3} - S_{1,2,2,-2} - S_{1,2,2,2} - S_{1,2,3,1} - S_{1,3,1,-2} \\
&- S_{1,3,1,2} - S_{1,3,2,1} - S_{2,-2,1,2} - S_{2,-2,2,1} - S_{2,1,1,3} - S_{2,1,2,-2} - S_{2,1,2,2} \\
&- S_{2,1,3,1} - S_{2,2,1,-2} - S_{2,2,1,2} - S_{2,2,2,1} - S_{2,3,1,1} - S_{3,1,1,-2} - S_{3,1,1,2} - S_{3,1,2,1} \\
&- S_{3,2,1,1}) + 13\,S_{2,-2,3} - 14\,S_{2,-2,1,-2} + 15\,(S_{2,3,-2} + S_{3,2,-2}) \\
&+ 16\,(S_{-4,1,-2} + S_{-2,1,-4} - S_{-2,-2,1,2} - S_{-2,-2,2,1} - S_{-2,1,-2,2} - S_{-2,1,1,-3} \\
&- S_{1,-3,1,2} - S_{1,-3,2,1} - S_{1,-2,-2,2} - S_{2,-2,-2,1} + S_{-2,1,1,-2,1} + S_{1,1,-2,1,-2} \\
&+ S_{1,1,-2,1,2} + S_{1,1,-2,2,1}) - 17\,S_{-5,2} + 18\,(-S_{4,-3} - S_{6,1} + S_{1,-3,3}) \\
&+ 20\,(-S_{1,-6} - S_{1,6} - S_{4,3} + S_{-5,1,1} + S_{-4,-2,1} + S_{-3,-2,2} + S_{-2,-4,1} \\
&+ S_{-2,-3,2} + S_{1,3,3} + S_{3,1,3} + S_{3,3,1} - S_{1,1,-2,3} - S_{1,2,-2,-2} - S_{2,1,-2,-2}) \\
&- 21\,S_{3,4} + 22\,(S_{1,-2,-4} + S_{2,2,3} + S_{2,3,2} + S_{3,-2,2} + S_{3,2,2}) + 23\,(-S_{-3,-4} \\
&- S_{5,2} + S_{2,-2,-3}) + 24\,(-S_{-4,-3} + S_{1,-4,-2} - S_{1,-3,1,-2} - S_{1,1,1,4} - S_{1,1,4,1} \\
&- S_{1,3,-2,1} - S_{1,4,1,1} - S_{3,-2,1,1} - S_{3,1,-2,1} - S_{4,1,1,1} + S_{-2,-2,1,1,1} + S_{-2,1,-2,1,1} \\
&+ S_{1,-2,-2,1,1} + S_{1,-2,1,-2,1} + S_{1,1,-2,-2,1} + S_{1,1,1,-2,-2} + S_{1,1,2,-2,1} + S_{1,2,1,-2,1} \\
&+ S_{2,1,1,-2,1}) + 25\,S_{2,-3,-2} + 26\,(-S_{2,5} + S_{1,4,2} + S_{2,4,1} + S_{4,1,2} + S_{4,2,1}) \\
&+ 28\,(S_{1,2,4} + S_{2,1,4} - S_{-3,1,-2,1} - S_{-2,1,-3,1} - S_{1,-2,1,-3}) + 30\,S_{-3,1,-3} \\
&+ 32\,(S_{1,5,1} + S_{5,1,1} - S_{-3,-2,1,1} - S_{-2,-3,1,1} - S_{1,-3,-2,1} - S_{1,-2,-3,1} \\
&- S_{2,2,-2,1} + S_{1,2,-2,1,1} + S_{2,1,-2,1,1} - S_{1,1,1,-2,1,1}) + 36\,(S_{1,1,5} + S_{1,3,-3} \\
&+ S_{3,1,-3} - S_{1,1,-3,-2} - S_{1,1,-2,-3} - S_{1,1,2,-3} - S_{1,2,-2,2} - S_{1,2,1,-3} - S_{2,1,-2,2} \\
&- S_{2,1,1,-3}) + 38\,S_{-3,-3,1} + 40\,(-S_{1,-4,1,1} - S_{2,-3,1,1} + S_{1,1,1,-2,2}) \\
&- 41\,S_{3,-4} + 42\,(-S_{2,-5} + S_{1,-4,2} + S_{1,-3,-3}) + 44\,(S_{1,-5,1} + S_{2,-3,2} + S_{3,-3,1}) \\
&+ 46\,S_{2,2,-3} + 48\,S_{1,1,-3,1,1} + 60\,(S_{1,1,-5} - S_{1,1,-3,2}) + 62\,S_{2,-4,1} + 64\,S_{1,1,1,-3,1} \\
&+ 68\,(S_{1,2,-4} + S_{2,1,-4} - S_{1,2,-3,1} - S_{2,1,-3,1}) - 72\,S_{1,1,1,-4} - 80\,S_{1,1,-4,1} \\
&\boldsymbol{- \zeta(3) S_1 (S_3 - S_{-3} + 2\,S_{-2,1})}.
\end{aligned}
$$

Transcendentality, $\mathcal{N} = 4$ and QCD

Kotikov and Lipatov obtained the corresponding two-loop result for $\mathcal{N} = 4$ gauge theory [Kotikov, Lipatov '03]. They noticed that the answer may be extracted from the QCD result by focusing on the "most complicated terms". These are the ones of highest degree of transcendentality.

In 2004 a many-year effort to compute three-loop (NNLO) anomalous dimensions Δ of leading twist-two operators at finite spin $S_1 = M$ in QCD was completed. [Moch,Vermaseren,Vogt '04]. The result fills pages ...

The above conjecture for the three-loop dimensions of the analog of these operators in (planar) $\mathcal{N} = 4$ was put forward by KLOV [Kotikov,Lipatov,Onishchenko,Velizhanin '04], and agrees with the asymptotic Bethe ansatz.

The BFKL Pomeron in $\mathcal{N} = 4$

Similar to QCD, we have the $\mathcal{N} = 4$ BFKL equation for reggeized gluons:

[Balitsky-Fadin-Kuraev-Lipatov '76 - '78]

$$\frac{\omega}{-g^2} = \Psi\left(-g^2 E(g)\right) + \Psi\left(1 + g^2 E(g)\right) - 2\,\Psi\,(1)\ ,$$

encoding the pomeron resonance of high energy scattering amplitudes. In $\mathcal{N} = 4$ the pomeron may be described, with $M = \omega - 1$, by

[Kotikov, Lipatov, Rej, MS, Velizhanin, '07]

$$\boxed{\text{pomeron} = \operatorname{Tr}\left(\mathcal{Z}\,\mathcal{D}^{-1+\omega}\,\mathcal{Z}\right).}$$

The spin M of twist operators is to be continued to $M = -1$.
$\Psi(x) = \frac{d}{dx}\log\Gamma(x)$ is the logarithmic derivative of the gamma function.

Conformal Algebra $\mathfrak{so}(2,4)$

$$[M_{ab}, M_{cd}] = i\,(\eta_{ac}\,M_{bd} - \eta_{bc}\,M_{ad} - \eta_{ad}\,M_{bc} + \eta_{bd}\,M_{ac})$$

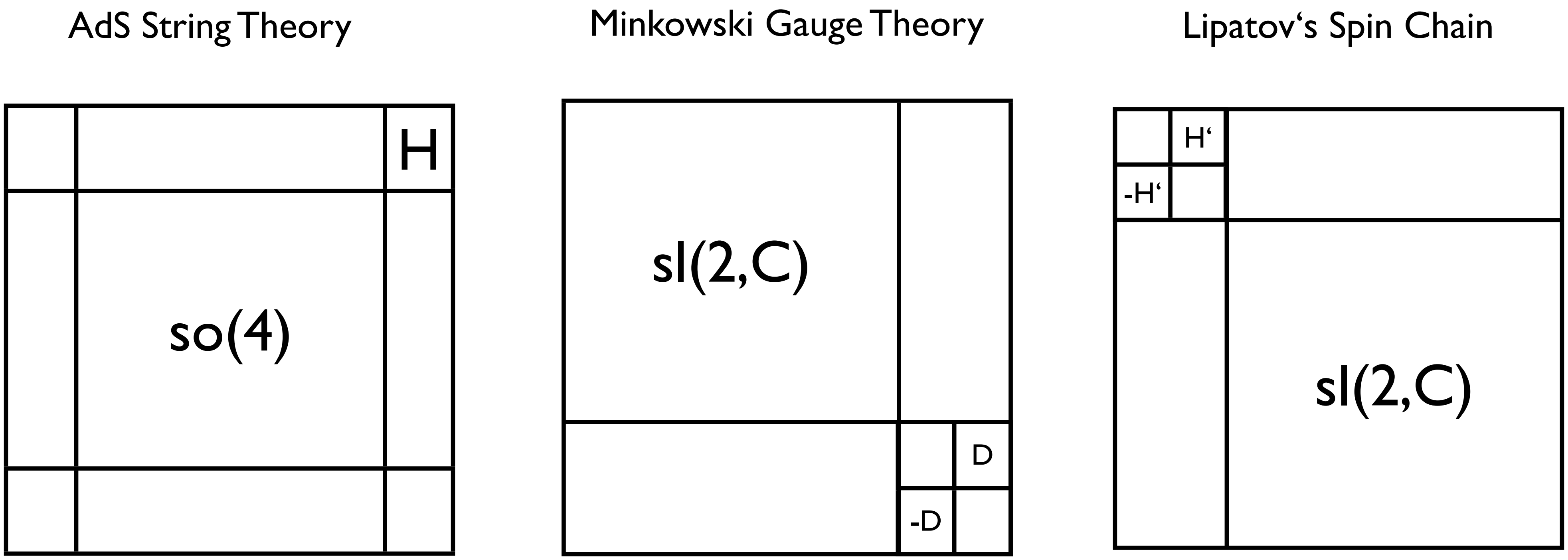
	-	+	+	+	+	-
-	0	+ boost	+ boost	+ boost	$-\frac{\mathfrak{P}_0-\mathfrak{K}_0}{2}$	$-\frac{\mathfrak{P}_0+\mathfrak{K}_0}{2}$
+	- boost	0	+ rot	+ rot	$-\frac{\mathfrak{P}_1-\mathfrak{K}_1}{2}$	$-\frac{\mathfrak{P}_1+\mathfrak{K}_1}{2}$
+	- boost	- rot	0	+ rot	$-\frac{\mathfrak{P}_2-\mathfrak{K}_2}{2}$	$-\frac{\mathfrak{P}_2+\mathfrak{K}_2}{2}$
+	- boost	- rot	- rot	0	$-\frac{\mathfrak{P}_3-\mathfrak{K}_3}{2}$	$-\frac{\mathfrak{P}_3+\mathfrak{K}_3}{2}$
+	$+\frac{\mathfrak{P}_0-\mathfrak{K}_0}{2}$	$+\frac{\mathfrak{P}_1-\mathfrak{K}_1}{2}$	$+\frac{\mathfrak{P}_2-\mathfrak{K}_2}{2}$	$+\frac{\mathfrak{P}_3-\mathfrak{K}_3}{2}$	0	$+\mathfrak{D}$
-	$+\frac{\mathfrak{P}_0+\mathfrak{K}_0}{2}$	$+\frac{\mathfrak{P}_1+\mathfrak{K}_1}{2}$	$+\frac{\mathfrak{P}_2+\mathfrak{K}_2}{2}$	$+\frac{\mathfrak{P}_3+\mathfrak{K}_3}{2}$	$-\mathfrak{D}$	0

$\mathfrak{H} \leftarrow$ energy

$\leftarrow$ dilatation

AdS Spin Chains

[MS, work in progress]



BFKL and the Four-Loop Breakdown of the ABA

To four-loop order the one-loop BFKL equation predicts

$$E=\left(\frac{-4\,g^2}{\omega}\right)-0\,\left(\frac{-4\,g^2}{\omega}\right)^2+0\,\left(\frac{-4\,g^2}{\omega}\right)^3-2\,\zeta(3)\,\left(\frac{-4\,g^2}{\omega}\right)^4\pm\ldots\,.$$

This may be compared to the result of the asymptotic Bethe ansatz (ABA). After continuing to negative values of the spin M, and expanding in ω around the pole at $M=-1$ one finds

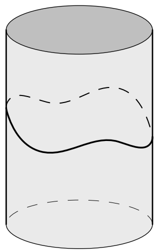
$$E^{\mathrm{ABA}}=\left(\frac{-4\,g^2}{\omega}\right)-0\,\left(\frac{-4\,g^2}{\omega}\right)^2+0\,\left(\frac{-4\,g^2}{\omega}\right)^3-\frac{(-4\,g^2)^4}{\omega^7}\pm\ldots,$$

One observes maximal violation of the BFKL prediction: The leading singularity in ω should be a pole of fourth order. Instead, one finds a seventh order pole! This establishes that the long-range asymptotic Bethe ansatz breaks down at four-loop order.

[Kotikov, Lipatov, Rej, MS, Velizhanin, '07]

IIB Superstring on $AdS_5 \times S^5$

Two-dimensional worldsheet with coordinates σ, τ :



Embedded into the coset space (Fermions act like "staples")

$$\frac{\widetilde{\mathrm{PSU}}(2,2|4)}{\mathrm{SO}(4,1)\times\mathrm{SO}(5)} = \overbrace{AdS_5 \times S^5} \,.$$

The IIB Superstring σ-Model on $AdS_5 \times S^5$

Action:

[Metsaev, Tseytlin '98]

$$\boxed{S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau\, d\sigma\ \left(\partial_a Z^M \partial^a Z_M + \partial_a Y_N \partial^a Y_N\right) + \mathrm{Fermions}\,.}$$

with

$$\begin{aligned} \mathrm{AdS}_5 : \quad & -Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 - Z_5^2 = -R^2 \\ \mathrm{S}^5 : \quad & Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = R^2 \end{aligned}$$

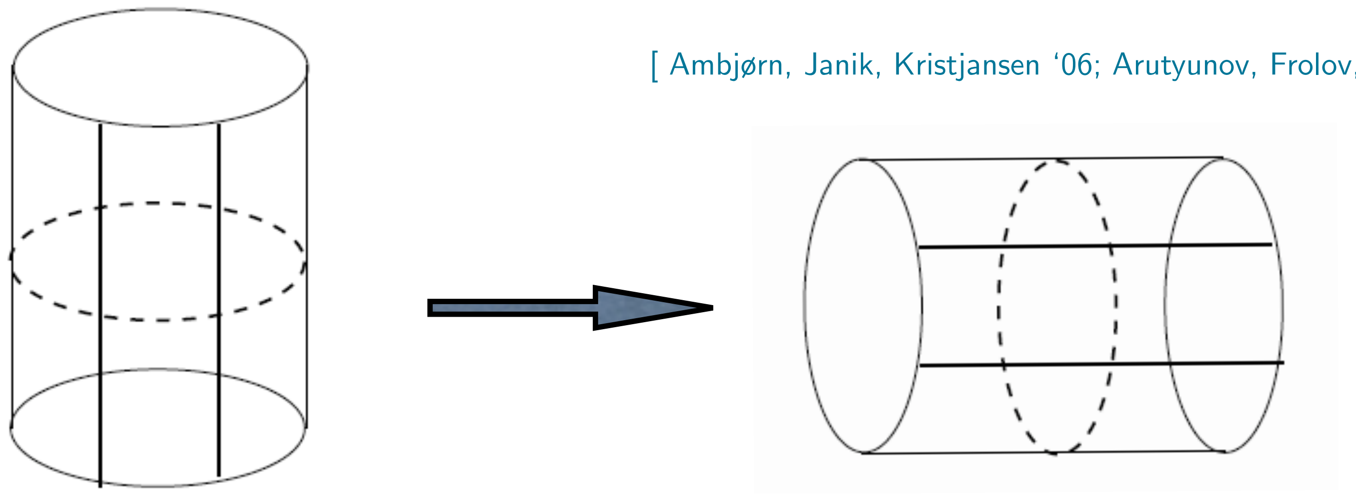
The quantization of this model has not yet been understood.

However, see below ...

The Thermodynamic Bethe Ansatz

[Al. Zamolodchikov '90]

[Ambjørn, Janik, Kristjansen '06; Arutyunov, Frolov, '07, '08, '09; Bajnok, Janik '08]



In the TBA, one "turns around" the world sheet cylinder of the string σ-model, and considers scattering in the cross channel. This takes into account virtual field-theoretic corrections.

TBA and BFKL

[Bajnok, Janik, Łukowski '08]

Using the thermodynamic Bethe ansatz, the wrapping correction to the ABA result was recently computed:

$$\begin{aligned}\Delta E_6^{\mathrm{TBA}}(M) = 128\, S_1^2 \Big(& - S_5 + S_{-5} + 2\, S_{4,1} - 2\, S_{3,-2} + 2\, S_{-2,-3} - 4\, S_{-2,-2,1} \Big) \\ & -256\, \zeta(3)\, S_1^2\, S_{-2} \\ & -320\, \zeta(5)\, S_1^2 \,,\end{aligned}$$

Adding $E_6^{\mathrm{ABA}}(M) + \Delta E_6^{\mathrm{TBA}}(M)$, and expanding around $M = -1$, the BFKL pole structure is confirmed to two orders!

Incidentally, the complete (ABA + TBA) five-loop anomalous dimension is currently being computed. [Łukowski, Rej, Velizhanin, work in progress]

It will be very interesting to see whether it passes the BFKL pole test.

The Konishi Operator

The simplest long multiplet in $\mathcal{N} = 4$ gauge theory is the Konishi field. One multiplet member is the $M = 2$ twist-two operator. The ABA + TBA result for its anomalous dimension reads (four loop: [Bajnok, Janik '08])

$$\boxed{\gamma = 12\, g^2 - 48\, g^4 + 336\, g^6 - 96\, \left(26 - 6\,\zeta(3) + 15\,\zeta(5)\right)\, g^8 + \ldots\,.}$$

The four-loop result was confirmed last year by a field theory calculation by computing all "wrapping" Feynman diagrams [Fiamberti, Santambrogio, Sieg, Zanon '07, '08, Velizhanin '08].

Recently, the TBA 5-loop result was also obtained: [Bajnok, Hegedus, Janik, Łukowski '09]

$$\boxed{\gamma_5 = +96\, \left(158 + 72\,\zeta(3) - 54\,\zeta(3)^2 - 90\,\zeta(5) + 315\,\zeta(7)\right)}$$

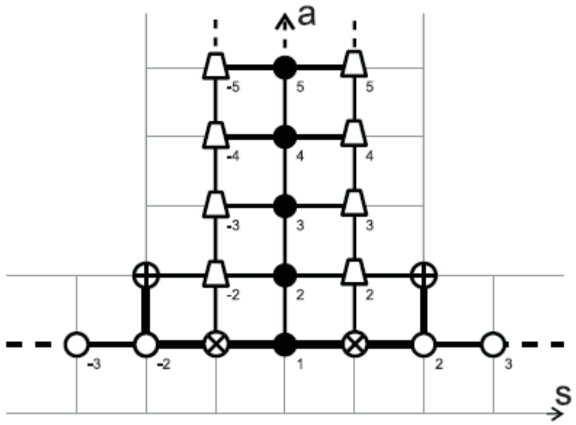
Curious signs. It would be important to check this in field theory!

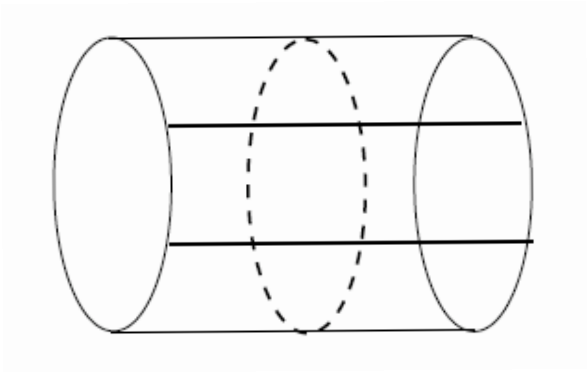
All-loop TBA equations, Y-System

Ground State TBA: [Bombardelli, Fioravanti, Tateo '09, Arutyunov, Frolov '09, Gromov, Kazakov, Kozak, Vieira '09]

Bold claim: Exact spectrum of planar $\mathcal{N} = 4$. [Gromov, Kazakov, Kozak, Vieira '09]

Infinite system of integral equations for "Y-functions" living on a lattice:



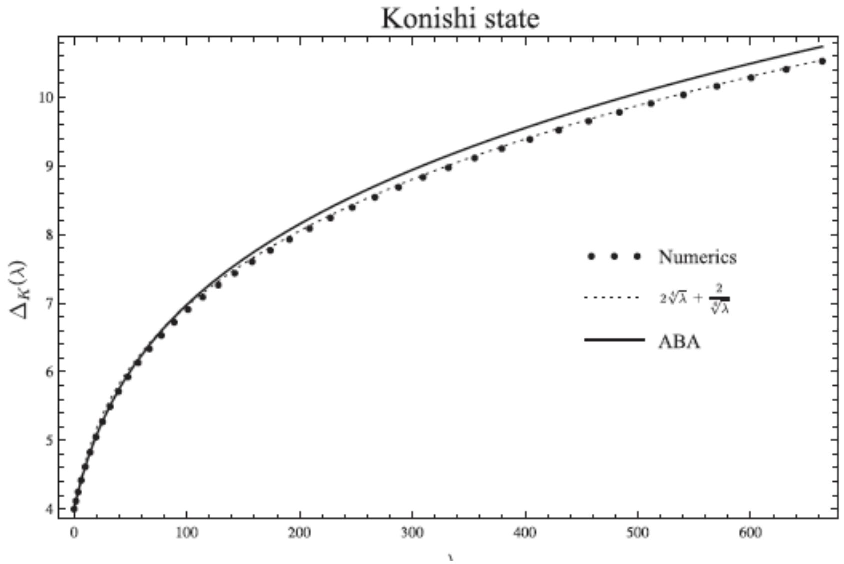


Supposed to take into account infinitely many virtual corrections!

Crucial future test: Does the Y-system agree to all orders with BFKL?

If true, the Y-system must yield the exact BFKL equation of $\mathcal{N} = 4$.

Predictions from the Y-System



[Gromov, Kazakov, Vieira '09]

- A numerical plot for the Konishi field was obtained.

- It fits well weak coupling, but is not yet accurate enough to test the novel 5-loop TBA result.

- It predicts the strong coupling behavior $2\,\lambda^{\frac{1}{4}} + 2\,\lambda^{-\frac{1}{4}} + \ldots$.
  Currently there is a discrepancy with string theory: $2\,\lambda^{\frac{1}{4}} + 1\,\lambda^{-\frac{1}{4}} + \ldots$.

[Roiban, Tseytlin '09]

Crucial Open Problems

- Actually, what exactly is this system we are solving?
  How can we define it, and prove its integrability?
  In other words, what is it we have been/currently are diagonalizing?
  Did we already identity the correct elementary excitations of the model?

- How can we derive this system from the planar $\mathcal{N} = 4$ gauge theory?
  How does Yang-Baxter symmetry emerge from planar gauge theory?

- And the same question remains open for the σ-model on $AdS_5 \times S^5$.
  How does Yang-Baxter symmetry emerge from the string σ-model?

- Are the current proposals (Y-System, all-loop TBA) already the correct, final solution? If so, how to simplify and prove this solution?

Solvable Structures in the (Planar) AdS/CFT System

- Spectral Problem
- High Energy Scattering (BFKL)
- Gluon Amplitudes
- Wilson Loops

These are all related! (E.g. recall the universal scaling function.)

- Recently much progress with $\mathcal{N}=4$ gluon amplitudes.
- Exciting hints at Yangian structures in planar $\mathcal{N}=4$ amplitudes.

[Plefka, Drummond, Henn '09]