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Path Integral Methods for Soft Gluon Resummation

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QCD: The Modern View of the Strong Interactions

Overview

What is the structure of soft gluon corrections at next-to-eikonal order?

- Review of soft gluon resummation.
- Exponentiation in (non-)abelian gauge theories - webs.
- New approach using path integral methods.
- Classification of next-to-eikonal contributions.
- Outlook.

Soft resummation

- Multiple soft gauge boson emission can lead to large corrections to cross-sections.
- If ξ is the energy carried by soft bosons, typically get contributions:

$$\frac{d\sigma}{d\xi} = \sum_{n,m} \alpha^n \left[c^0_{nm} \frac{\log^m(\xi)}{\xi} + c^1_{nm} \log^m(\xi) + \ldots \right]$$

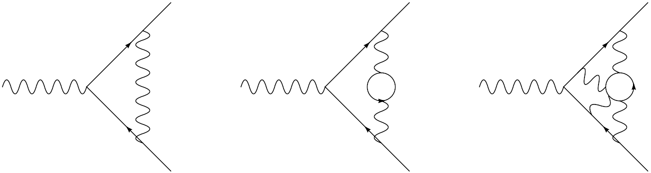
- First set of terms corresponds to *eikonal approximation*, in which momenta $k_i \to 0$ for all (soft) emissions.
- Second set of terms is *next-to-eikonal* (NE) limit i.e. first order in k_i.
- Happens in abelian and non-abelian theories.

Soft resummation - abelian case

- When ξ is small, perturbation theory breaks down - must resum problem logarithms.
- At eikonal order, have a simple result for the amplitude in abelian theories

$$\mathcal{A} = \mathcal{A}_0 \exp\left[\sum G_c\right],$$

where $\mathcal{A}_0$ is the Born amplitude, and G_c are connected subdiagrams.



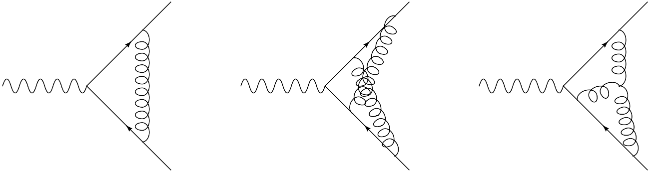
- Gives eikonal logarithms at all orders in α.

Soft resummation - nonabelian case

- Exponentation generalisable to non-abelian theories, but structure is more complicated:

$$\mathcal{A} = \mathcal{A}_0 \exp\left[\sum \bar{C}_W W\right],$$

where W are *webs* (two-eikonal line irreducible subdiagrams).
- Webs have modified colour weights $\bar{C}_W$.



- More effort than abelian case, but still predicts eikonal logs to all orders.

Generalisation to NE order

Question: Can this be extended to NE order?

- Will now introduce new framework for soft resummation.
- Old results are recovered, and can be easily generalised to sub-eikonal approximation.
- Based on key observation:

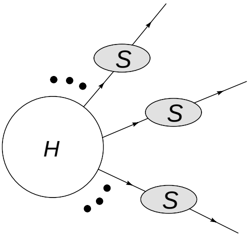
  Exponentiation of connected subdiagrams looks like exponentiation of connected diagrams in QFT (a textbook result!).
- Are they by any chance related?
- Answer: yes, after rewriting of the problem.
- Let's first look at abelian case (with scalar emittors) in detail...

Path integral method

- Consider a Green's function with a number of hard external lines, each of which may emit soft radiation.
- Can write this as:

$$G(p_1, \ldots p_n) = \int \mathcal{D}A_s^\mu H(x_1, \ldots, x_n) S(p_1, x_1) \ldots S(p_n, x_n),$$

where H is hard interaction, and S are propagators for the emitting particles in the presence of a soft gauge field A_s^μ, sandwiched between states $|p_i\rangle$, $|x_i\rangle$.



- Propagator factors $S(p_i, x_i)$ can now be re-expressed as first-quantised path integrals...

Propagators as path integrals

- Can write the scalar free particle propagator factor as

$$S(x,p) = \int \mathcal{D}x\mathcal{D}p \exp\left[-ip(T)x(T) + i\int_0^T dt(p\dot{x} - H(p,x))\right].$$

- This is a first-quantised path integral, where $x(t)$ is the trajectory of the particle.
- For an emitting particle in a background soft gauge field, this becomes

$$S(p,x,A_s^\mu) = \int_{x(0)=0}^{p(T)=0} \mathcal{D}p\mathcal{D}x \exp\left[i\int_0^T dt(p\dot{x} - \frac{1}{2}p^2 \right.$$
$$+ (p_f + p)\cdot A_s(x_i + p_f t + x) + \frac{i}{2}\partial\cdot A_s(x_i + p_f t + x)$$
$$\left. - A_s^2(x_i + p_f t + x))\right].$$

Soft photon exponentiation

- One now substitutes the propagator factors into the expression for the Green's function.
- Can carry out the path integrals over p_i (for each hard external line).
- Result has the form

$$G(p_1, \dots p_n) = \int \mathcal{D}A_s^\mu H(x_1, \dots x_n) \prod_i \mathcal{D}x_i e^{-ip_i \cdot x_i}$$

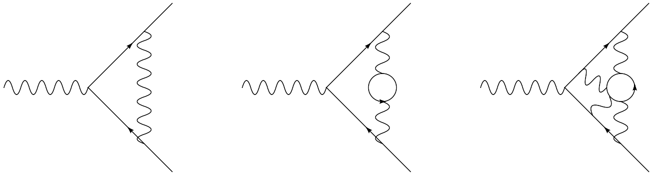
$$\exp\left[i \int_0^\infty dt \left(\frac{1}{2}\dot{x}^2 + (p_f + \dot{x}) \cdot A(x_i + p_f t + x(t)) + \frac{i}{2}\partial \cdot A(x_i + p_f t + x) \right) \right].$$

- This is a quantum field theory for the soft gauge field! Terms in exponent act as *sources* for A_s^μ.

Soft photon exponentiation

- These sources are localised on the hard external lines.
- All possible soft photon diagrams are generated, which span the external lines.
- Field theory, so disconnected diagrams exponentiate.

$\Rightarrow$ Soft photon corrections exponentiate.



Path integral picture - summary

- Factorise Green's functions into hard interactions with outgoing (hard) legs emitting soft radiation.
- Rewrite propagators for these legs in terms of first quantised path integrals involving worldines x_i^μ.
- Get a field theory with source terms localised on the external lines.
- Exponentiation of disconnected diagrams in this field theory $\equiv$ exponentiation of soft photon subdiagrams.
- Have considered scalar external lines, and abelian gauge fields, but framework generalises...

Generalisation

- Extension to fermion emitting particles is straightforward.
- Get extra terms in classical action, which have spinor structure (magnetic moment vertices).
- Can also consider non-abelian theories (see later).
- Clear physical interpretation allows extension of exponentiation beyond eikonal order.
- To understand this, let's look at the method in more detail...

- Green's function with many soft emissions has the form

$$G(p_1, \dots p_n) = \int \mathcal{D}A_s^\mu H(x_1, \dots x_n) \prod_x \mathcal{D}x e^{-ip_i \cdot x}$$
$$\exp\left[i \int_0^\infty dt \left(\frac{1}{2}\dot{x}^2 + (p_f + \dot{x}) \cdot A(x_i + p_f t + x(t)) \right.\right.$$
$$\left.\left. + \frac{i}{2} \partial \cdot A(x_i + p_f t + x) \right) \right].$$

- Here $\{x\}$ are the worldline trajectories of the hard emitting particles.
- The eikonal approximation corresponds to neglecting recoil i.e. x is the straight-line classical trajectory.
- Above result simplifies in this limit, in which one sets fluctuations to zero ($x = x_i + p_f t$, $p = p_f$).

- One finds

$$G(p_1,\dots p_n) = \int \mathcal{D}A_s^\mu H(x_1,\dots x_n)\prod_x \mathcal{D}x e^{-ip_f\cdot x_i}$$
$$\exp\left[i\int_0^\infty dt p_f\cdot A(x_i+p_f t)\right]$$
$$= \int \mathcal{D}A_s^\mu H(x_1,\dots x_n)\prod_x \exp\left[\int dx\cdot A_s(x)\right].$$

- This is the well-known result that eikonal corrections can be treated via Wilson lines (Korchemsky, Marchesini).
- Momentum space Feynman rule for soft gauge field follows from Fourier transform

$$i\int_0^\infty dt p_f^\mu A_\mu(p_f t) = -\int \frac{d^d k}{(2\pi)^d}\frac{p_f^\mu \tilde{A}_\mu(k)}{p_f\cdot k}.$$

- To go to next-to-eikonal order, one systematically expands about the classical trajectory.
- Outgoing momenta are lightlike, so one can set $p_f = \lambda n$, where $n^2 = 0$, for each external line.
- Then each external line factor in the Green's function becomes

$$\int \mathcal{D}x \exp\left[i\int_0^\infty dt \left(\frac{1}{2}\dot{x}^2 + (\lambda n + \dot{x})\cdot A(\lambda n t + x)\right.\right.$$
$$\left.\left.+\frac{i}{2}\partial\cdot A(\lambda n t + x)\right)\right]$$

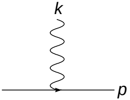
  $\Rightarrow \lambda \to \infty$ gives the eikonal approximation.
- Expanding to first subleading order in λ gives next-to-eikonal contribution.

- After rescaling $t \to t/\lambda$ get

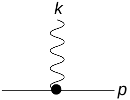
$$\int \mathcal{D}x \exp\left[i\int_0^\infty dt\left(\frac{\lambda}{2}\dot{x}^2 + (n+\dot{x})\cdot A(nt+x)\right.\right.$$
$$\left.\left. + \frac{i}{2\lambda}\partial\cdot A(nt+x)\right)\right]$$

  for each external line.
- Putting this into the expression for the Green's function, the x path integrals can be done perturbatively, keeping all terms $\mathcal{O}(1/\lambda)$.
- The result is a set of new Feynman rules at NE level, which generalise the rules of eikonal perturbation theory...

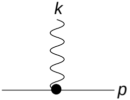
NE Feynman Rules



$$\frac{p^\mu}{p \cdot k},$$



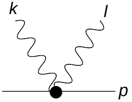
$$\frac{k^\mu}{2p \cdot k},$$



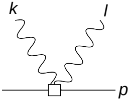
$$-\,k^2 \frac{p^\mu}{2(p \cdot k)^2}.$$

- One also finds two-gluon vertices...

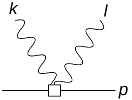
NE Feynman Rules



$$+\frac{\eta^{\mu\nu}}{p\cdot(k+l)},$$



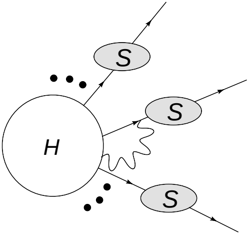
$$-\frac{l^{\mu}p^{\nu}p\cdot k+k^{\nu}p^{\mu}p\cdot l}{p\cdot(k+l)p\cdot kp\cdot l},$$



$$+\frac{p^{\mu}p^{\nu}k\cdot l}{p\cdot(k+l)p\cdot kp\cdot l}.$$

Comments

- A given Feynman diagram will have at most one NE Feynman rule in it.
- Connected subdiagrams exponentiate using the same argument as in the eikonal case.
- This is not the whole story - one also gets NE corrections from soft gauge bosons which land inside the hard interaction.



- These contributions are fixed by gauge invariance.

Internal emissions

- Separation of the gauge field into hard and soft modes leaves a residual gauge invariance:

$$A^{\mu}_{h,s}(k) \to A^{\mu}_{h,s}(k) + k^{\mu}\xi_{h,s}(k).$$

- Gauge invariance of the Green's function leads (after some work) to the condition

$$H^{\mu}(p_1, \ldots, p_n; k) = -\sum_{j=1}^{n} q_j \frac{\partial}{\partial p_{j\mu}} H(p_1, \ldots, p_n),$$

  where H^{μ} is the subamplitude for emission of a soft photon of momentum k^{μ} from within the hard interaction.
- This is essentially a rederivation of the well-known Low-Burnett-Kroll theorem.

Internal emissions

- The contributions from graphs with internal soft emissions are next-to-eikonal, due to the derivative in hard momentum.
- They do not formally exponentiate, but have an iterative structure to all orders in perturbation theory.
- Thus, the complete structure of matrix elements up to NE order is

$$\mathcal{M} = \mathcal{M}_0 \exp\left[\mathcal{M}^{E} + \mathcal{M}^{NE}\right] \times \left[1 + \mathcal{M}_{rem.}\right] + \mathcal{O}(NNE).$$

- External emission graphs contribute to the exponent, and internal graphs to the remainder.

Summary so far

- Have introduced path integral method for investigating soft gluon resummation.
- We have seen how it is applied to abelian theories.
- Get effective NE Feynman rules.
- Can classify which diagrams formally exponentiate at NE order and which do not.

What about non-abelian theories?

Non-abelian theories

- The exponentiation of soft photon corrections followed naturally from the path integral for the soft gauge field, after writing the external propagators as path integrals over x.
- This generated source terms for A_s localised on the hard external lines.
- The argument does not carry over straightforwardly to non-abelian theory, as the source terms are matrix-valued in colour space.
- Thus, they do not commute, and the usual combinatorics of the path integral do not apply.
- Can make progress using the *replica trick* of statistical physics.

The replica trick

- Consider a theory with N copies of the soft gauge bosons.
- Now consider the Green's function raised to the power N:

$$G^N = 1 + N \log G + \mathcal{O}(N^2)$$

- Crucially, only a subset of diagrams have a term linear in N.
- Then one has:

$$G = G_0 \exp \left[\sum C_i G_i \right],$$

  where G_i are subgraphs linear in N, and C_i their corresponding colour factors.
- Finally, one sets $N = 1$.

Comments

- We have considered the simplest case of a colour-singlet hard inteaction, with two external lines.
- In that case, one can find the subset of diagrams which a linear in the replica number N.
- This subset W have the property of being two-eikonal line irreducible.
- Furthermore they have modified colour factors $\bar{C}_W$ corresponding exactly the webs of GFT!
- A slightly more elegant solution for the colour factors results from the new technique.

Non-abelian exponentiation

- The extension to NE order proceeds similarly to in the abelian case.
- The structure of matrix elements (based on the simple hard interaction considered) has the same form:

$$\mathcal{M} = \mathcal{M}_0 \exp\left[\mathcal{M}^{E} + \mathcal{M}^{NE}\right] \times \left[1 + \mathcal{M}_{rem.}\right] + \mathcal{O}(NNE).$$

- The remainder comes from internal emission graphs.
- The exponent receives contributions from both eikonal and next-to-eikonal webs.

Applications

- It is known in many processes that NE logarithms are potentially sizeable.
- Prediction / resummation of these would be useful in any such process.
- Our technique potentially allows one to calculate these logarithms.
- Before phenomenological studies can take place, need to consider phase-space of emitted gluons.
- One expects:

$$\sigma^{\mathsf{NE}} = \int d\mathsf{PS}^{\mathsf{E}} |\mathcal{M}^{\mathsf{NE}}|^2 + \int d\mathsf{PS}^{\mathsf{NE}} |\mathcal{M}^{\mathsf{E}}|^2.$$

Conclusions

- Have developed a new framework for examining soft gluon resummation.
- Uses path integral methods to relate exponentiation to known exponentation of field theory diagrams.
- Works for all spins of emitting particles, and for (non)-abelian gauge theories.
- Old results are recovered (i.e. webs), with more elegant solution for $\bar{C}_W$.
- Extension to next-to-eikonal corrections straightforward.
- Structure of NE corrections in matrix elements classified.

Outlook

- Have so far looked at a simple non-abelian case (two external lines only). Can extend method to more complex systems.
- In cross-sections, need corrections to phase space as well as matrix elements. Under investigation.
- Phenomenological applications: What are the $\ln(1-x)$ terms in various circumstances?
- Can the new methods say anything about recent developments in $N = 4$ SYM?