04/25/17
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Workshop Seminar on Gravitational Waves:

Fundamentals of Gravitational Waves:

0. Conventions:

metric signature: $(-,+,+,+)$

units: $\hbar = c = M_{pl} = 1$

where $(M_{pl})^2 = \frac{\hbar c}{8\pi G}$ & $G = 6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$

1. Recap: Electromagnetic Waves:

1.1 $S_{EM} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

equations of motion: $\partial_\nu F^{\mu\nu} = 0$

In Lorenz gauge $\partial_\mu A^\mu = 0$

$\rightsquigarrow$ $\boxed{\partial^2 A_\mu = 0}$ free wave equation $(*)$

What is the number of propagating degrees of freedom?

- General solution of $(*)$ is superposition of plane waves $A_\mu = \epsilon_\mu e^{ikx} + c.c.$ with $k^2 = 0$ and polarization ϵ_μ
- $\partial_\mu A^\mu = 0 \implies k_\mu \epsilon^\mu = 0$
- Can decompose $\epsilon_\mu = \epsilon_\mu^L + \epsilon_\mu^T$ with $\epsilon_\mu^L \propto k_\mu$ „longitudinal polarization" and „transverse polarization" ϵ_μ^T with $\epsilon_0^T = 0$ & $\epsilon_i^T \cdot k^i = 0$

- Longitudinal part of ϵ_μ can be gauged away: $\partial_\mu A^\mu = 0$ leaves „residual" gauge freedom

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$$
$$\text{with } \partial^2 \Lambda = 0$$

Can use it to set $A_0 = 0$

<u>reason:</u> A_0 satisfies $\partial^2 A_0 = 0$

Then, the longitudinal polarization vanishes.

- For definiteness, take $k_\mu = (k, k, 0, 0)$

$$\Rightarrow \epsilon_\mu = (0, 0, \epsilon_2, \epsilon_3)$$

i.e. there are two propagating d.o.f.

<u>Important remark:</u> This no longer holds when the gauge symmetry is spontaneously broken!

1.2 Electromagnetism coupled to complex scalar:

$$S = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - D_\mu \phi D^\mu \bar{\Phi} - V(|\phi|) \right\}$$

gauge symm.:
$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda$$
$$\phi \longrightarrow e^{iq\Lambda} \phi$$

$$D_\mu \phi \equiv (\partial_\mu - iq A_\mu) \phi$$

$\rightsquigarrow$ equations of motion:

$$\partial_\mu F^{\mu\nu} = -iq \, \phi D^\nu \bar{\Phi} + \text{c.c.}$$
$$D^2 \phi = \frac{\partial}{\partial \bar{\Phi}} V$$

If $\langle\phi\rangle \equiv \phi_0 = 0$: Same as before.
If $\phi_0 = v/\sqrt{2} \neq 0$: $U(1)$ spont. broken.

- Isolate Goldstone mode θ:

$$\phi = \frac{1}{\sqrt{2}}(v+\varphi)e^{i\theta}, \qquad \varphi, \theta \text{ real}$$

- Can fix gauge by setting $\theta = 0$. "Unitary Gauge"

$\rightsquigarrow$ linearized field equations:

$$\partial_\mu F^{\mu\nu} = M^2 A^\nu, \quad M = q \cdot v$$

- These <u>imply</u> $\partial \cdot A = 0 \Rightarrow (\partial^2 - M^2)A^\nu = 0$

$\Rightarrow A_\mu$ has become massive by 'eating up' the would-be Goldstone mode θ
("Higgs mechanism")

- There are now <u>3</u> propagating d.o.f.
- Again: General sol. of (massive) wave eqn. is superposition of plane waves

$$A_\mu = \epsilon_\mu e^{ikx} + c.c.$$

but with $k^2 = -M^2$ & $k_\mu \epsilon^\mu = 0$

in rest frame $k_\mu = (M, 0, 0, 0)_\mu$

$\rightsquigarrow \epsilon_\mu = (0, \epsilon_1, \epsilon_2, \epsilon_3)_\mu$

2. Gravitational Waves:

$$S_{EH} = \frac{1}{2}\int d^4x \sqrt{g}\, R$$

$$S = S_{EH} + S_{rest}$$

Einstein eqn.: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$

where $T_{\mu\nu} = -2 \frac{\delta S_{rest}}{\sqrt{g}\,\delta g^{\mu\nu}}$

is the energy momentum tensor

goal: Find the linearized equations of motion of small metric perturbations around ~~the~~ Minkowski space time.

metric pert. h: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$.

From now on: Raise/lower indices with $\eta_{\mu\nu}$

Christoffel symbols:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

$$= \frac{1}{2}\left(\partial_\nu h^{\rho}{}_{\mu} + \partial_\mu h^{\rho}{}_{\nu} - \partial^{\rho} h_{\mu\nu}\right) + \dots$$

Ricci tensor: $R_{\mu\nu} = \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\rho\mu} + \dots$

$$= \frac{1}{2}\left(\partial_\rho \partial_\nu h^{\rho}{}_{\mu} + \partial_\rho \partial_\mu h^{\rho}{}_{\nu} - \partial^2 h_{\mu\nu} - \partial_\mu \partial_\nu h\right)$$

$+ \dots$ where $h \equiv h^{\rho}{}_{\rho}$

useful definition: $\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h\, \eta_{\mu\nu}$

Field equations:

$$\partial^\rho \partial_\nu \gamma_{\rho\mu} + \partial^\rho \partial_\mu \gamma_{\rho\nu} - \eta_{\mu\nu} \partial^\rho \partial^\sigma \gamma_{\rho\sigma} - \partial^2 \gamma_{\mu\nu} = 2 \cdot T_{\mu\nu} + \dots$$

gauge symmetry:

Under small diffeos $x^\mu \to x^\mu - \xi^\mu$

one has: $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

$\Rightarrow \gamma_{\mu\nu} \to \gamma_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial \cdot \xi$

$\Rightarrow$ Can enforce Hilbert gauge $\partial_\nu \gamma^{\mu\nu} = 0$

because $\boxed{\partial_\nu \gamma^{\mu\nu} \to \partial_\nu \gamma^{\mu\nu} + \partial^2 \xi^\mu}$

& from electromagnetism we know

$$0 = \partial_\nu \gamma^{\mu\nu} + \partial^2 \xi^\mu$$

can be solved

In this gauge: $\boxed{\partial^2 \gamma_{\mu\nu} = -2 T_{\mu\nu}}$

application 1: Newtonian limit: $T_{00} \gg T_{0i}, T_{ij}$

$\Rightarrow$ only γ_{00} non-vanishing

$$\Rightarrow \left| \begin{aligned} g_{00} &= -1 + \tfrac{1}{2}\gamma_{00} \\ g_{ij} &= \delta_{ij}\left(1 + \tfrac{1}{2}\gamma_{00}\right) \\ \gamma_{00} &= +\frac{1}{2\pi}\int d^3x' \, \frac{\rho(t,x')}{|x-x'|} \equiv -4\Phi \end{aligned} \right|$$

$$\rightsquigarrow \boxed{ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)d\vec{x}^2}$$

application 2: Gravitational Waves:

- For $T_{\mu\nu} = 0$, the field equations are

$$\boxed{\partial^2 \gamma_{\mu\nu} = 0} \quad \text{in Hilbert gauge} \quad (*)$$

- Again $\exists$ residual gauge symmetry

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

with ξ_μ subject to $\partial^2 \xi_\mu = 0$

- Can use it to set $\gamma \equiv \gamma^\mu{}_\mu = 0$

because $\gamma^\mu{}_\mu \longrightarrow \gamma^\mu{}_\mu - 2\, \partial \cdot \xi$

& $\gamma^\mu{}_\mu = 2\, \partial \cdot \xi$ can be solved

because $\partial^2 \gamma = 0$.

- Gauge fixing constraints so far:

$$\partial_\mu \gamma^{\mu\nu} = 0 \quad \& \quad \gamma^\mu{}_\mu = 0$$

$$\Rightarrow \gamma_{\mu\nu} = h_{\mu\nu}$$

$$\Rightarrow \boxed{\partial_\mu h^{\mu\nu} = 0 \quad \& \quad h^\mu{}_\mu = 0} \quad (**)$$

originally, $h_{\mu\nu}$ had 10 indep. components

(**) eliminates 5 comp.

- General solution to (*) given by superpositions of plane waves

$$h_{\mu\nu} = \epsilon_{\mu\nu}\, e^{ikx} + c.c. \quad , \quad k^2 = 0$$

with polarisation tensor $\epsilon_{\mu\nu}$

- After imposing (**) it can be written as

$$\epsilon^{\mu\nu} = \begin{pmatrix} \epsilon^{00} & -\epsilon^{00} & \epsilon^{02} & \epsilon^{03} \\ & \epsilon^{00} & -\epsilon^{02} & -\epsilon^{03} \\ & & -\epsilon^{33} & \epsilon^{23} \\ & & & \epsilon^{33} \end{pmatrix}$$

for $k_\mu = (k, k, 0, 0)_\mu$

- There still $\exists$ residual gauge symm. parametrized by ξ^μ subject to $\partial^2 \xi^\mu = 0 = \partial \cdot \xi$

$\Rightarrow$ Can choose $\xi^\mu = -i\lambda^\mu e^{ikx} + c.c.$

with $\lambda^\mu = (-\lambda^1, \lambda^1, \lambda^2, \lambda^3)^\mu$

$$\lambda^1 = -\frac{\epsilon^{00}}{2k} \,,\quad \lambda^2 = \frac{\epsilon^{02}}{k} \,,\quad \lambda^3 = \frac{\epsilon^{03}}{k}$$

- This transforms $\epsilon^{\mu\nu}$ to $\epsilon^{\mu\nu}_{TT} = \epsilon^{\mu\nu} + k^\mu\lambda^\nu + k^\nu\lambda^\mu$

$$\epsilon^{\mu\nu}_{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon^{33}_{TT} & \epsilon^{23}_{TT} \\ 0 & 0 & \epsilon^{23}_{TT} & \epsilon^{33}_{TT} \end{pmatrix}$$

„transverse traceless gauge"

$\Rightarrow$ 2 propagating d.o.f.

3. Physical effect of passing GW a.k.a What's the TT gauge?

1) We have fixed the gauge in an abstract manner. But in GR, every choice of gauge corresponds to a coordinate frame that a local observer may set up.
<u>So</u>: Which coordinate frame does the TT-gauge correspond to?

2) How would we notice a (sufficiently strong) gravitational wave passing by?

It turns out: We can answer 1) by answering 2):
How does a test mass respond to passing GW?

geodesic equation: $\boxed{\nabla_u u = 0}$

where u is the 4-velocity

in components: $\boxed{u^\mu \partial_\mu u^\nu + \Gamma^\nu_{\mu\rho} u^\mu u^\rho = 0}$

If at $t = t_0$ test mass is at rest
$$u^\mu = (1, 0, 0, 0)^\mu$$

$$\Rightarrow \quad \partial_t u^\nu + \Gamma^\nu_{00} = 0$$

but $\Gamma^\nu_{00} = \frac{1}{2}\left(2\partial_t h_0{}^\mu - \partial^\mu h_{00}\right) = 0$
in TT-gauge!

$$\Rightarrow \boxed{\partial_t u = 0}$$

$\Rightarrow$ test mass at rest initially stays at rest!
(if no other forces act between them)

This answers 2):

$\boxed{\text{The TT-gauge corresponds to the coordinate frame in which (non-interacting) test masses remain at their coordinate position}}$

What about physical distances?

Let's change notation slightly:

$$\epsilon^{ij}_{TT} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon^+ & \epsilon^\times \\ 0 & \epsilon^\times & -\epsilon^+ \end{pmatrix}$$

1) Physical distance along direction of propagation:

length of space-like curve $\gamma(\lambda)$:

$$l = \int ds = \int d\lambda \sqrt{g(\gamma', \gamma')}$$

$\gamma = (0, \lambda, 0, 0)$

$$\Rightarrow l_x = \int d\lambda \sqrt{g_{11}} = \int d\lambda \sqrt{1 + h_{11}} = \Delta\lambda \Rightarrow \underline{\text{no effect}}!$$

2) Physical distance transversal to direction of prop.:

$\underline{y\text{-dir.}}$: $$l_y = \int d\lambda \sqrt{1 + h_{22}} \simeq \int d\lambda \left(1 + \tfrac{1}{2} h_{22}\right)$$
$$= \Delta\lambda + \tfrac{1}{2}\int d\lambda \left(\epsilon^+ e^{ikx} + c.c.\right)$$

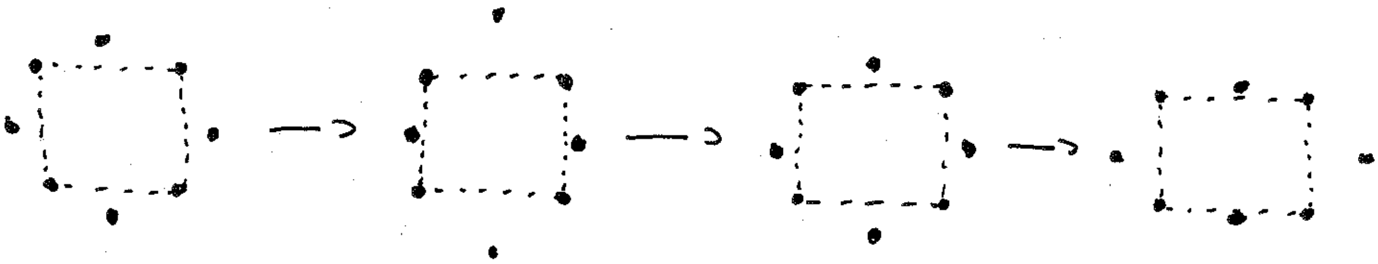
$\underline{z\text{-dir.}}$: $$l_z = \Delta\lambda - \tfrac{1}{2}\int d\lambda \left(\epsilon^+ e^{ikx} + c.c.\right)$$

3) Angle between $\vec{e}_z$ & $\vec{e}_y$:

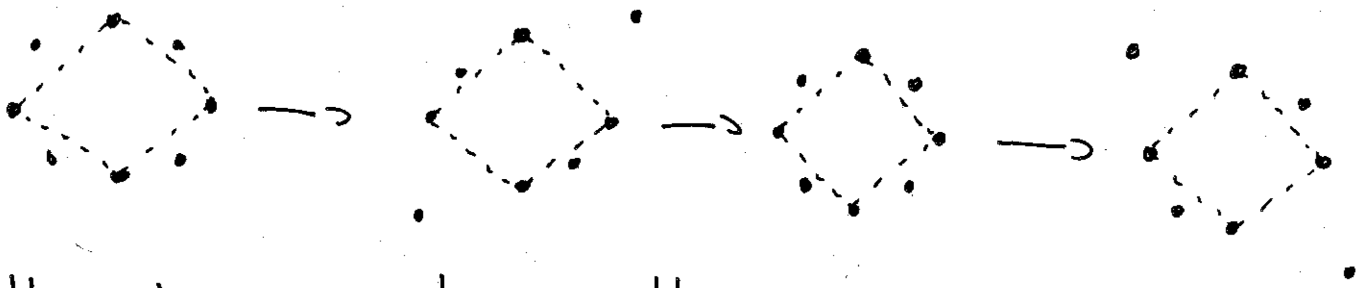
$$\cos(\alpha) = \frac{g(e_z, e_y)}{|e_z| \cdot |e_y|} = \frac{h_{32}}{\sqrt{(1+h_{22})(1-h_{22})}}$$
$$= h_{32} + \mathcal{O}(h^2) = \epsilon^\times e^{ikx} + c.c.$$

This means that test masses initially aligned on circle react as:

ϵ^+ - mode:

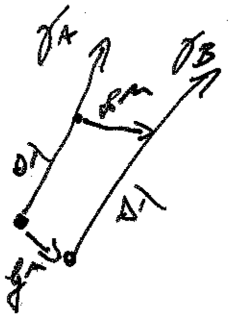


$\epsilon^\times$ - mode:



Alternative way to see this: geodesic deviation:

idea: - Consider two neighboring geodesics $\gamma_A^\mu(\lambda), \gamma_B^\mu(\lambda)$ param. by affine parameter λ that:
- are parallel at $\lambda = 0$
- have small „connecting vector" $\xi^\mu = \gamma_B^\mu - \gamma_A^\mu$



- Adopt local inertial frame at γ_A

i.e. $g_{\mu\nu}(\gamma_A) = \eta_{\mu\nu}$

$\Gamma^\rho_{\mu\nu}(\gamma_A) = 0$

- In these coord., have A & B initially at rest.

- geodesic equations: $\frac{d^2}{d\lambda^2}\gamma_A^\mu = 0$

$$\frac{d^2}{d\lambda^2}\gamma_B^\mu + \Gamma^\mu_{00}\Big|_B = 0$$

$$\Gamma^\mu_{00}\Big|_B = \xi^\nu \partial_\nu \Gamma^\mu_{00}\Big|_A \Rightarrow \boxed{\frac{d^2}{d\lambda^2}\xi^\mu + \xi^\nu \partial_\nu \Gamma^\mu_{00} = 0}$$

- Let $u^\mu = \frac{d}{d\lambda}\gamma_A^\mu$

$$\Rightarrow \nabla_u \nabla_u \xi^\mu = \nabla_u \left(\frac{d}{d\lambda}\xi + \Gamma^\mu_{0\nu}\xi^\nu \right)$$

$$= \frac{d^2}{d\lambda^2}\xi^\mu + \partial_0 \Gamma^\mu_{0\nu}\xi^\nu$$

$$\Rightarrow \nabla_u \nabla_u \xi^\mu + \xi^\nu \underbrace{\left(\partial_\nu \Gamma^\mu_{00} - \partial_0 \Gamma^\mu_{0\nu}\right)}_{= -R^\mu{}_{00\nu}} = 0$$

$$\Rightarrow \boxed{\nabla_u \nabla_u \xi^\mu = R^\mu{}_{\rho\sigma\nu} u^\rho u^\sigma \xi^\nu}$$

i.e. $\nabla_u \nabla_u \xi = R(u, \xi) u$

- Why is this better?

For plane wave in TT-gauge $R^i{}_{00j} = \frac{1}{2}\ddot{h}^{TT\,i}{}_j$

$$\Rightarrow \boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{TT\,i}{}_j \xi^j} \quad (*)$$

in local inertial frame of γ_A („proper detector frame")

[Note that we have cheated a bit:
$R^i{}_{00j} = \frac{1}{2}\ddot{h}^{TT\,i}{}_j$ valid in TT-frame
while (*) is in proper detector frame.
Since they are related by gauge transformation of $\mathcal{O}(h)$, (*) is correct.]

$\Rightarrow$ In proper detector frame, effect of passing gravitational wave captured by Newtonian force: $\boxed{F_i = \frac{m}{2}\ddot{h}_{ij}\xi^j}$ (**)

The energy & momentum carried by a GW:

Question: What's the energy momentum tensor of a gravitational wave?

Naive answer: This does not make sense!

reason: Can always go to local inertial frame at point p $\Rightarrow$ $g_{\mu\nu}|_p = \eta_{\mu\nu}$

$$\Gamma^\delta_{\mu\nu}|_p = 0$$

$\Rightarrow$ no local effect of gravitational wave.

But: Eqn. (**) shows that it acts as a force between ~~two~~ two nearby test masses!

$\Rightarrow$ energy transfer

Moreover: Expanding $S_{EH} = \frac{1}{2}\int d^4x \sqrt{-g}\, R$

to second order in $h_{\mu\nu}$ (first order piece is total derivative)

gives special relativistic theory of tensor $h_{\mu\nu}$

$$\Rightarrow \text{Can assign } T_{\mu\nu} = -2 \frac{\delta S^{(2)}_{EH}}{\delta \eta^{\mu\nu}}$$

$$= -\left(R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R^{(2)} \right)$$

that is conserved $\partial_\mu T^{\mu\nu} = 0$

Crucially, energy transfer to e.g. system of two test masses connected by a spring

can only be measured by averaging over many periods!

$$\rightsquigarrow \boxed{T^{GW}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle + \frac{1}{2}\eta_{\mu\nu}\langle R^{(2)} \rangle \underset{\text{gauge}}{\overset{TT}{=}} \frac{1}{4}\langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle}$$

4. Einstein's confusion about GW:

- Already in 1905, Poincaré realized that gravitational waves had to exist!
- 1915: Final version of field equations.
- 1916: Einstein derives wave equation in linearized theory.
- But: In 1936: Letter by Einstein to Max Born:

  "Together with a young collaborator I arrived at the interesting result that gravitational waves do not exist... "

- He even submitted a paper to Physical Review with Nathan Rosen: "Do Gravitational Waves exist?"

  Their point: Wave solutions to non-linear equations are singular. (1936)

- Referee report (probably Howard Robertson): singulirities are coordinate singularities.
- Einstein's reply:

  "Dear Sir,

  We (Mr. Rosen and I) had sent you our manuscript for publication and had not authorized you to show it to a specialist before it is printed. I see no reason to address the – in any case erroneous – comments of your anonymous expert. On the basis of this incident I prefer to publish the paper elsewhere.

  Respectfully, ... "

- They instead published in "Journal of the Franklin Institute" under the name "On Gravitational Waves."
They had abandoned their original conclusion...
(see "Einstein Versus the Physical Review" by Daniel Kennefick for more)

<u>An exact wave solution:</u>

$$ds^2 = L^2\left(e^{2\beta}dx^2 + e^{-2\beta}dy^2\right) + dz^2 - dt^2$$

$$u \equiv t - z \,, \quad v \equiv t + z \,, \quad L = L(u) \,, \quad \beta = \beta(u)$$

$\rightsquigarrow$ only non-vanishing comp. of Einstein equ.:

$$R_{uu} = -2L^{-1}\big[\underbrace{L'' + (\beta')^2 L}_{\overset{!}{=}\,0}\big]$$

linearized version: $L = 1$, $\beta(u)$ arbitrary (& small)
$\Rightarrow$ ϵ^+-pol. prop. in z-dir.

Consider pulse of duration $2T$, $|\beta' \cdot T| \ll 1$

$\Rightarrow$ <u>For $u < -T$:</u> $\quad \beta = 0 \,, \quad L = 1$

<u>For $-T < u < T$:</u> $\quad L(u) = 1 - \int_{-T}^{u}\left(\int_{-T}^{\hat{u}}(\beta'(\hat{\hat{u}}))^2 d\hat{\hat{u}}\right)d\hat{u} + \mathcal{O}((\beta' T)^4)$

<u>For $u > T$:</u> $\quad \beta = 0 \,, \quad L = 1 - \frac{u}{a} \,, \quad a = \dfrac{1 + \mathcal{O}((\beta' T)^2)}{\int_{-T}^{T} du\, (\beta')^2}$

$\Rightarrow$ $L = 0$ at $u = a$!
Moreover: $a \gg T$

<u>i.e.:</u> $ds^2 = \left(1 - \frac{u}{a}\right)^2 (dx^2 + dy^2) - du\, dv$

But: With $x = \frac{\tilde{x}}{1-\tilde{u}/a}$, $y = \frac{\tilde{y}}{1-\tilde{u}/a}$, $u = \hat{u}$

& $v = \hat{v} + \frac{\tilde{x}^2 + \tilde{y}^2}{a - \tilde{u}}$

$\Leftrightarrow ds^2 = d\tilde{x}^2 + d\tilde{y}^2 - d\hat{u}\, d\hat{v}$

which is flat space.

References:

standard textbooks/lecture notes:

- 'Gravitation' by C. Misner, K. Thorne, J. Wheeler, Princeton Univ. Press (mostly Part VIII)
- 'A First Course in General Relativity' by B. Schutz Cambridge Univ. Press (mostly ch. 9)
- 'Gravitational Waves: Volume 1: Theory and Experiments' by M. Maggiore, Oxford Univ. Press (ch. 1)
- GR lecture notes by M. Bartelmann
- — " — A. Hebecker

Some history:

- A. Einstein: 'Näherungsweise Integration der Feldgleichungen der Gravitation' (June 1916) Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften Berlin
- A. Einstein, N. Rosen: 'On gravitational waves' (Jan. 1937) Journal of the Franklin Institute, Vol. 223, issue 1, pgs 43-54
- D. Kennefick: 'Einstein Versus the Physical Review' Physics Today, Sept. 2005