Astrophysical sources of GW and millisecond pulsar timing array.

Sebastian Bruggisser

References:

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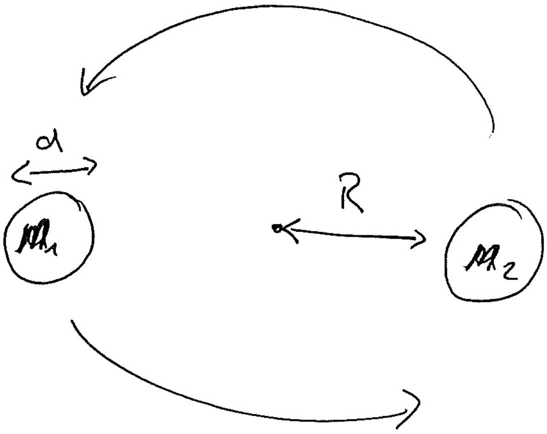
Binaries

The "Chirp"

Two massive objects in circular orbit.

$R \gg d$

$\Rightarrow$ point like mass approximation



We work in the non-relativistic regime

Orbital frequency $\quad \omega_s^2 = \dfrac{Gm}{R^3} \qquad m = m_1 + m_2$

Defining the chirp Mass:

$$\mathcal{M}_c = \mu^{3/5} m^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$$

$\Rightarrow$ from the quadrupole formula we get:

$$h_+(t) = \frac{4}{r}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\frac{1+\cos^2\theta}{2}\cos\left(2\pi f_{gw} t_{ret} + 2\varphi\right)$$

$$h_\times(t) = \frac{4}{r}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\cos\theta \; \sin\left(2\pi f_{gw} t_{ret} + 2\varphi\right)$$

where $\quad f_{gw} = \omega_{gw}/2\pi \quad ; \quad \omega_{gw} = 2\omega_s$

Backreaction of GW emission on the orbit.

The source of energy for the GWs is given by:

$$E_{orbit} = E_{pot} + E_{kin} = -\frac{G m_1 m_2}{2R} \qquad (*)$$

$\Rightarrow$ as the system radiates energy through GWs R must decrease.

$\Rightarrow$ The masses spiral in. Can we still use our formulae for circular orbits?

from $\omega_s^2 = \frac{Gm}{R^3}$

$$\Rightarrow \dot{R} = -\frac{2}{3} R \frac{\dot{\omega}_s}{\omega_s} = -\frac{2}{3} \underbrace{(\omega_s R)}_{\text{orbital velocity}} \frac{\dot{\omega}_s}{\omega_s^2}$$

$\Rightarrow$ if $\dot{\omega}_s \ll \omega_s^2$ then $|\dot{R}| \ll |\omega_s R|$ and our approximation is good.

From last week:

$$P = \frac{32}{5} \frac{c^5}{G} \left(\frac{G \mathcal{M}_c \omega_{gw}}{2c^3} \right)^{10/3}$$

$$(*) \Rightarrow E = -\left(G^2 \mathcal{M}_c^5 \omega_{gw}^2 / 32\right)^{1/3}$$

$$P = -\frac{dE}{dt}$$

$\Rightarrow$

$$\dot{\omega}_{gw} = \frac{12}{5} 2^{1/3} \left(\frac{G\mathcal{M}_c}{c^3}\right)^{5/3} \omega_{gw}^{11/3}$$

diverges at finite t

$\Rightarrow$ coalescence

solve for $f_{gw} = \omega_{gw}/2\pi$

$\Rightarrow$

$$f_{gw}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8}\left(\frac{G\mathcal{M}_c}{c^3}\right)^{-5/8}$$

where $\tau = t_{coal} - t$

t_{coal} = time at coalescence

Numerically:

$$f_{gw}(\tau) \simeq 134\,\text{Hz}\left(\frac{1.21\,M_\odot}{\mathcal{M}_c}\right)^{5/8}\left(\frac{1\text{s}}{\tau}\right)^{3/8}$$

$$\Rightarrow \tau \simeq 2.18\,\text{s}\left(\frac{1.21\,M_\odot}{\mathcal{M}_c}\right)^{5/3}\left(\frac{100\,\text{Hz}}{f_{gw}}\right)^{8/3}$$

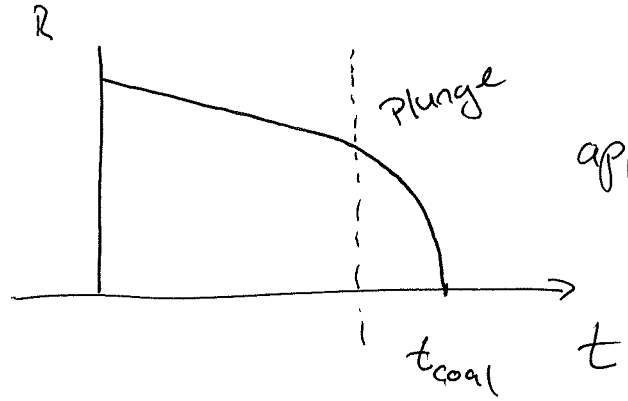
for LISA: $f_{gw} \sim 10^{-4}$ Hz

$\Rightarrow$ τ for a binary of supermassive BH $(m \sim O(10^6 M_\odot))$ at LISA is of $O(10\text{ days})$

$\Rightarrow$ Observe merger with othe telescopes and antenas!

Using $\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{\omega}_{gw}}{\omega_{gw}} = -\frac{1}{4\tau}$

$$\Rightarrow \quad R(\tau) = R_0\left(\frac{\tau}{\tau_0}\right)^{1/4}$$



approximation breaks down.

The trajectory of an inspiraling system is given by:

$$x(t) = R(t)\cos\left(\frac{\Phi(t)}{2}\right)$$
$$y(t) = R(t)\sin\left(\frac{\Phi(t)}{2}\right)$$

where $\Phi(t) = 2\int_{t_0}^{t} dt'\,\omega_s(t) = \int_{t_0}^{t} dt'\,\omega_{gw}(t')$

How does the waveform of the GWs change w.r.t the circular case?

In the quadrupole formula:

- replace $\omega_{gw} t$ with $\Phi(t)$ in trigonometric functions
- replace ω_{gw} with $\omega_{gw}(t)$ in amplitude
- We should also expect contributions from derivatives of $R(t)$ and $\omega_{gw}(t)$

however: $\dot{R}$ negligible if $\dot{\omega}_s \ll \omega_s^2$ (working assumption)

For the evolution before the plunge this is the case.

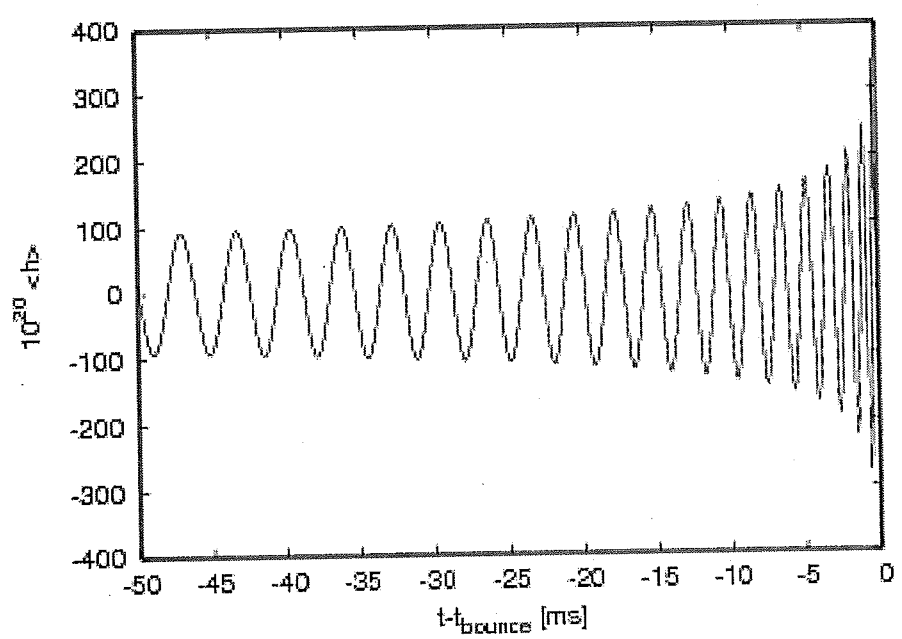
$\Rightarrow$ neglect contributions from derivatives of $R(t)$ and $\omega_{gw}(t)$.

We can write the waveforms (usin $f_{gw}(\tau)$)

$$h_+(\tau) = \frac{1}{r}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(\Phi(\tau))$$

$$h_\times(\tau) = \frac{1}{r}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/4}\left(\frac{5}{c\tau}\right)^{1/4}\cos\theta\,\sin(\Phi(\tau))$$

with $\Phi(\tau) = -2\left(\frac{5G\mathcal{M}_c}{c^3}\right)^{-5/8}\tau^{5/8} + \Phi_0$



Examples of binary systems:

BH / BH ← stellar mass $\quad f \sim 40 - 200$ Hz (main sensitivity band for ligo/virgo)

- Largest ~~single~~ signal strength

$\Rightarrow$ testing GR

needs very high precision prediction of the waveform. This is achieved through a combination of precise post-newtonian expansion calculations and simulation.

NS / NS $\quad f \sim$ kHz

- the waveform depends on the equation of state of the nuclear matter in the NS

  K. Thorn claims that LIGO / VIRGO will be able to study collisions of atomic nuclei with nucleon numbers $A \sim 10^{57}$!

- NS/NS coalescence is a popular explanation of γ-ray bursts. If this really is the primary production mechanism of γ-ray bursts, GW detectors could provide detailed information about the production of γ-rays.

NS / BH $\quad f \sim$ kHz

- If $M_{BH} \gtrsim 10\, M_\odot$ and $M_{NS} \sim M_\odot \Rightarrow$ NS is swallowed whole.

- If $M_{BH} \lesssim 10\, M_\odot$ NS will tidally disrupt. Hard to predict, because NS is torn apart when in the very relativistic regime.

- should also give information about the equation of state in the NS.

Other sources

Stellar core collaps and SN

- spherical collaps
  - naively: no GWs emmitted (quadrupole formula)

  however: huge amount of convection expected

  $\Rightarrow$ NS "boils"

  heat gets transfered from hot core to the ν-sphere where the energy is radiated of via ν emission.

  $\Rightarrow$ This boiling can generate GWs of $f \sim 100$ Hz ; $h \sim 3 \times 10^{-22} \left(\frac{3 kpc}{r} \right)$ for ~ 10 cicles.

  Interesting if managed to have correlated obs. from GW- & ν- telescopes.
- Non-axisymmetric collaps.

  Fast spinning cores can prevent the final collapsed core to be axisymmetric.

  $\Rightarrow$ formation of a bar or even two (or more) Lumps

  $\Rightarrow$ similar signal as binary coalescence.

Pulsars - Spinning neutron stars

Reasons for GW emission:

- Small ellipticity in its equatorial plane ε_e
- Principal axis of moment of inertia tensor not alligned with spin axis by some angle.
  $\Rightarrow$ wobble

Compact bodies spiraling into massive BHs

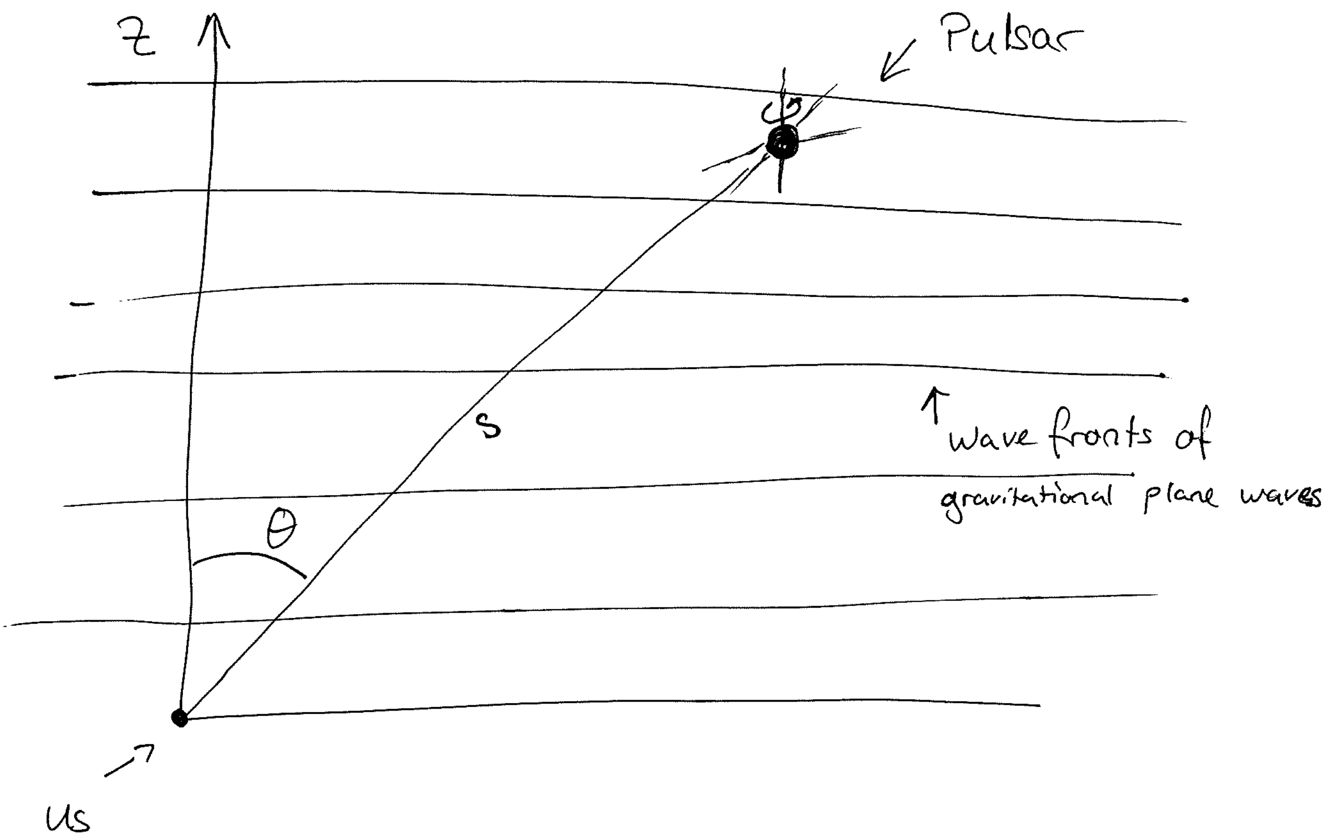
$\sim 10^5$ cycles during last year of infall

$\Rightarrow$ good mapping of BH's space-time metric (see later seminar)

Millisecond Pulsar timing array

Pulsars are extremely stable clocks if integrated over a couple of months. They are actually more precise then latest technology maser clocks.

Idea: use Doppler shift of a passing GW.



Doppler shift: $$\frac{\Delta\nu(t)}{\nu} = \frac{1}{2}(1+\cos\theta)\left[\cos 2\psi\, h_+\left(t-\frac{s}{c}\right) + \sin 2\psi\, h_\times\left(t-\frac{s}{c}\right)\right]$$

ψ depends on the choice of coordinate system in which we define h_+ and $h_\times$ $\Rightarrow$ not relevant.

Comments:

- Limited frequency band due to finite observation time. GWs with periods larger than the observation time can not be detected and neither can GWs with a period < few months because Pulsars are not stable enough on this short time scales.

$\Rightarrow$ sensitivity band $\sim$ $f \sim 10^{-9} - 10^{-8}$ Hz

- Current results based on observation of single pulsars allow to constrain GW stochastic background

$$h_0^2 \, \Omega_{GW}(f_*) < 10^{-8} \qquad (95\% \text{ C.L})$$

at $f_* \simeq 4.4 \times 10^{-9}$ Hz

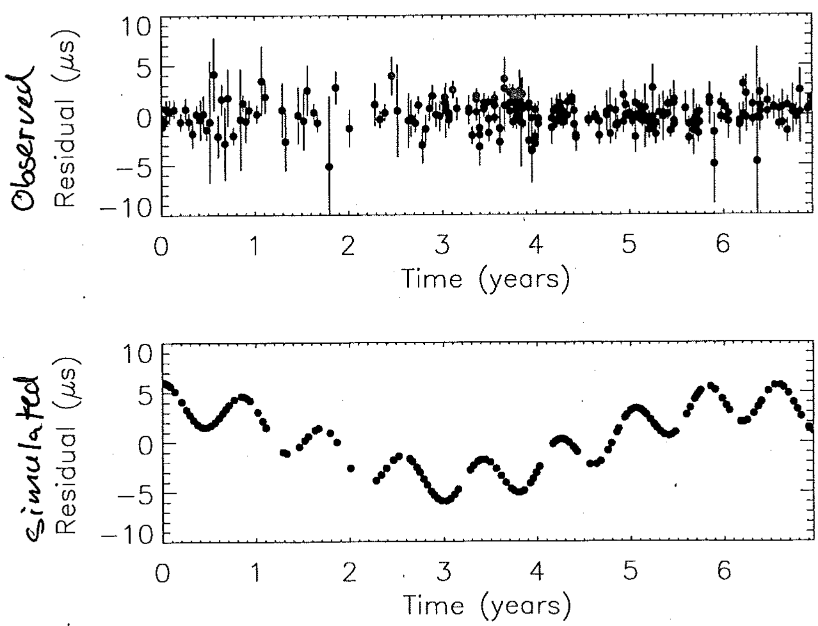
- sensitivity on h increases as $1/T$ (T = observation time)

relevant frequencies $f \sim 1/T$

and the energy density in GWs $h_0^2 \Omega_{GW} \sim h^2$

$$\Rightarrow h_0^2 \, \Omega_{GW}(f) < 10^{-8} \left(\frac{f}{f_*}\right)^2 \qquad (95\% \text{ C.L.})$$

- Rule out supermassive BH binary ($10^{10} M_\odot$) as source for observed periodic motion in 3C66B (Radio Galaxy)



Limitations:

- major uncertainties from motion of the earth.
- Clocks used to time-stamp observations not precise enough.

Way out:

Simultaneously observe multiple pulsars isotropically distributed over the sky.

$\Rightarrow$ <u>Pulsar</u> <u>timing</u> <u>arrays</u>

Advantages:

- Use 3 pulsars to track the motion of the earth.
- Use 1 pulsar as extremely precise clock.
- use 5th pulsar to detect GW!
- use cross correlation to get rid of uncorrelated timing noise individual to each pulsar.

generally: $\frac{\Delta\nu_i(t)}{\nu} = \alpha_i\, S(t) + n_i(t)$

geometrical factor for i^{th} pulsar

intrinsic noise of i^{th} pulsar

$\Rightarrow$ cross correlation

$$\alpha_i \alpha_j \langle S^2(t)\rangle + \alpha_i \langle S\, n_i\rangle + \alpha_j \langle S\, n_j\rangle + \langle n_i n_j\rangle$$

$$\xRightarrow{T\to\infty} \alpha_i \alpha_j \langle S^2(t)\rangle$$

Binaries with elliptic trajectories

In non-rel mechanics and in the com the problem reduces to a one-body problem of a particle with mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$

2 first integrals

$$L = \mu r^2 \dot{\psi}$$

$$E = \frac{1}{2} \mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{G\mu \overset{m_1+m_2}{m}}{r}$$

Integrate

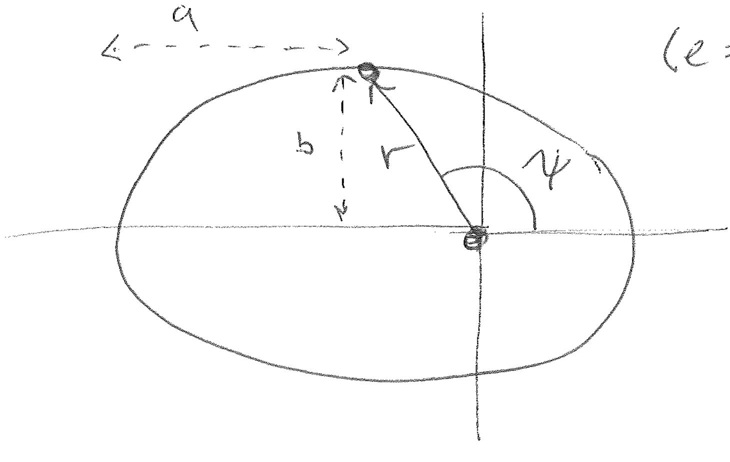
$$\Rightarrow \frac{1}{r} = \frac{1}{R}(1 + e \cos \psi)$$

$R = \frac{L^2}{G m \mu^2}$ characteristic length scale

$e^2 = 1 + \frac{2EL^2}{G^2 m^2 \mu^3}$ eccentricity

($e = 0 \Rightarrow$ circle)

($e = 1 \Rightarrow$ parabola)



$$a = \frac{R}{1 - e^2}$$

$$b = \frac{R}{(1 - e^2)^{1/2}}$$

$$\Rightarrow \quad r = \frac{a(1-e^2)}{1+e\cos\psi}$$

$$\dot{\psi} = \frac{(GmR)^{1/2}}{r^2}$$

Integrade to get:

$$r = a(1-e\cos u)$$

$$\cos\psi = \frac{\cos u - e}{1-e\cos u}$$

where $u - e\sin u = \omega_0 t = \beta \qquad (*)$

$$\omega_0^2 = \frac{GM}{a^3}$$

u is called eccentric anomaly.

$\Rightarrow$ in cartesion:

$$x = a(\cos u(t) - e)$$
$$y = b\sin u(t)$$

observe that $(*) \Rightarrow$ if $t \to t + \frac{2\pi}{\omega_0}$

then $\beta \to \beta + 2\pi$

and $u \to u + 2\pi$

$\Rightarrow$ $x(\beta)$ & $y(\beta)$ are periodic in β

$\Rightarrow$ discrete Fourier transform

$$x(\beta) = \sum_{n=-\infty}^{\infty} \tilde{x}_n \, e^{-in\beta}$$

$$y(\beta) = \sum_{n=-\infty}^{\infty} \hat{y}_n \, e^{-in\beta}$$

From $x(-\beta) = x(\beta)$ & $y(-\beta) = -y(\beta)$

$$\Rightarrow \quad x(\beta) = \sum_{n=0}^{\infty} a_n \cos(n\beta)$$

$$y(\beta) = \sum_{n=1}^{\infty} b_n \sin(n\beta)$$

from $\beta = \omega_0 t \Rightarrow n\beta = \omega_n t$ where $\omega_n = n \cdot \omega_0$

can compute a_n and b_n (Bessel functions)

However what is interesting to us is

$x^2(t)$; $y^2(t)$; $x(t)\,y(t)$

similar arguments apply here

$$x^2(t) = \sum_{n=0}^{\infty} A_n \cos \omega_n t$$

$$y^2(t) = \sum_{n=0}^{\infty} B_n \cos \omega_n t$$

$$x(t)y(t) = \sum_{n=1}^{\infty} C_n \sin \omega_n t$$

$$A_n = \frac{a^2}{n}\left[J_{n-2}(ne) - J_{n+2}(ne) - 2e\left(J_{n-1}(ne) - J_{n+1}(ne)\right)\right]$$

$$B_n = \frac{b^2}{n}\left[J_{n+2}(ne) - J_{n-2}(ne)\right]$$

$$C_n = \frac{ab}{n}\left[J_{n+2}(ne) + J_{n-2}(ne) - e\left(J_{n+1}(ne) + J_{n-1}(ne)\right)\right]$$

and

$$M_{ab} = \mu \sum_{n=0}^{\infty} \begin{pmatrix} A_n \cos\omega_n t & C_n \sin\omega_n t \\ C_n \sin\omega_n t & B_n \cos\omega_n t \end{pmatrix}$$

Remember $P \sim \langle \dddot{M}\, \dddot{M} \rangle$

$\Rightarrow$ we have to calculate t-averages

$$\langle \sin\omega_n t \sin\omega_m t \rangle$$

which vanishes for $n \neq m$

$$\Rightarrow P = \sum_{n=1}^{\infty} P_n$$

Final result:

$$P_n = \frac{32\, G^4 \mu^2 m^3}{5 c^5 a^5}\, g(n,e)$$

$$g(n,e) = \frac{n^6}{96\, a^4}\left[A_n^2(e) + B_n^2(e) + 3C_n^2(e) - A_n(e)B_n(e)\right]$$