Gravitational waves from inflation

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Outline

1. Introduction

- Inflation predicts not only scalar density perturbations but also tensor perturbations, which are potentially observable as a cosmological GW background.
- Why is the observation of inflationary GWs interesting?
  - It determines the energy scale of inflation. (assuming the single-field slow-roll paradigm)
  - Probe of the thermal history after inflation.
  - It discriminates among the variety of scenarios such as those based on modified gravity, additional matter fields, etc. (This will not be covered in this talk, see, e.g., Ref. [1].)

2. Tensor perturbations from inflation

(Derivation based on [1] and [2].)

Tensor perturbation around spatially flat FRW background

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]$$

τ : conformal time

Here and hereafter, we work in the transverse traceless (TT) gauge,

$$h_{0\mu} = 0, \quad h^i{}_i = 0, \quad \partial^i h_{ij} = 0.$$

(In the linear perturbation theory, TT metric perturbations are gauge invariant.)

Expanding the Einstein-Hilbert action up to the second order in h_{ij}, we find

$$S_T^{(2)} = \frac{M_{Pl}^2}{8}\int d\tau dx^3\, a^2\left[(h'_{ij})^2 - (\partial_k h_{ij})^2\right]. \qquad (\,' \equiv \partial/\partial\tau)$$

Let us decompose h_{ij} into plane waves:

$$h_{ij} = \int\frac{d^3k}{(2\pi)^3}\sum_{\lambda=+,\times}\epsilon^{\lambda}_{ij}(k)\, h^{\lambda}_{k}(\tau)\, e^{ik\cdot x}$$

where $\epsilon^{\lambda}_{ii} = k^i\epsilon^{\lambda}_{ij} = 0$ and $\epsilon^{\lambda}_{ij}\epsilon^{\lambda'}_{ij} = 2\delta_{\lambda\lambda'}$.

Furthermore, we define the canonically normalized field

$$v^{\lambda}_{k} \equiv \frac{a}{2}M_{Pl}\, h^{\lambda}_{k}.$$

Then,

$$S_T^{(2)} = \sum_{\lambda}\frac{1}{2}\int d\tau\, d^3k\left[(v^{\lambda\prime}_{k})^2 - \left(k^2 - \frac{a''}{a}\right)(v^{\lambda}_{k})^2\right].$$

The equation derived from this action describes the propagation of cosmological perturbations. (Mukhanov-Sasaki equation)

In quasi-de Sitter background with $\epsilon \equiv -\dot{H}/H^2 \neq 0$ but $\epsilon \ll 1$,

$$aH \simeq -(1+\epsilon)\frac{1}{\tau}\,, \quad \frac{a''}{a} = (2-\epsilon)a^2H^2 \simeq (2+3\epsilon)\frac{1}{\tau^2}.$$

The general solution reads

$$v_k(\tau) = \sqrt{-k\tau}\left[C_1 H_\nu^{(1)}(-k\tau) + C_2 H_\nu^{(2)}(-k\tau)\right]$$

with $\nu \simeq \frac{3}{2} + \epsilon$ $\qquad$ $H_\nu^{(1)}, H_\nu^{(2)}$: Hankel functions

Initial condition is fixed by Bunch Davis vacuum:

Minkowski vacuum in the sub-horizon limit $v_k \xrightarrow{|k\tau|\gg 1} e^{-ik\tau}/\sqrt{2k}$.

$$\rightarrow \quad v_k = \frac{\sqrt{\pi}}{2} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} \sqrt{-\tau}\, H_\nu^{(1)}(-k\tau)$$

These modes are frozen in the super-horizon limit:

$$v_k'' - \frac{a''}{a} v_k = 0 \quad \text{for} \quad |k\tau| \ll 1$$

$$\rightarrow \quad v_k \propto a \qquad \left(\text{and} \quad v_k \propto a\int\frac{d\tau}{a^2} \text{ : decaying solution}\right)$$

$$\rightarrow \quad h_k \propto a^{-1} v_k \propto \text{const}.$$

We define tensor power spectrum

$$\langle h_{ij}(\tau,x)\, h^{ij}(\tau,x)\rangle = \int \frac{dk}{k} P_T(k).$$

where

$$P_T(k) = \frac{k^3}{\pi^2}\sum_\lambda \langle |h_k^\lambda|^2\rangle$$

Substituting the super-horizon limit of h_k^λ, we obtain

$$\boxed{\begin{aligned} P_T(k) &= \frac{8}{M_{Pl}^2}\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{-2\epsilon} \\ &\simeq \frac{8}{M_{Pl}^2}\left(\frac{H_*}{2\pi}\right)^2 \end{aligned}}$$

(evaluated at horizon crossing $k = a_* H_*$).

- Almost scale invariant power spectrum.

At 1st order in the slow-roll expansion, tilt is given by

$$n_T(k_*) = -2\epsilon_*.$$

Note that $\epsilon = -\dot{H}/H^2$

→ If $\dot{\rho} \leq 0$ and $H > 0$, Friedman eq. implies $\dot{H} \leq 0$.

→ $\epsilon \geq 0$ and hence $n_T \leq 0$: red spectrum.

- Tensor-to-scalar ratio

$$r(k_*) \equiv \frac{P_T(k_*)}{P_s(k_*)},$$ where $$P_s(k_*) = \frac{1}{8\pi^2}\frac{H^2}{M_{Pl}^2}\frac{1}{\epsilon}\bigg|_{k=aH}$$

is the scalar power spectrum.

From CMB obs. we have $P_s \sim 10^{-9}$

Combining with $P_T \propto H^2 \propto V_{inf}$, we obtain

$$\boxed{V_{inf} \sim \left(\frac{r}{0.01}\right)(10^{16}\,\mathrm{GeV})^4}$$

c.f. $r < 0.10$ (95% CL, Planck TT + low P) at $k = 0.002\,\mathrm{Mpc}^{-1}$ [3].

Measurement of r provides the energy scale of inflation.

- Consistency relation

We can also write $r = 16\epsilon$

$$\rightarrow \quad \boxed{r = -8n_T}$$

The violation of the consistency relation implies a departure from the standard single field slow roll scenario.

3. Transfer function and energy density

After horizon exit, the amplitude of GWs is frozen ($h^\lambda_k \approx$ const).

It starts to evolve again when the mode re-enters the horizon after inflation.

Hence we can write the amplitude of GWs after inflation as

$$\boxed{h^\lambda_k(\tau) \equiv h^\lambda_{k,\mathrm{prim}}\, \mathcal{T}(\tau, k)}$$

where $h^\lambda_{k,\mathrm{prim}}$ is the primordial GW mode satisfying

$$P_T(k) = \frac{k^3}{\pi^2} \sum_\lambda \langle |h^\lambda_{k,\mathrm{prim}}|^2 \rangle = \frac{8}{M_{Pl}^2}\left(\frac{H_*}{2\pi}\right)^2$$

and $\mathcal{T}(\tau, k)$ is the <u>transfer function</u> describing the subhorizon evolution.

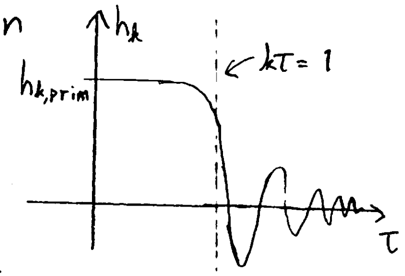
- $\mathcal{T}(\tau, k)$ is obtained by solving the wave equation

$$\ddot{h}^\lambda_k + 3H\dot{h}^\lambda_k + \frac{k^2}{a^2} h^\lambda_k = 16\pi G\, \Pi^\lambda_k$$

  with the initial conditions $\mathcal{T} \to 1$ and $\mathcal{T}' \to 0$ as $k\tau \to 0$.

- If there is no anisotropic stress $\Pi^\lambda_k = 0$, sub-horizon evolution is approximated by the WKB solution

$$\mathcal{T}(\tau, k) \propto a^{-1} e^{\pm ik\tau}.$$



3.1. Energy density of gravitational waves

Energy density of GWs is given by [4]

$$\rho_{gw}(\tau) = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle = \frac{\langle h'_{ij} h'^{ij}\rangle}{32\pi G\, a^2(\tau)}.$$

Substituting $h_k = h_{k,\mathrm{prim}}\, \mathcal{T}(\tau, k)$, we find

$$\rho_{gw}(\tau) = \frac{1}{32\pi G a^2} \int d\ln k \; P_T(k) \left[\mathcal{T}'(\tau, k) \right]^2 .$$

The spectrum of GWs is described in terms of the following quantity :

$$\boxed{\begin{aligned} \Omega_{gw}(\tau, k) &= \frac{1}{\rho_c(\tau)} \frac{d\rho_{gw}(\tau, k)}{d\ln k} \\ &= \frac{1}{12 a^2(\tau) H^2(\tau)} P_T(k) \left[\mathcal{T}'(\tau, k) \right]^2 \end{aligned}}$$

where $\rho_c = 3H^2/8\pi G$ is the total energy density of the universe.

Let us focus on the modes entering the horizon during the radiation dominated era, and estimate the typical amplitude of GWs.

To this end, we use the approximation that $\mathcal{T}(\tau, k)$ is replaced by the WKB solution just after the horizon crossing $\tau = \tau_{hc}$,

$$\left[\mathcal{T}'(\tau, k) \right]^2 \approx \frac{k^2}{2} \left(\frac{a(\tau_{hc})}{a(\tau)} \right)^2 \underset{k = a(\tau_{hc}) H(\tau_{hc})}{=} \frac{a^4(\tau_{hc}) H^2(\tau_{hc})}{2 a^2(\tau)}$$

$\mathcal{T} \propto a^{-1} e^{\pm ik\tau}$ and average over rapidly oscillating factor to obtain $1/2$.

Note : This oscillation is not a spurious feature but a genuine feature. Inflationary GW is coherent in temporal phase. (Modes are "synchronized" at inflationary epoch.) However, direct detection experiments might not resolve this oscillation since $k\tau_0 \gg 1$. (But it might be relevant at the CMB scale.)

We also note that ("0" means the quantity at the present time)

$$\frac{H_{hc}^2}{H_0^2} = \Omega_R \frac{g_{*,hc} T_{hc}^4}{g_{*,0} T_0^4} = \Omega_R \left(\frac{g_{*,hc}}{g_{*,0}} \right) \left(\frac{g_{*,hc}}{g_{*s,0}} \right)^{-4/3} \left(\frac{a_0}{a_{hc}} \right)^4$$

where $\Omega_R = \rho_R(\tau_0)/\rho_c(\tau_0) = 4.15 \times 10^{-5} h^{-2}$

Then we obtain

$$\Omega_{gw}(\tau_0, k) \approx \frac{1}{24}\Omega_R \left(\frac{g_{*,hc}}{g_{*,0}}\right)\left(\frac{g_{*s,hc}}{g_{*s,0}}\right)^{-4/3} P_T(k).$$

The spectrum is almost flat for modes entering the horizon during the RD era.

Furthermore, since $P_T \propto H_*^2 \propto V_{inf}$, we obtain

$$\boxed{\Omega_{gw}(\tau_0, k)h^2 \sim 10^{-17} \left(\frac{g_{*,hc}}{106.75}\right)\left(\frac{g_{*s,hc}}{106.75}\right)^{-4/3}\left(\frac{V_{inf}^{1/4}}{10^{16}\,\mathrm{GeV}}\right)^4}$$

The frequency of GWs is given by

$$\boxed{f = \frac{k}{2\pi a_0} = \frac{H_{hc}}{2\pi}\frac{a_{hc}}{a_0} = 2.65\,\mathrm{Hz}\left(\frac{g_{*,hc}}{106.75}\right)^{1/2}\left(\frac{g_{*s,hc}}{106.75}\right)^{-1/3}\left(\frac{T_{hc}}{10^8\,\mathrm{GeV}}\right)}$$

3.2 Equation of state of the early universe

Inflationary GW spectrum is sensitive to the e.o.s. of the early universe [5,6]

Let us differentiate Ω_{gw} with respect to the scale factor.

$$\frac{d\ln\Omega_{gw}}{d\ln a} = \frac{d\ln\rho_{gw}}{d\ln a} - \frac{d\ln\rho_c}{d\ln a}$$

The evolution of the total energy density is determined by the e.o.s.

$$\frac{d\ln\rho_c}{d\ln a} = -3(1+w), \qquad w = \frac{p}{\rho_c}$$

The energy density of GWs after the horizon entry evolves as

$$\rho_{gw} \propto a^{-2}[\mathcal{I}']^2 \propto a^{-4} \rightarrow \frac{d\ln\rho_{gw}}{d\ln a} = -4$$

Combining them, we obtain

$$\Omega_{gw} = \exp\left[\int_{a_{hc}}^{a}(3w-1)\,d\ln a\right]\Omega_{gw}(\tau_{hc}, k)$$

where $\Omega_{gw}(\tau_{hc}, k) \sim P_T(k)$.

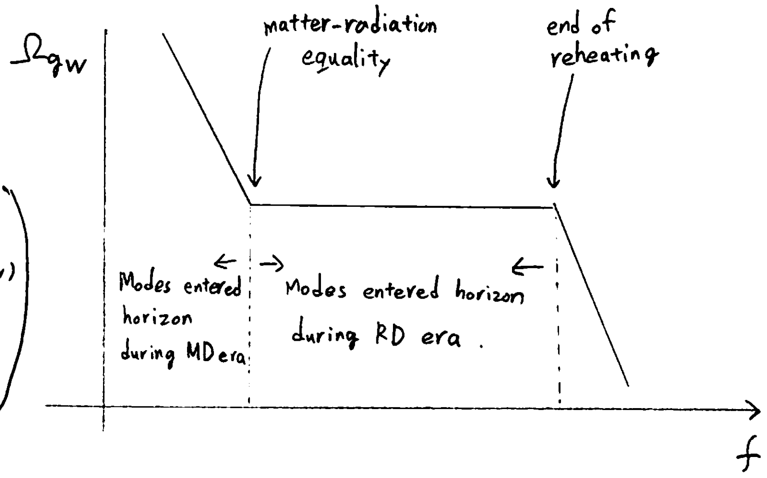
There is a frequency dependence even if $P_T(k) = \text{const}$.

$$\frac{\Omega_{gw}(k_2)}{\Omega_{gw}(k_1)} = \exp\left[\int_{a_{hc,2}}^{a_{hc,1}} (3w-1)\, d\ln a\right]$$

If w is constant for $\tau_{hc,1} \le \tau \le \tau_{hc,2}$,

$$\Omega_{gw} \propto k^{\frac{2(3w-1)}{3w+1}}$$

$$\left(\begin{array}{l} \Omega_{gw} \propto a^{-(3w-1)} \\ a \propto H^{-\frac{2}{3(1+w)}} \leftarrow \rho \propto H^2 \propto a^{-3(1+w)} \\ \quad \propto k/H \\ \rightarrow H \propto k^{\frac{3(1+w)}{3w+1}} \end{array}\right)$$



For instance,

$\Omega_{gw} \propto k^0$ for modes entered horizon during RD era ($w = 1/3$)

$\Omega_{gw} \propto k^{-2}$ for modes entered horizon during MD era ($w = 0$)

Comments

- Spectrum falls off at the mode entering into the horizon at the end of reheating

$$f_{RH} \sim \mathcal{O}(1)\,\mathrm{Hz}\left(\frac{T_{RH}}{10^8\,\mathrm{GeV}}\right)$$

$\rightarrow$ This feature can be used to probe the reheating temperature [7].

- Change of the relativistic d.o.f. leads to the non-trivial spectrum [8].

$$\Omega_{gw} \propto g_{*,hc}\, g_{*s,hc}^{-4/3} \sim g_{*,hc}^{-1/3}$$

For instance,

- At e^+e^- annihilation $T \sim 0.5\,\mathrm{MeV}$ ($f \sim 10^{-11}\,\mathrm{Hz}$)

$$\left(\frac{3.91}{10.75}\right)^{1/3} \sim 0.7$$

- At QCD phase transition $T \sim 100\,\mathrm{MeV}$ ($f \sim 10^{-9}\,\mathrm{Hz}$)

$$\left(\frac{10}{80}\right)^{1/3} \sim 0.5$$

4. Collisionless damping due to neutrino free-streaming

Neutrinos decouple from the thermal bath when $T \lesssim 2\,\mathrm{MeV}$, and subsequently they behave as free-streaming particles.

The existence of inflationary GWs causes tensor type perturbations in the neutrino distribution function $F(t, x^i, P_i)$, which give rise to the anisotropic stress and the energy flow between neutrinos and GWs:

Weinberg (2004) [9]

$$h_{ij} \longrightarrow F = \bar{F} + \delta F \longrightarrow \Pi_{ij}$$

Consider a free streaming neutrino with a momentum P^μ.

Its evolution is described by the geodesic equation:

$$\frac{dP^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta \qquad \Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\nu}}{2}\left[\partial_\beta g_{\alpha\nu} + \partial_\alpha g_{\beta\nu} - \partial_\nu g_{\alpha\beta}\right]$$

At the first order in h_{ij}, we find

$$\frac{1}{P^0}\frac{dP^0}{dt} = -\frac{\dot{a}}{a} - \frac{1}{2}\left(\frac{a}{P^0}\right)^2 p^i p^j \frac{\partial h_{ij}}{\partial t}$$

Redshift of neutrino energy

Energy flow:
Neutrino loses energy if $\frac{\partial h_{ij}}{\partial t} > 0$.
Neutrino gains energy if $\frac{\partial h_{ij}}{\partial t} < 0$

- The energy flow occurs around the time of the horizon crossing:

  $\frac{\partial h_{ij}}{\partial t} = 0$ before the horizon crossing.

  $\frac{\partial h_{ij}}{\partial t} \rightarrow$ rapidly oscillate after the horizon crossing.

- Neutrinos mostly gain energy and GWs lose energy, since $\partial h_{ij}/\partial t < 0$ just after the horizon crossing.

  $\rightarrow$ damping of Ω_{gw}.

The evolution of the neutrino distribution function after their decoupling is described by the collisionless Boltzmann equation

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{dx^i}{dt}\frac{\partial F}{\partial x^i} + \frac{dP_i}{dt}\frac{\partial F}{\partial P_i} = 0$$

where $F = \bar{F} + \delta F(t, x^i, P_i)$, $\bar{F} = \dfrac{g_\nu}{e^{P^0/T}+1}$, $g_\nu = 6$.

At linear order in perturbation theory, one can find the following solution [8, 9] :

$$\delta F = \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^3} f_{\lambda,k}(\tau, q, \mu)\, e^{ik\cdot x} \epsilon^\lambda_{ij}\gamma^i\gamma^j$$

$$f_k = \frac{q}{2}\frac{\partial \bar{F}}{\partial q}\int_{\tau_{\nu dec}}^{\tau} d\tau' h'_k(\tau')\, e^{-i\mu k(\tau-\tau')}$$

$q \equiv q^0 = aP^0$

γ^i : direction cosines of P^i

$\mu \equiv \gamma^i k_i/k$.

$\tau_{\nu dec}$: conformal time at the neutrino decoupling.

($\delta F = 0$ for $\tau \leq \tau_{\nu dec}$ is assumed.)

Substituting this solution into the neutrino stress-energy tensor,

$$T^{(\nu)}_{\mu\nu} = \frac{1}{\sqrt{-g}}\int d^3P\, F\, \frac{P_\mu P_\nu}{P^0}$$

we obtain the anisotropic stress

$$\Pi_{ij} = \sum_{\lambda=+,\times}\int \frac{d^3k}{(2\pi)^3}\Pi^\lambda_k\, e^{ik\cdot x}\epsilon^\lambda_{ij}$$

$T_{ij} = P g_{ij} + a^2\Pi_{ij}$

$$\boxed{\Pi^\lambda_k = -4\rho_\nu(\tau)\int_{\tau_{\nu dec}}^{\tau} d\tau' \frac{j_2[k(\tau-\tau')]}{k^2(\tau-\tau')^2}h'^\lambda_k(\tau')}$$

$\rho_\nu(\tau)$: energy density of neutrinos

$j_n(x)$: spherical Bessel function

$$j_2(x) = \frac{1}{x}\left[\left(\frac{3}{x^2}-1\right)\sin x - \frac{3}{x}\cos x\right]$$

This contributes to the right-hand side of the propagation equation of GWs :

$$\ddot{h}_k^\lambda + 3H\dot{h}_k^\lambda + \frac{k^2}{a^2}h_k^\lambda = 16\pi G \Pi_k^\lambda$$

- Modes that entered the horizon before the neutrino decoupling ($f \gtrsim 10^{-10}$ Hz) are not suppressed, since h_k' is rapidly oscillating already at $\tau = \tau_{\nu dec}$.

- Modes that entered the horizon after the matter-radiation equality ($f \lesssim 10^{-16}$ Hz) are also not suppressed, since the neutrino energy density $\rho_\nu(\tau)$ becomes much smaller than the total energy density $\rho_c(\tau)$.

$$\mathrm{RHS} = 16\pi G \Pi_k \propto G\rho_c \cdot \frac{\rho_\nu}{\rho_c} \sim H^2 \frac{\rho_\nu}{\rho_c} \ll H^2 .$$

- As a result, Ω_{gw} is damped only in the range 10^{-16} Hz $\lesssim f \lesssim 10^{-10}$ Hz by 35.5%.

- The suppression factor depends on the fraction of free-streaming particles $f_\nu = \rho_\nu/\rho_c$ [6].

References

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Wave equation

- The evolution of tensor perturbations is governed by the integro-differential equation: Watanabe and Komatsu (2006) [8]

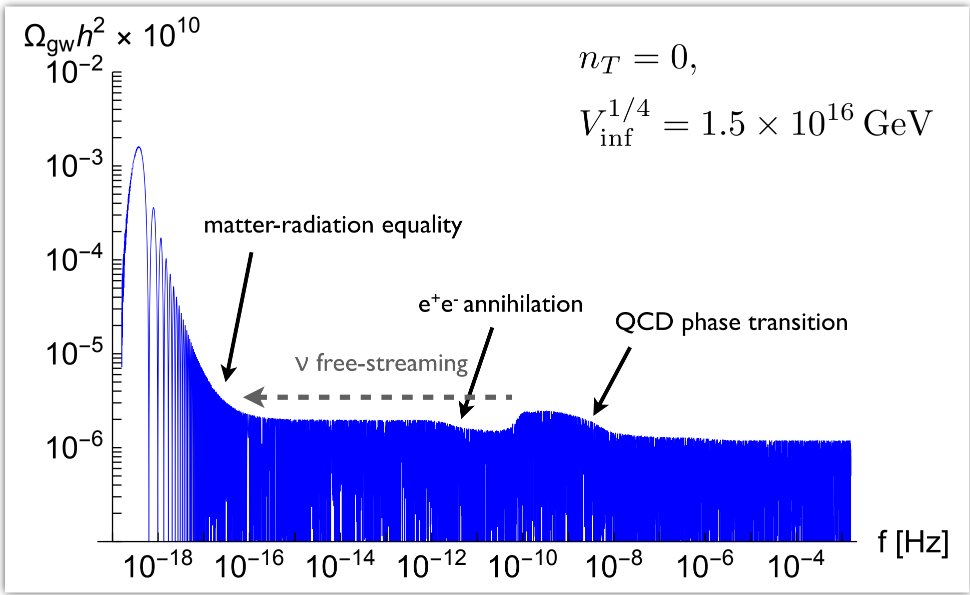
$$h_k'' + \left(\frac{2a'}{a}\right) h_k' + k^2 h_k$$
$$= -24 f_\nu(\tau) \left[\frac{a'(\tau)}{a(\tau)}\right]^2 \int_{\tau_{\nu\,\mathrm{dec}}}^{\tau} d\tau' \left[\frac{j_2[k(\tau-\tau')]}{k^2(\tau-\tau')^2}\right] h_k'(\tau')$$

where $f_\nu(\tau) \equiv \rho_\nu(\tau)/\rho_c(\tau)$

- This can be solved numerically with initial conditions:

$$h_k^\lambda(0) = h_{k,\mathrm{prim}}^\lambda \quad \text{and} \quad \frac{\partial h_k^\lambda}{\partial \tau}(0) = 0$$

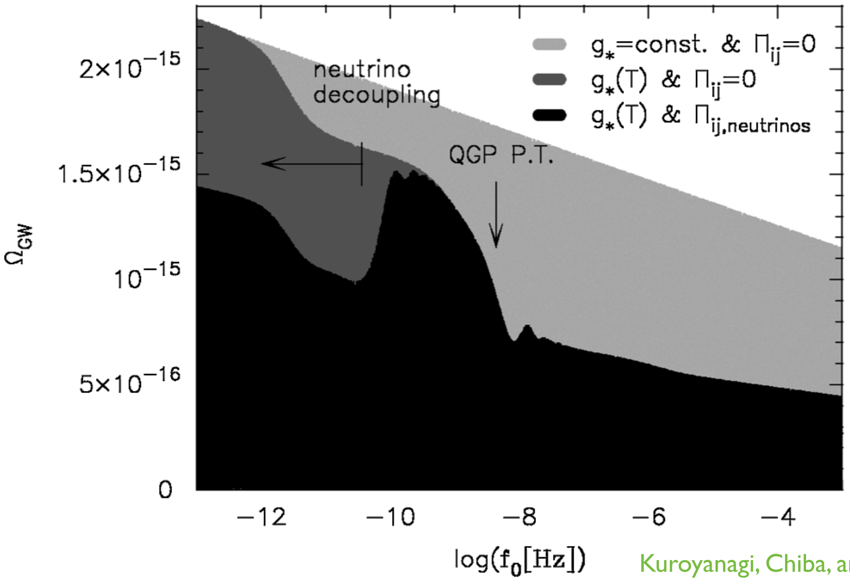
Gravitational wave spectrum



Various events in the early universe are imprinted on the spectrum of GWs.

$$f = \frac{H_{\mathrm{hc}}}{2\pi}\frac{a_{\mathrm{hc}}}{a_0} = 2.65\,\mathrm{Hz}\left(\frac{g_{*\rho,\mathrm{hc}}}{106.75}\right)^{1/2}\left(\frac{g_{*s,\mathrm{hc}}}{106.75}\right)^{-1/3}\left(\frac{T_{\mathrm{hc}}}{10^8\,\mathrm{GeV}}\right)$$

Effects of relativistic d.o.f. and neutrino anisotropic stress



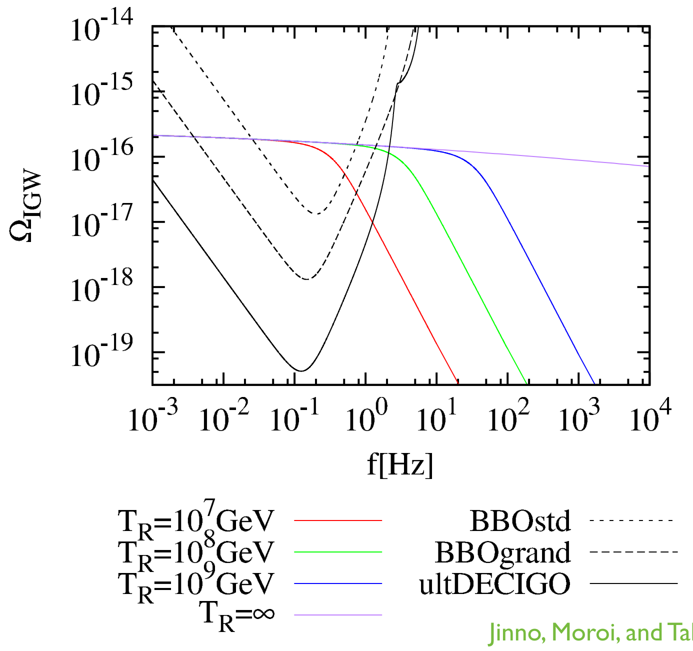
Kuroyanagi, Chiba, and Sugiyama (2009) [10]

- Frequency range relevant to pulser timing observations:

$$f \simeq \mathcal{O}(10^{-9}\text{–}10^{-8})\,\mathrm{Hz}$$

- Difficult to observe as the amplitude is too small.

Probing reheating temperature



Jinno, Moroi, and Takahashi (2014) [11]

- A detailed statistical analysis was performed to estimate how well we can determine the shape of GWs, taking Ω_{gw}, n_T, and T_R as fundamental parameters.
- T_R can be determined with the error of $\lesssim 30\%$ if $T_R \sim 10^{6.5}\text{–}10^{7.5}$ GeV.