Modular discretization, fast scrambling and eigenstate thermalization for holographic systems

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1. Work based on arXiv : math-ph/9805012, 1306.5670 [hep-th], 1504.00483 [hep-th], 1608.07845 [hep-th] and in progress

Outline

Introduction–Motivations

- Construct a natural lattice formulation for the AdS/CFT correspondence
- Study the dynamics of probes of bulk and boundary
- Focus on some salient properties of the quantum dynamics : scrambling and thermalization.
- These properties are of interest beyond the subject of black holes as such and can be probed in many other physical systems.

Work done in collaboration with Minos Axenides and Emmanuel Floratos

A lattice approach to holographic systems

This is based on arXiv : 1306.5670 [hep-th] (cf. arXiv : 1504.00483 [hep-th] for a summary) The idea is to set up a lattice on the phase space of the system of interest–but in a way that preserves the isometries of the space and to use group elements to represent the dynamics. The scaling limit is a much more delicate issue than for lattices on a fixed spacetime geometry and topology.

In previous work we studied compact phase spaces, namely tori ; now we have extended the approach to non–compact phase spaces. We've focused on the single–sheet hyperboloid, that describes the AdS_2 manifold, because it describes the radial and temporal part of the near horizon geometry of extremal black holes, that factorizes from the charge manifold.

A modular lattice : the points

$$x_0^2 + x_1^2 - x_2^2 \equiv 1 \bmod N \quad (1)$$

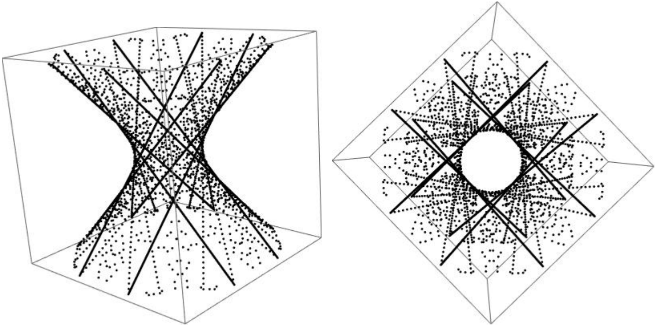

Figure – The rational points on $AdS_2[N]$–side view and top view, for $N = 47$.

A modular lattice : the links

The way for going from one point to another is by the Weyl map–mod N :

$$\mathsf{X}_{n+1} = \mathsf{A}\mathsf{X}_n\mathsf{A}^{-1} = \mathsf{A}^n\mathsf{X}_0\mathsf{A}^{-n} \bmod N$$

with $\mathsf{A} \in SL(2, \mathbb{Z}_N)/SO(1,1,\mathbb{Z}_N)$.
The elementary transformations are translations along the two light cone generators, L and R

$$\mathsf{L} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \mathsf{R} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

These, also, generate the braid group. Their products describe more complicated motion. The product $\mathsf{L}\mathsf{R}^{-1}$ is the Arnol'd cat map. And any A of interest can be written as a product of the L's and the R's and their inverses.

The Arnol'd cat map

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

It generates the Fibonacci numbers! And mod N it has a period, N

$$A^{T(N)} \equiv I \bmod N$$

Fast quantum maps on the torus and the hyperboloid

This is based on math-ph/9805012 and arXiv : 1608.07845 [hep-th]

The matrix $\mathrm{A} \in SL(2,\mathbb{Z}_N)/SO(1,1,\mathbb{Z}_N)$ is the classical evolution operator, for point–like probes of the AdS_2 geometry.

It is possible to construct explicitly and effectively the corresponding quantum evolution operator, $U(\mathrm{A})$ for wavepackets :

$$\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod p \qquad \Leftrightarrow$$

$$U(\mathrm{A})_{k,l} = \frac{(-2c|p)}{\sqrt{p}} \times \left\{ \begin{array}{c} 1 \\ -\mathrm{i} \end{array} \right\} \omega_p^{-\frac{ak^2-2kl+dl^2}{2c}}$$

The fundamental property of $U(\mathrm{A})$ is

$$U(\mathrm{AB}) = U(\mathrm{A})U(\mathrm{B}) \Rightarrow U(\mathrm{A}^k) = [U(\mathrm{A})]^k$$

Random products of matrices

When the classical evolution operator, A depends explicitly on time, then A^n, in fact, means

$$\prod_{k=0}^{n} \mathsf{A}_k$$

The quantum evolution operator, still factorizes

$$U\left(\prod_{k=0}^{n} \mathsf{A}_k\right) = \prod_{k=1}^{n} U(\mathsf{A}_k)$$

but if A is a random matrix, the average over the product can lead to subtle effects, that have been extensively studied in the context of deterministic chaotic systems.

When $N = p_1^{r_1} \cdots p_k^{r_k}$

When the number of points isn't a prime, the representation factorizes over the prime factors, since the group factorizes, also :

$$SL(2, \mathbb{Z}_N) = SL(2, \mathbb{Z}_{p_1^{r_1}}) \times \cdots \times SL(2, \mathbb{Z}_{p_k^{r_k}})$$

A prime example of a tensor network, since which of the N states go into which grouping is not globally defined and does lead to entanglement.

Fast scrambling

What's fast scrambling ?

- The **scrambling time**, $t_{\text{scrambling}}$: The time required for a wavepacket to attain uniform spreading.
- This time is bounded from above by (half) the period of the map :

$$t_{\text{scrambling}} = t_{\text{mixing}} \leq \frac{T(N)}{2}$$

  We find, numerically, that this bound is saturated.
- If $N = f_{2k}$, then $T(N) = 2k$ (Falk and Dyson). Short periods mean non–trivial conservation laws. This highlights the sensitive dependence of the dynamics on the arithmetic properties of the discretization–that, for the dynamics of probes of black holes, can be of physical import.

The period of the Arnol'd cat map

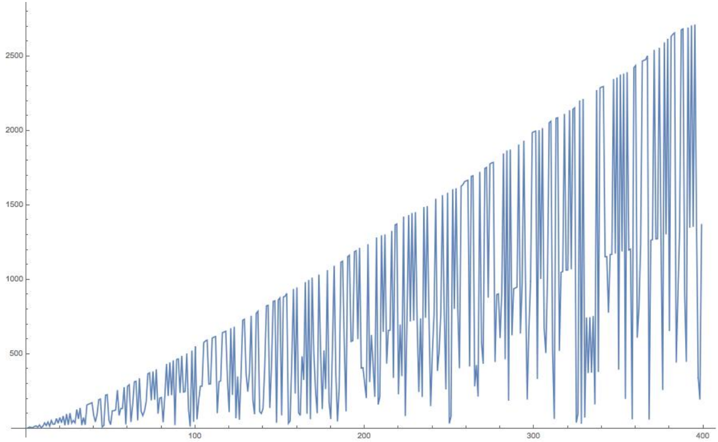

Figure – The period of the Arnol'd cat map as a function of the order, k, of the prime p_k, for the first 400 primes.

The random walk along the space–like direction

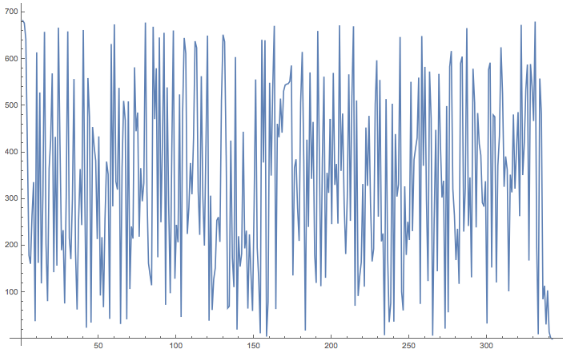

Figure – $x_2^{(n)}$ mod 683 as a function of n. The period is found to be equal to $684 = N + 1$.

The quantum cat on extremal black hole horizons

This is based on arXiv : 1608.07845 [hep-th]

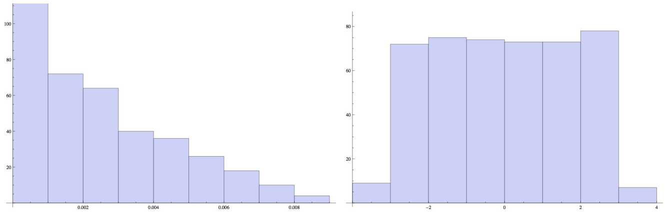

Figure – The bin values for the squared amplitude and of the phases of the ground state, for $p = 461$.

Evolution of a Gaussian wavepacket of short period ($T(N) = 23$)

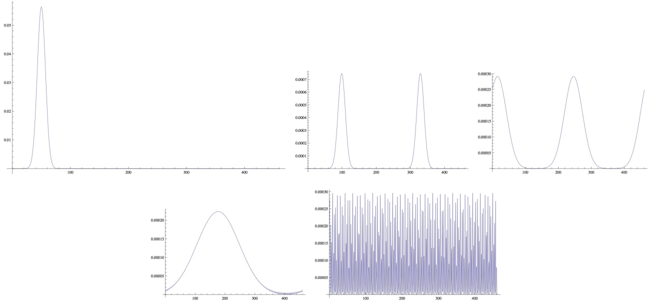

Figure – Evolution of a Gaussian wavepacket, for $t = 1, 2, 3, 4$. For $t = 3$ we remark the appearence of oscillations, precursors to chaotic behavior. For $11 > t \geq 4$, we observe complete delocalization, but remark that patterns still persist.

Conclusions–Perspectives

- We have constructed consistent lattice formulations for the dynamics of the single-particle probes of extremal black hole spacetimes and provided a concrete example of the AdS_2/CFT_1 correspondence. The corresponding lattice models are generalized Eguchi–Kawai models.
- Tensor networks appear naturally, in the context of the dynamics of point–like probes, of extremal black hole spacetimes.
- We have shown that the dynamics realizes fast scrambling, consistent with deterministic chaos.
- We have found evidence for the validity of the eigenstate thermalization hypothesis, by studying the quantum cat map, that describes the deterministic, but random, way a wavepacket spreads in the AdS_2 bulk.

Conclusions–Perspectives

- When the number of degrees of freedom is a an odd integer, $N = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, the evolution operator factorizes over the prime factors ; entanglement appears, when focusing on how a p^r size block factorizes into separate factors ; or, when trying into sum over a block, whose size isn't equal to any particular prime power factor.