Holography and Quantum Information:

from Novel Probes of Bulk Matter (and Geometry of Causal Diamonds) to Complexity of Quantum Fields

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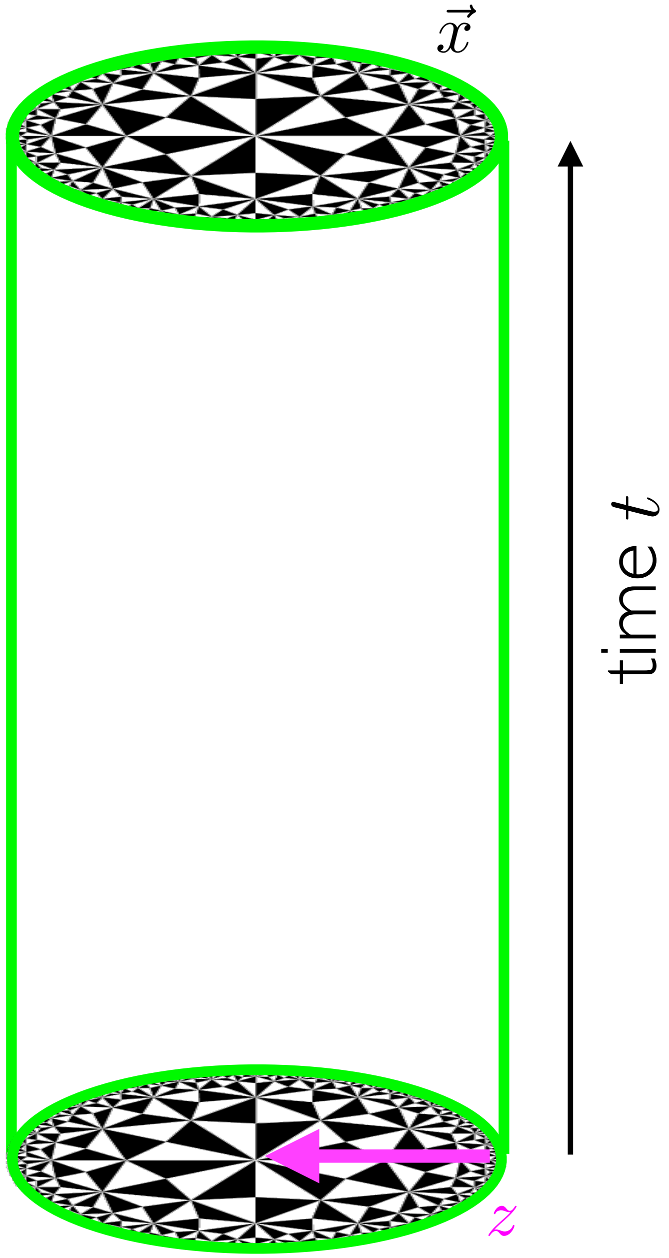
based on **1509.00113** and **1606.05344** with de Boer, Haehl, Myers and Neiman

and **1707.08582** with Chapman, Marrochio and Pastawski

Introduction

Holography (a.k.a. AdS$_{d+2}$/CFT$_{d+1}$ or Gauge-Gravity Duality)



$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left(dz^2 - dt^2 + d\vec{x}^2\right)$$

certain QFTs$_{d+1}$ (in particular CFTs$_{d+1}$) with large number of strongly interacting dofs defined on the bdry

$$Z_{QFT}[J] = e^{i\, S_{\rm gravity}[J]}$$

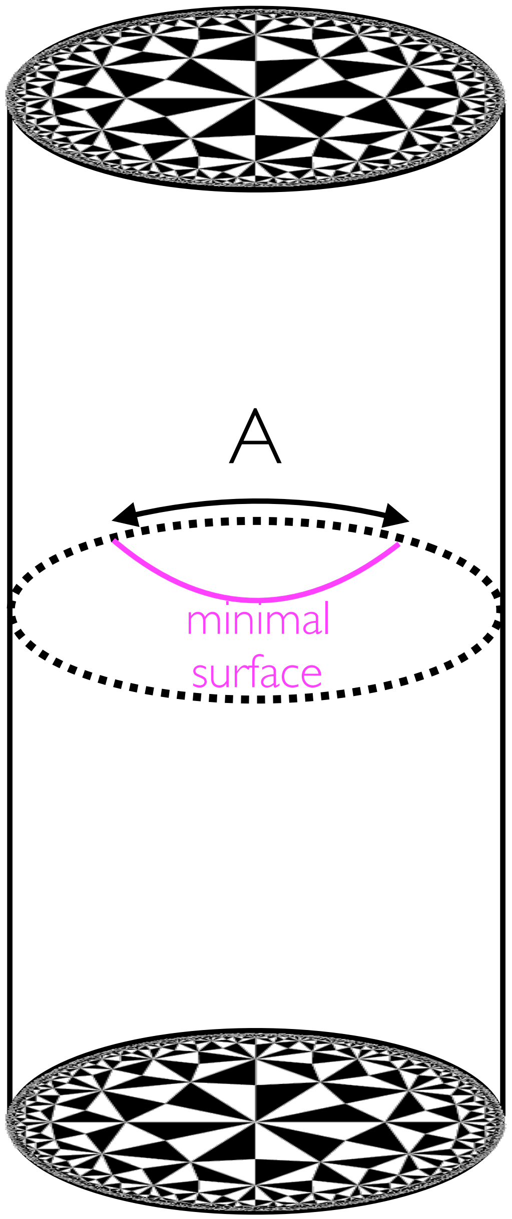
Maldacena 1997

GR + cosmological constant + certain matter on anti-de Sitter (AdS$_{d+2}$)

Holography and Quantum Info: Entanglement Entropy

time slice defines
a state, e.g. $|0\rangle$

$$(,,\frac{1}{\sqrt{2}}|\uparrow\rangle_A|\downarrow\rangle_{\bar{A}} - \frac{1}{\sqrt{2}}|\downarrow\rangle_A|\uparrow\rangle_{\bar{A}}'')$$



Ryu & Takayanagi 2006:

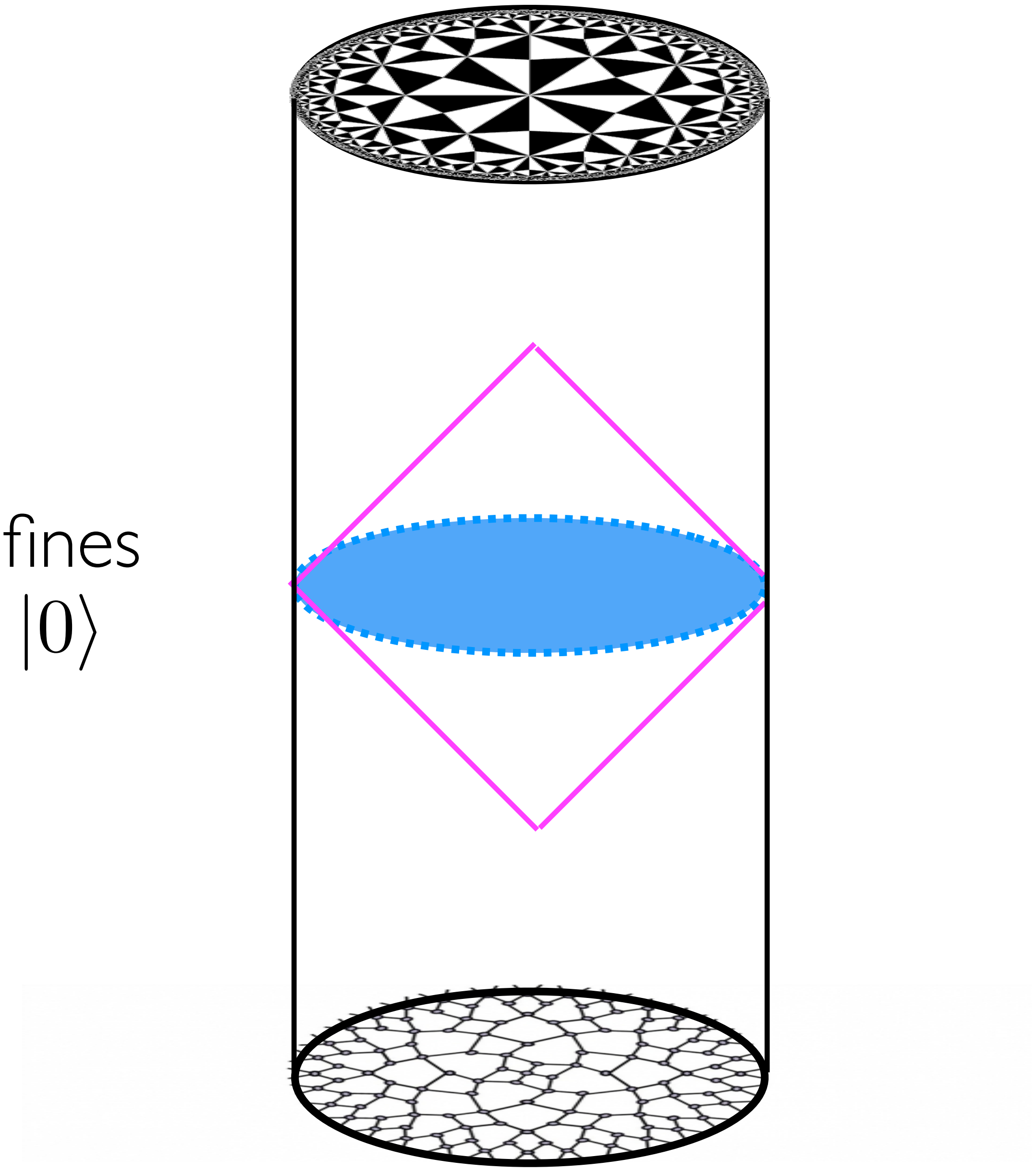
$$S_{EE} = -\mathrm{tr}\,\rho_A \log \rho_A \equiv \frac{\mathrm{Area}}{4G_N}$$

holography geometrizes
spatial entanglement in QFTs

Do Ryu-Takayanagi surfaces act also as probes of bulk matter field configurations?

Holography and Quantum Info: Complexity

Complexity of a state $\mathcal{C}$: min. number of elem. unitary operations δU s.t. $|0\rangle \approx \overset{\# \equiv \mathcal{C}}{\overleftrightarrow{\delta U \ldots \delta U}} | \uparrow \ldots \uparrow\rangle$



time slice defines a state, e.g. $|0\rangle$

Susskind et al. 2014-2017:

$$\mathcal{C} \sim \text{volume}_{\text{codim}-1}$$

$$\mathcal{C} = (\text{bulk action})_{\text{WdW}}$$

How to define complexity in continuous quantum-many body systems (QFTs)?

Novel Probes of Bulk Matter
(and Geometry of Causal Diamonds)

based on **1509.00113** and **1606.05344** with de Boer, Haehl, Myers and Neiman

see also **1604.03110** by Czech et al. for similar results

Setup

any conformal field theory (CFT) in d+1 spacetime dimensions

+

spatial subregions = spheres on some constant time slice

+

$\rho = |0\rangle\langle 0| + \gamma\,\delta\rho$ with $\gamma \ll 1$

Entanglement first law in CFTs I

Consider small perturbation of some reference density matrix $\rho = \rho_0 + \delta\rho$

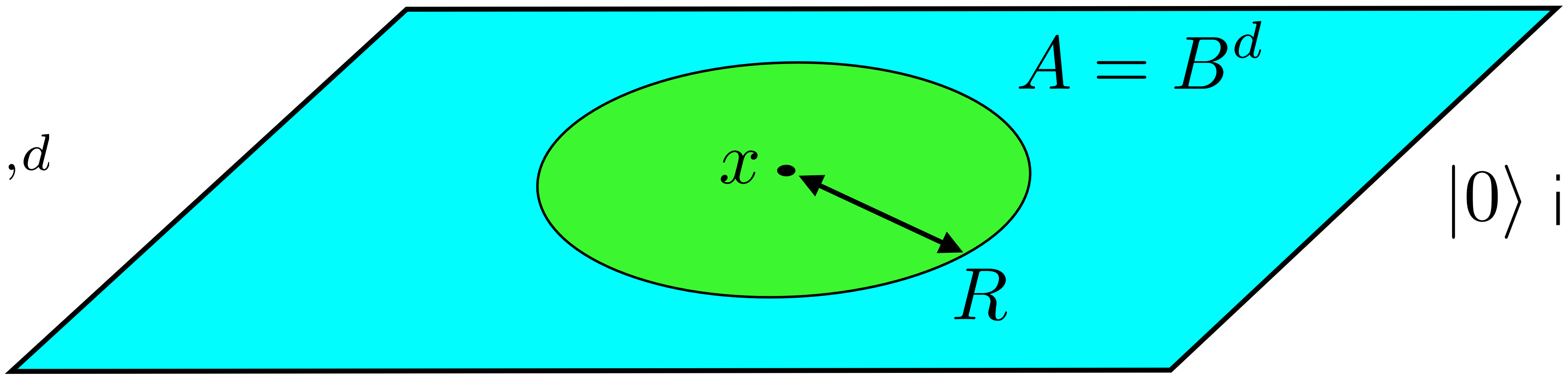
The change in the entropy is equal to the change in <the modular Hamiltonian>

$$\delta S = -\mathrm{tr}\,(\rho \log \rho) - S_0 = \delta\langle H_{mod}\rangle$$

In general, we expect $H_{mod} \equiv \log \rho_0$ to be nonlocal, but for $\rho_0 = \mathrm{tr}_{\bar{A}}\,|0\rangle\langle 0|$:

$$H_{mod} = c' + 2\pi \int_{|\vec{x}-\vec{x}'|^2 \le R^2} d^d x' \quad \frac{R^2 - |\vec{x}-\vec{x}'|^2}{2R} T_{tt}(x')$$

$t = 0$ of $\mathbb{R}^{1,d}$



$|0\rangle$ is the vacuum$_{\text{CFTd+1}}$

Casini, Huerta & Myers 1102.0440

Entanglement first law in CFTs 2

As a result, the change in the entanglement entropy for small perturbations of $|0\rangle$ is

$$\delta S_B = 2\pi \int_{|\vec{x}-\vec{x}'|^2 \leq R^2} d^d x' \quad \boxed{\frac{R^2 - |\vec{x}-\vec{x}'|^2}{2R}} \langle T_{tt} \rangle (x')$$

This equation (and its generalizations) is the main player of this part of my talk

Key insights of 1509.00113

δS obeys a wave equation on de Sitter$_{d+1}$:

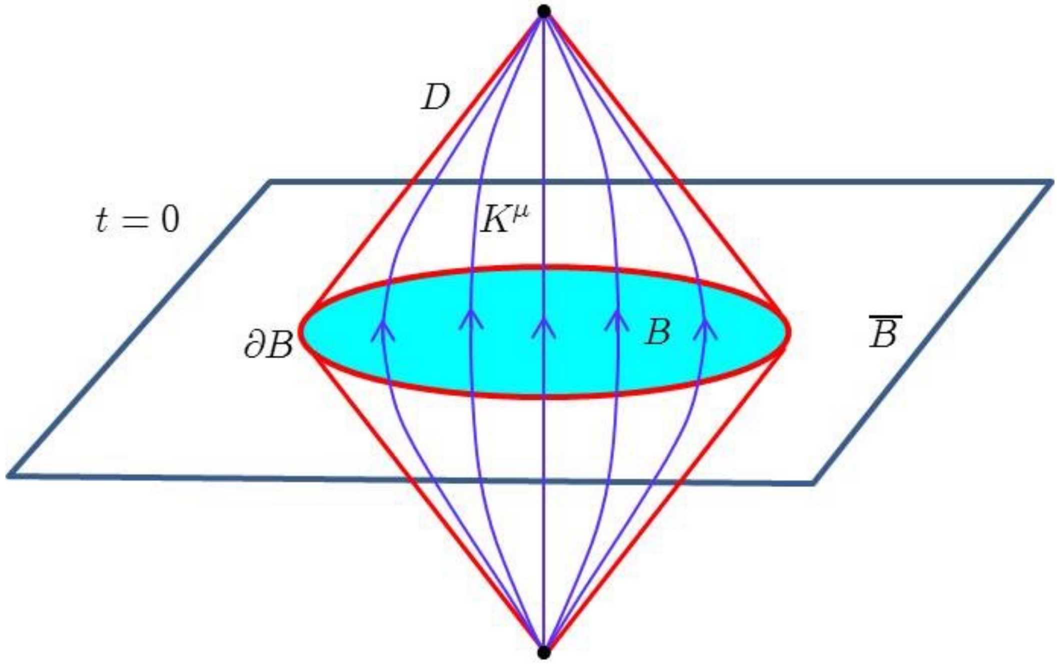
$$\frac{L^2}{R^2}\left(-dR^2 + d\vec{x}^2\right)$$

de Sitter appears as a moduli space of spheres

From $T_{\mu\nu}$ to $\mathcal{O}_{\mu_1\ldots\mu_l}$

We can write $\delta S_B = 2\pi \int_{|\vec{x}-\vec{x}'|^2 \leq R^2} d^d x' \; \frac{R^2 - |\vec{x}-\vec{x}'|^2}{2R} \langle T_{tt}\rangle(x') \sim \int_{\diamond} d^{d+1}\xi' |K|^{-2} \, K^{\mu} K^{\nu} \, \langle T_{\mu\nu}(\xi)\rangle$

which suggests to think about δS_B as a natural causal diamond observable associated with $T_{\mu\nu}$



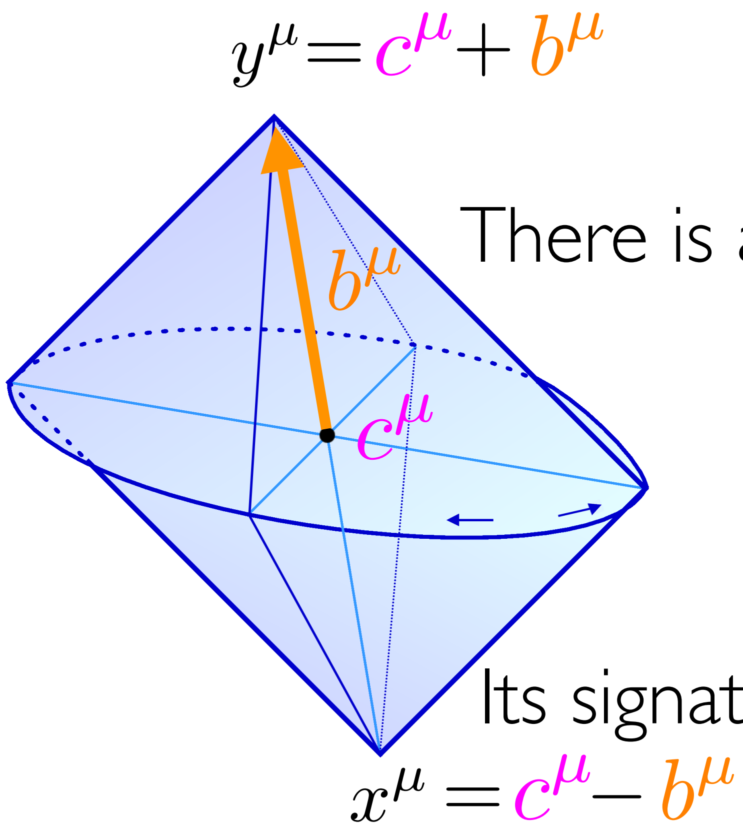
How about other primaries in a CFT? We propose the following definition

$$Q(\mathcal{O}, x, y) \sim \int_{\diamond} d^{d+1}\xi \, |K|^{\Delta - l - d - 1} \, K^{\mu_1} \cdots K^{\mu_l} \, \langle \mathcal{O}_{\mu_1\ldots\mu_l}(\xi)\rangle$$

This quantity has a holographic interpretation, but in general it does not "live" in dS$_{d+}$

Moduli space of causal diamonds (kinematic space)

General spherical surface on some constant-time slice is specified by the coordinates of the tips of the corresponding causal diamond: x^μ & y^μ :



There is a unique SO(2,d+1)-invariant metric parametrized by x^μ & y^μ:

$$\frac{L^2}{(-b^2)}\left(\eta_{\mu\nu}+\frac{2}{(-b^2)}b_\mu b_\nu\right)(-db^\mu db^\nu + dc^\mu dc^\nu)$$

Its signature is (d+1,d+1) and arises as SO(2,d+1) / [SO(1,d)xSO(1,1)]

$Q(\mathcal{O},x,y) \sim \int_\diamond d^{d+1}\xi\, |K|^{\Delta-l-d-1}\, K^{\mu_1}\cdots K^{\mu_l}\, \langle \mathcal{O}_{\mu_1\ldots\mu_l}(\xi)\rangle$ obey now a set of intricate local EOMs

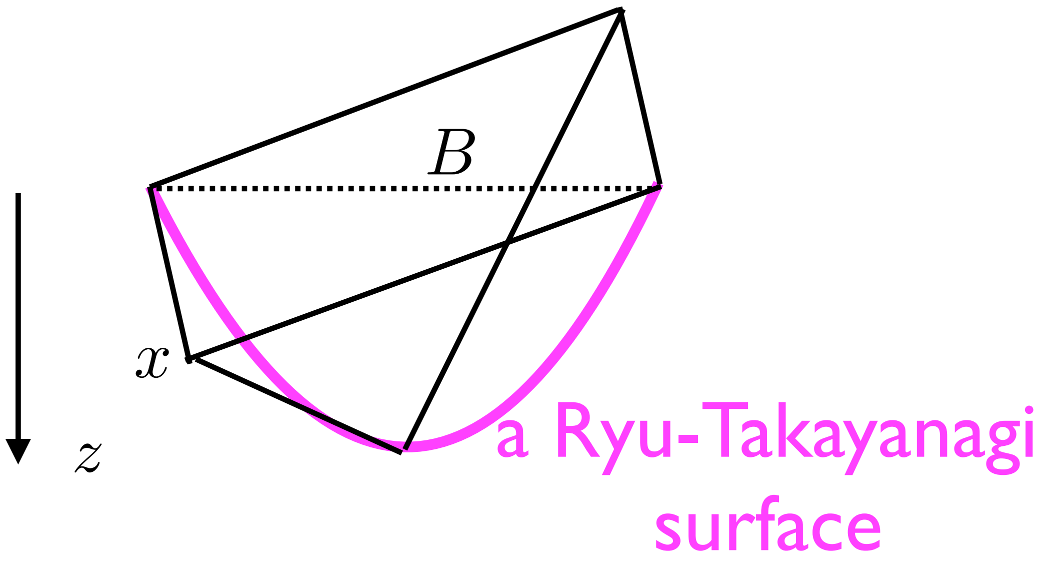
dS$_d$ is a particular submanifold of this larger moduli space: $c^0 = 0$ & $b^\mu \partial_\mu = R\,\partial_0$

RT-surfaces averages of bulk fields

In holographic theories for scalar operator $\mathcal{O}$ dual to free bulk field ϕ we have

$$\nabla^2_{AdS_{d+2}}\phi - m^2\,\phi = 0$$

with $\phi = \langle\mathcal{O}\rangle \cdot z^\Delta + \dots$



$$Q(\mathcal{O}, x, y)$$

$$\sim$$

$$\int_{\text{a Ryu-Takayanagi surface}} d^d\sigma\ \sqrt{g_{\text{induced}}}\,\phi$$

Can be proven/demonstrated in many different ways: group theory/ explicit sols for ϕ

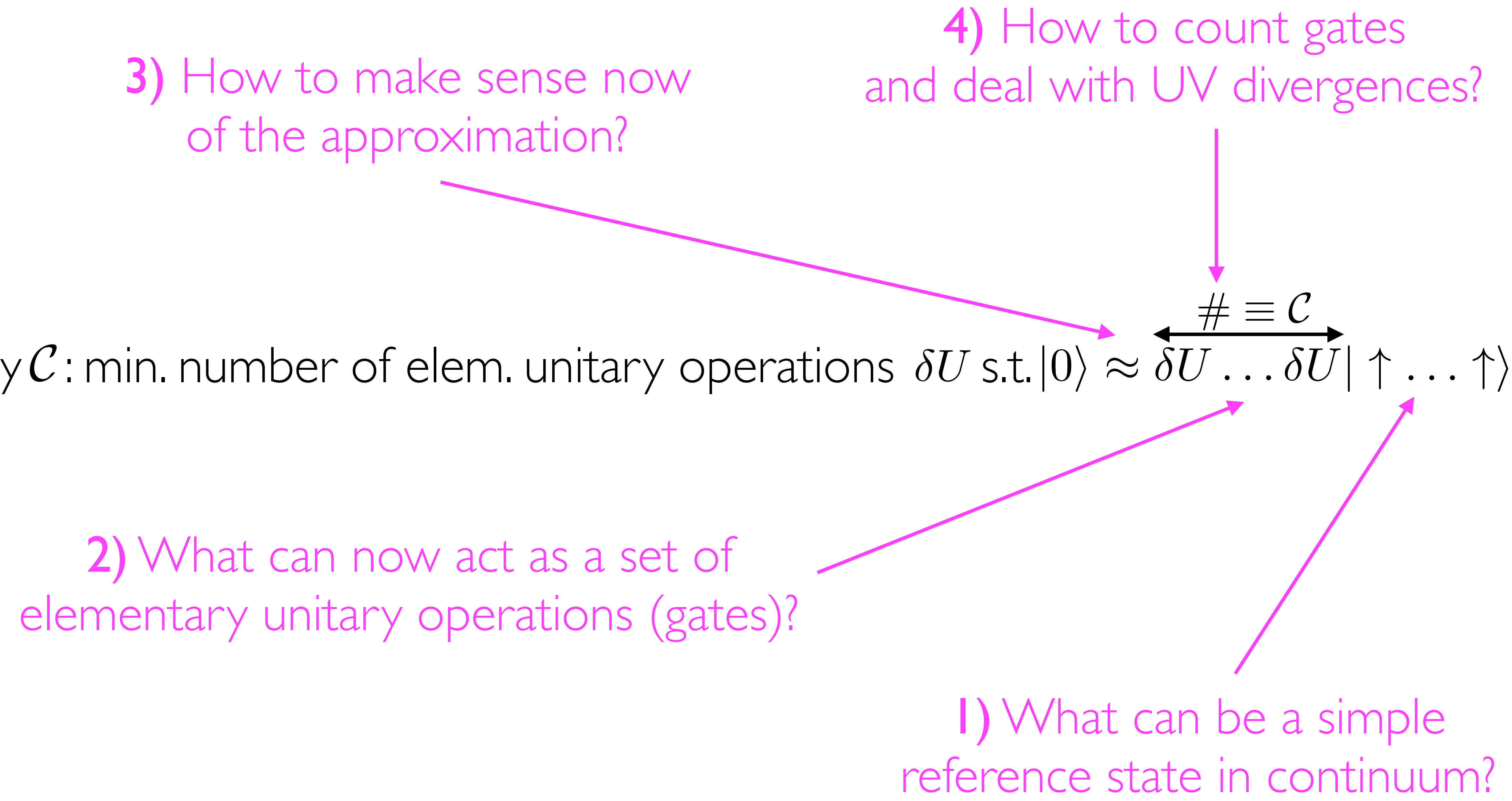
This provides a completely new set of nonlocal observables in holography

Complexity of Quantum Fields

1707.08582 with Chapman, Marrochio and Pastawski

see also **1707.08570** by Jefferson and Myers for related results

Complexity of QFT States — Challenges



3) How to make sense now of the approximation?

4) How to count gates and deal with UV divergences?

Complexity $\mathcal{C}$: min. number of elem. unitary operations δU s.t. $|0\rangle \approx \overset{\# \equiv \mathcal{C}}{\overleftrightarrow{\delta U \ldots \delta U}} | \uparrow \ldots \uparrow \rangle$

2) What can now act as a set of elementary unitary operations (gates)?

1) What can be a simple reference state in continuum?

5) We want an approach that is computable $\longrightarrow$ Gaussian States and free QFTs$_{d+1}$

What can be the continuum version of $|\uparrow\ldots\uparrow\rangle$?

Key feature of $|\uparrow\ldots\uparrow\rangle$: product state in real space ⟶ no spatial entanglement

C.f. ground state $|0\rangle$ of a free CFT $H = \int d^dx : \left\{\frac{1}{2}\pi^2 + \frac{1}{2}\left(\partial_{\vec{x}}\phi\right)^2\right\} :$

$$\langle 0|\,\phi(\vec{k})\phi(\vec{k}')\,|0\rangle = \boxed{\frac{1}{2k}}\,\delta^d(\vec{k}+\vec{k}')$$

this k-dependent factor leads to correlations in real space

Idea: define "$|\uparrow\ldots\uparrow\rangle$" in continuum as the groundstate of $\int d^dx : \left\{\frac{1}{2}\pi^2 + \frac{1}{2}M^2\phi^2\right\} :$

$\equiv$

$|R(M)\rangle$

free dimensionful parameter

Now: $\langle R(M)|\,\phi(\vec{k})\phi(\vec{k}')\,|R(M)\rangle = \frac{1}{2M}\delta^d(\vec{k}+\vec{k}') \longrightarrow \frac{1}{2M}\delta^d(\vec{x}-\vec{x}')$ ✓

The combined action of $\delta U \ldots \delta U$

starting point

$$\langle R(M)|\, \phi(\vec{k})\phi(\vec{k}')\, |R(M)\rangle = \frac{1}{2M}\delta^d(\vec{k}+\vec{k}')$$

vs.

end point after $\delta U \ldots \delta U\,|R(M)\rangle$

$$\langle 0|\, \phi(\vec{k})\phi(\vec{k}')\, |0\rangle = \frac{1}{2\,k}\;\delta^d(\vec{k}+\vec{k}')$$

In both cases no correlations other than between $\vec{k}$ and $(-\vec{k})$

So the only thing $\delta U \ldots \delta U$ need to do is to get us from $\frac{1}{2M}$ to $\frac{1}{2\,k}$

Our choice of elementary unitaries

One can show (cMERA, 2011) that the following transformation does the job:

$$\delta U = e^{-i\, F(\vec{k})\delta s} \Rightarrow \boxed{\begin{aligned} F(\vec{k}) &= \phi(\vec{k})\pi(-\vec{k}) + \pi(\vec{k})\phi(-\vec{k}) \\ &= i\left(a^\dagger_{\vec{k}}\, a^\dagger_{-\vec{k}} - a_{\vec{k}}\, a_{-\vec{k}}\right) \end{aligned}}$$

$\sim$ number of applications of δU

$$|0\rangle \approx |0^{(\Lambda)}\rangle = e^{-i \int_{k \leq \Lambda} d^d k\, F(\vec{k})\, \log \sqrt[4]{\frac{M}{k}}} |R(M)\rangle$$

cut-off in momentum space handles $\approx$ and UV divergences

How to count gates in continuum?

logs ~ number of times each $\delta U = e^{-iF(\vec{k})\delta s}$ acts

$$|0\rangle \approx |0^{(\Lambda)}\rangle = e^{-i\int_{k\leq\Lambda} d^dk\, F(\vec{k})\, \boxed{\log\sqrt[4]{\frac{M}{k}}}}|R(M)\rangle$$

Idea: one can quantify complexity by calculating L^n norm of the vector $\left\{\log\sqrt[4]{\frac{M}{k}}\right\}_{k\leq\Lambda}$:

$$\sqrt[n]{\int_{k\leq\Lambda} d^dk \left|\log\sqrt[4]{\frac{M}{k}}\right|^n}$$

This is dimensionful and one can improve upon it by also accounting for $|R(M)\rangle$

$$\sqrt{\langle R(M)|F(\vec{k})^2|R(M)\rangle - \langle R(M)|F(\vec{k})|R(M)\rangle^2} = \delta^d(\vec{k}-\vec{k}) = \mathrm{Vol}$$

Our proposal* for complexity in free CFTs is then $\mathcal{C}^{(n)} = \sqrt[n]{\mathrm{Vol}\int_{k\leq\Lambda} d^dk \left|\log\sqrt[4]{\frac{M}{k}}\right|^n}$

Optimal Unitaries and Fubini-Study metric

Let's think about unitary transformations as paths in the space spanned by $F(\vec{k})$:

$$|\Psi(\sigma)\rangle = Pe^{-i\int_{s_i}^{\sigma} ds \int_{k\leq\Lambda} d^dk\, F(\vec{k})\, Y'_{\vec{k}}(s)} |R(M)\rangle$$

Natural distance is defined by the Fubini-Study metric

$$ds_{FS}(\sigma) = d\sigma\sqrt{\Big|\partial_\sigma|\Psi(\sigma)\rangle\Big|^2 - \Big|\langle\Psi(\sigma)|\,\partial_\sigma|\Psi(\sigma)\rangle\Big|^2}$$

and **it defines a notion of complexity:**

$$\mathcal{C} = \min\Big|_{|\Psi(s_f)\rangle = |0^{(\Lambda)}\rangle} \int_{s_i}^{s_f} ds_{FS}(\sigma)$$

only one $F(\vec{k})$
per unit FS proper length

$$\mathcal{C}^{(1)} = \text{Vol}\int_{k\leq\Lambda} d^dk\,\left|\log\sqrt[4]{\frac{M}{k}}\right|$$

any number of $F(\vec{k})$
per unit FS proper length

$$\mathcal{C}^{(2)} = \sqrt{\text{Vol}\int_{k\leq\Lambda} d^dk\,\left|\log\sqrt[4]{\frac{M}{k}}\right|^2}$$

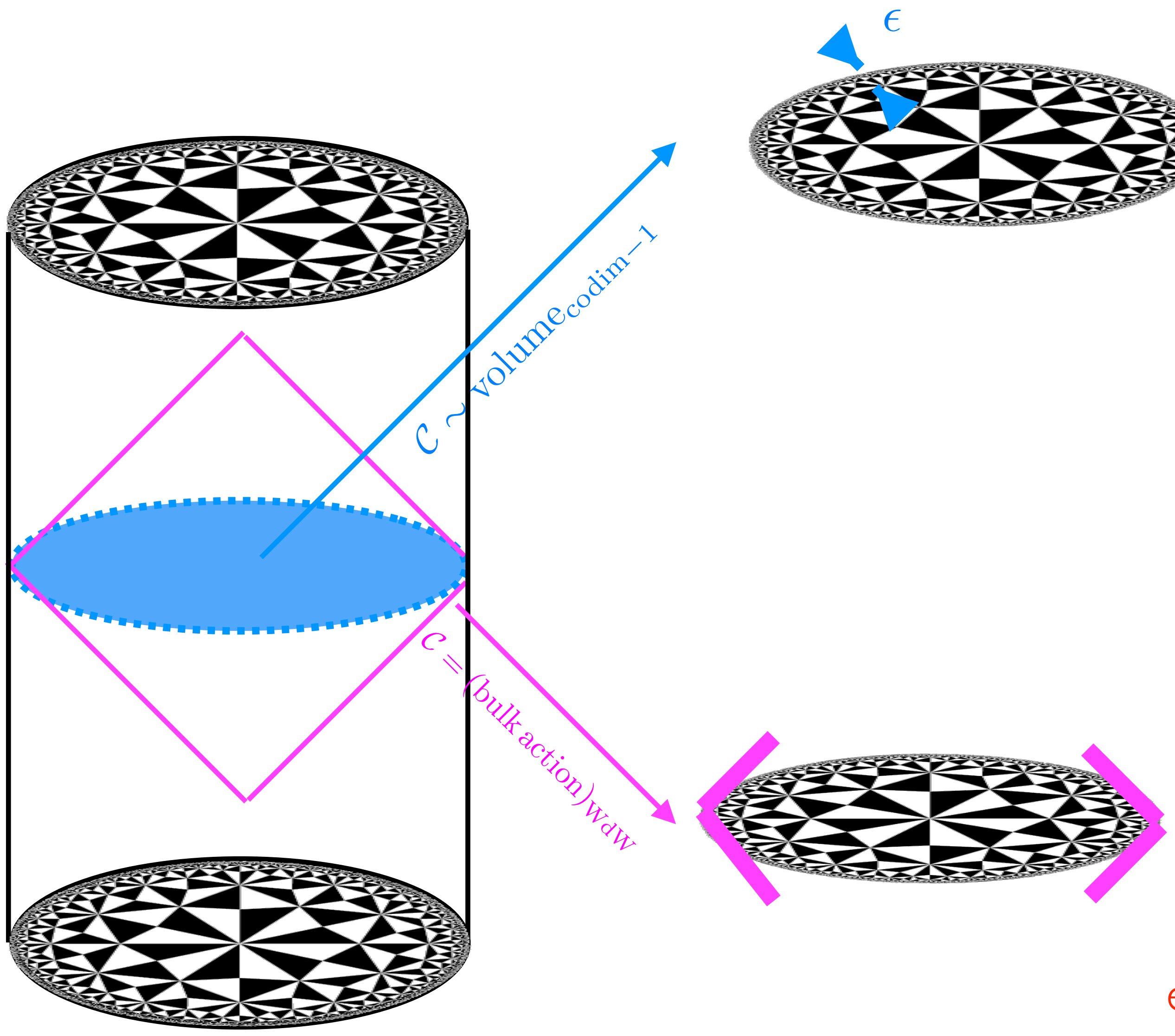
Results and comparison with holography

Our results for n = 1, 2:

$$\frac{C^{(1)}}{\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}\Lambda^d \,\mathrm{Vol}} = \begin{cases} \left|\log \sqrt[4]{\frac{M}{\Lambda}}\right| + \frac{M^d}{2\,d\,\Lambda^d} - \frac{1}{4\,d}, & M < \Lambda \\ \left|\log \sqrt[4]{\frac{M}{\Lambda}}\right| + \frac{1}{4\,d}, & M \geq \Lambda \end{cases}$$

$$\frac{\left(C^{(2)}\right)^2}{\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}\Lambda^d \,\mathrm{Vol}} = \left|\log \sqrt[4]{\frac{M}{\Lambda}}\right|^2 + \frac{\log \sqrt[4]{\frac{M}{\Lambda}}}{2d} + \frac{1}{8d^2}$$

In holography one gets remarkably similar results: to L^1 norm with $\Lambda \sim \frac{1}{\epsilon}$



$C \sim \frac{1}{\epsilon^d}\,\mathrm{Vol}$

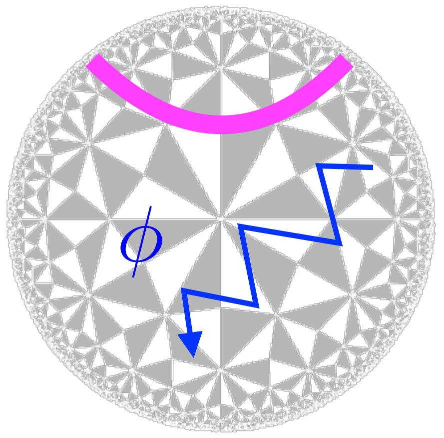
corner term ambiguity to the action gives:

$C \sim \frac{1}{\epsilon^d}\left|\log \frac{\frac{1}{\mathcal{L}}}{\frac{1}{\epsilon}}\right| \mathrm{Vol}$

e.g. Reynolds & Ross 2016

Summary

1509.00113
1606.05344
with de Boer, Haehl, Myers and Neiman



moduli space of $\Diamond$ & dS$_{d+1}$
(kinematic space)

$$\int_{RT} \phi \sim \int_{\Diamond} |K|^{\Delta-d-1} \langle O \rangle$$

codim-2 bulk surfaces

entanglement entropy

Holography + Quantum Information

codim-1 or codim-0 bulk objects

complexity of states ?

1707.08582
with Chapman, Marrochio and Pastawski

$$\mathcal{C}^{(n)} = \sqrt[n]{\mathrm{Vol} \int_{k \leq \Lambda} d^d k \left| \log \sqrt[4]{\frac{M}{k}} \right|^n}$$

$n = 1$ similar
to holography

$$\Lambda^d \left| \log \frac{M}{\Lambda} \right| \sim$$