Entanglement entropy and related observables in gauge/gravity duality and in tensor networks

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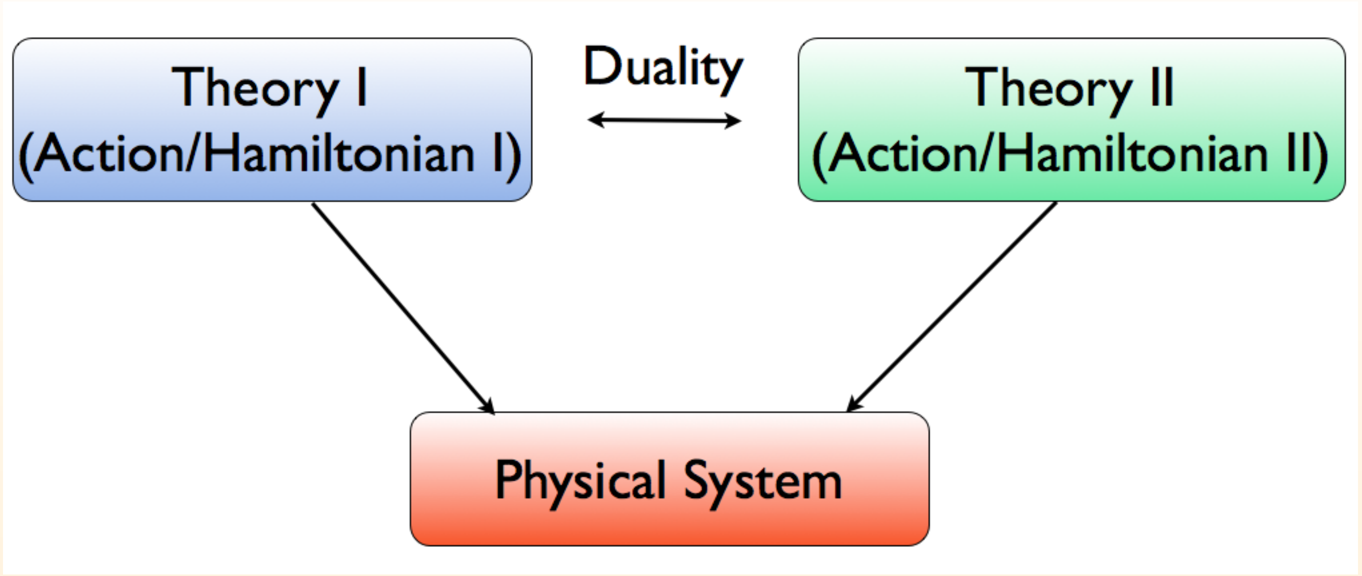
Motivation: Quantum theory and geometry

- Quantum gravity
- Quantum information
- Gauge/gravity duality (AdS/CFT correspondence)

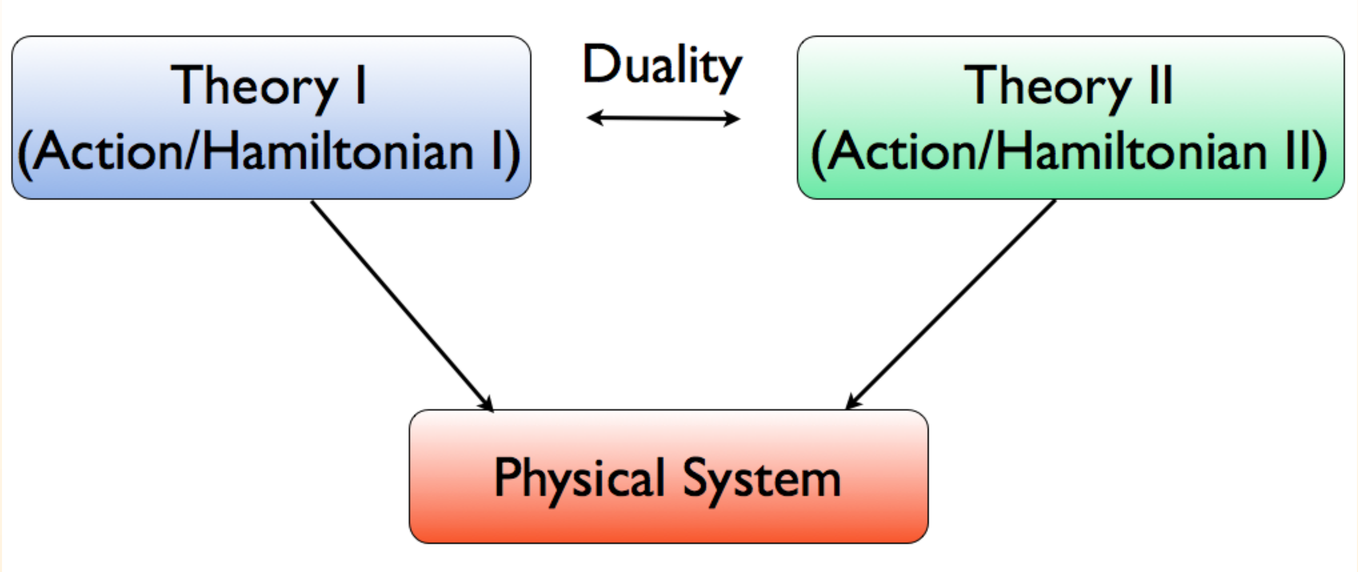
Duality

Duality:



Duality

Duality:



Gauge/gravity duality:

A quantum theory without gravity is related to a gravity theory

Gauge/Gravity Duality

- Conjecture which follows from a low-energy limit of string theory

- Duality:

  Quantum field theory at strong coupling

  $\Leftrightarrow$ Theory of gravitation at weak coupling

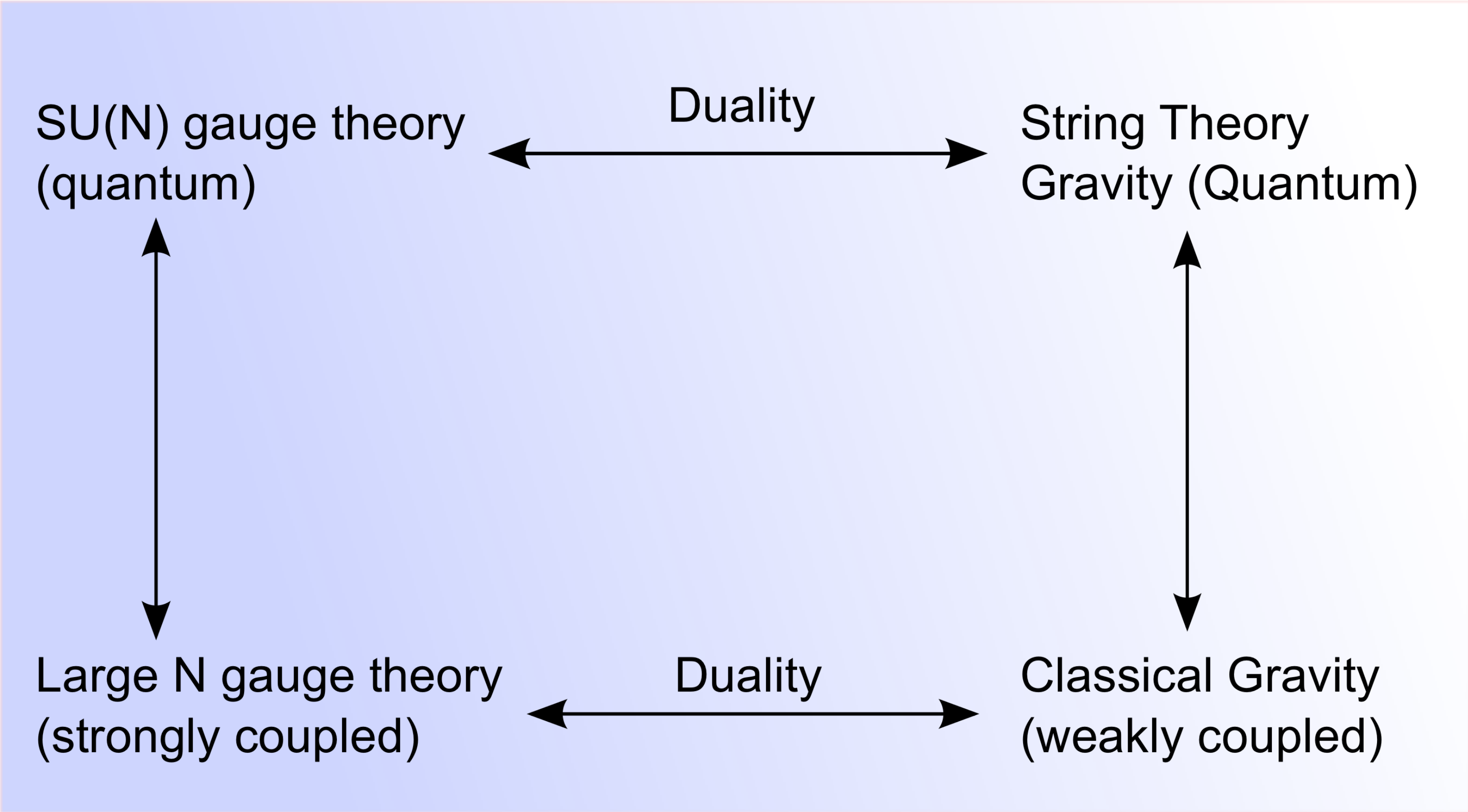
- Holography:

  Quantum field theory in d dimensions

  $\Leftrightarrow$ Gravitational theory in $d+1$ dimensions

  Quantum field theory defined on the boundary of the $d+1$-dimensional space

Gauge/Gravity Duality: String Theory Origin

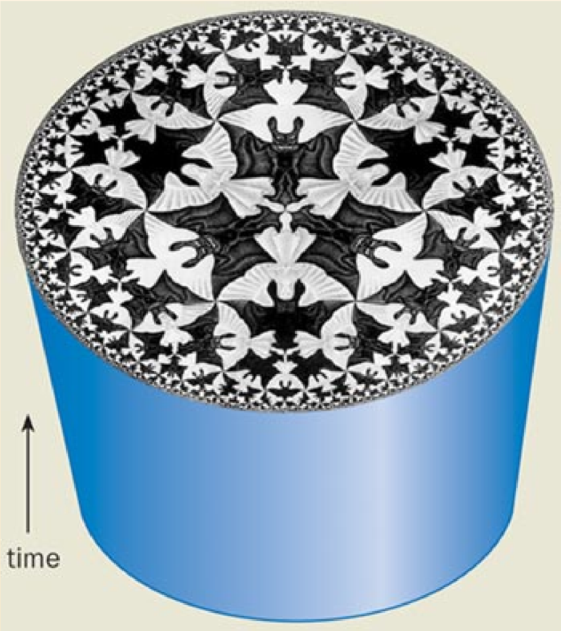


Example of gauge/gravity duality with huge amount of symmetry

AdS: Anti-de Sitter space:
Hyperbolic space
with constant negative curvature

CFT: Conformal field theory
Example: QFT at RG fixed point

$$ds^2 = \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right)$$

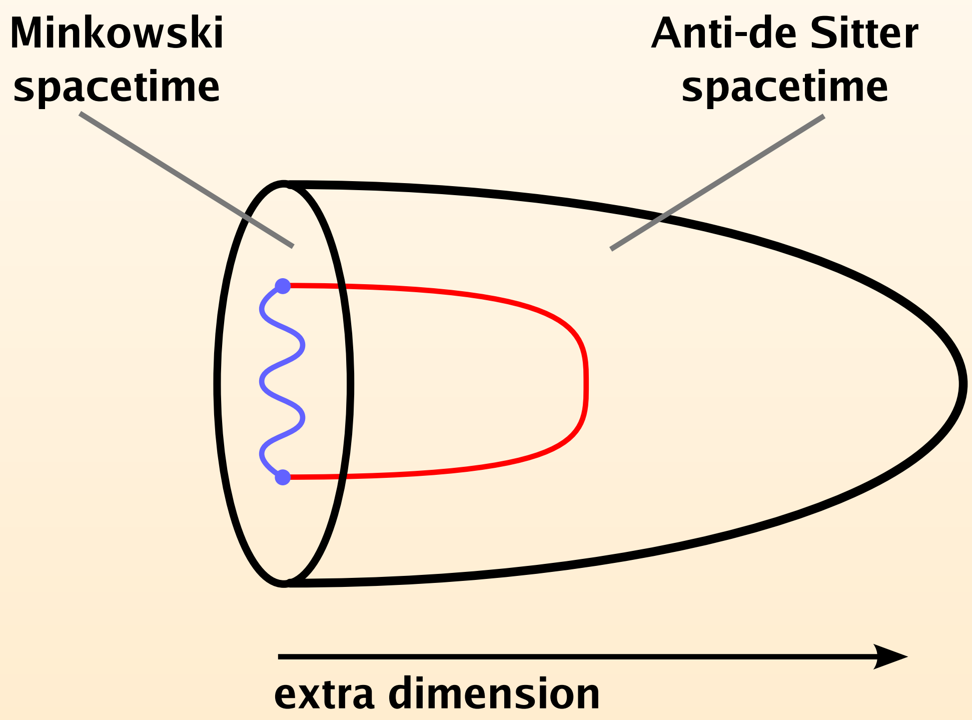


Quelle: Institute of Physics, Copyright: C. Escher

Gauge/Gravity Duality: Bulk-boundary correspondence

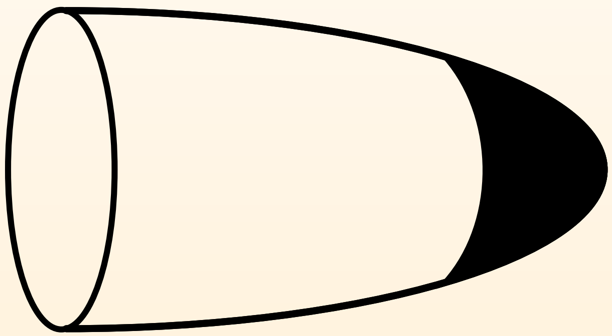
Quantum observables at the boundary of the curved space

may be calculated from propagation through curved space



Quantum theory at finite temperature:

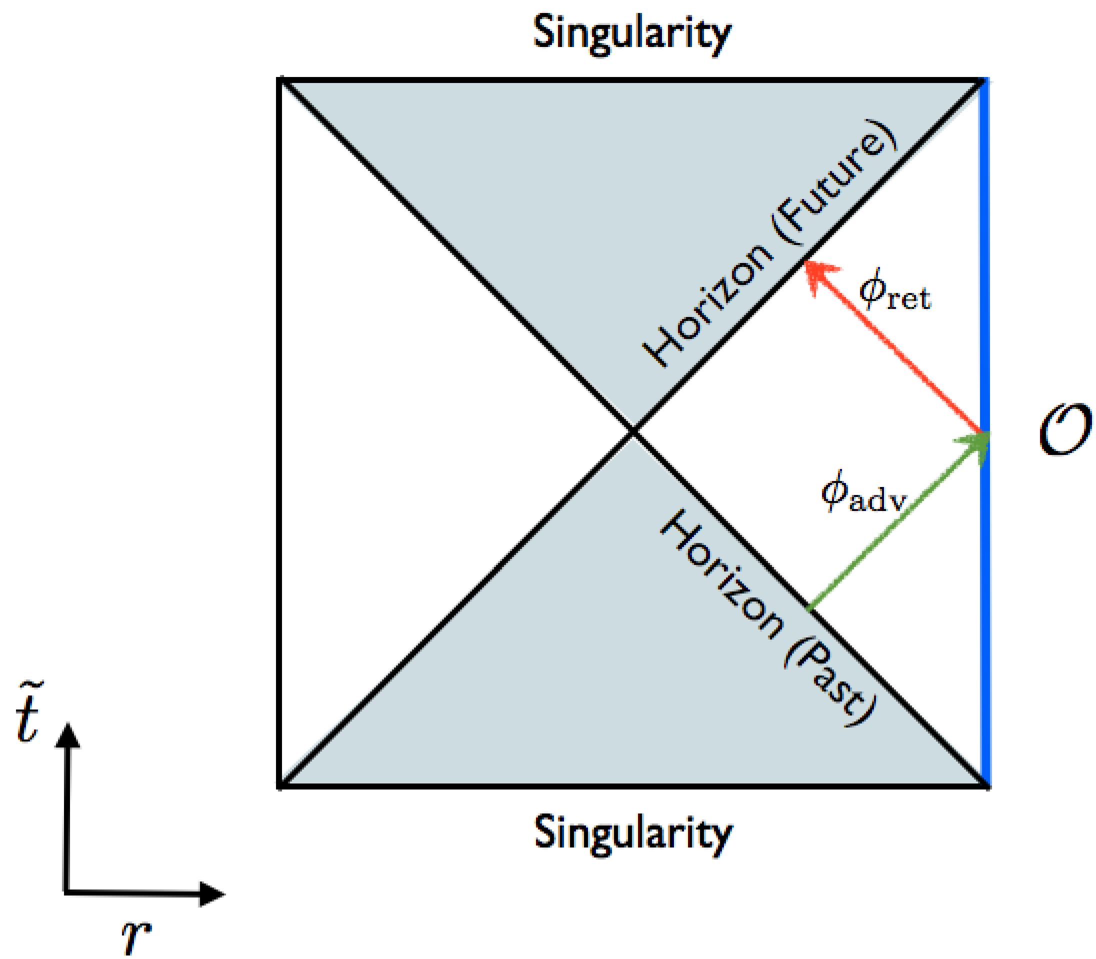
Dual to gravity theory with black hole



Hawking temperature identified with temperature in the dual field theory

Retarded Green's Functions in Strongly Coupled Systems

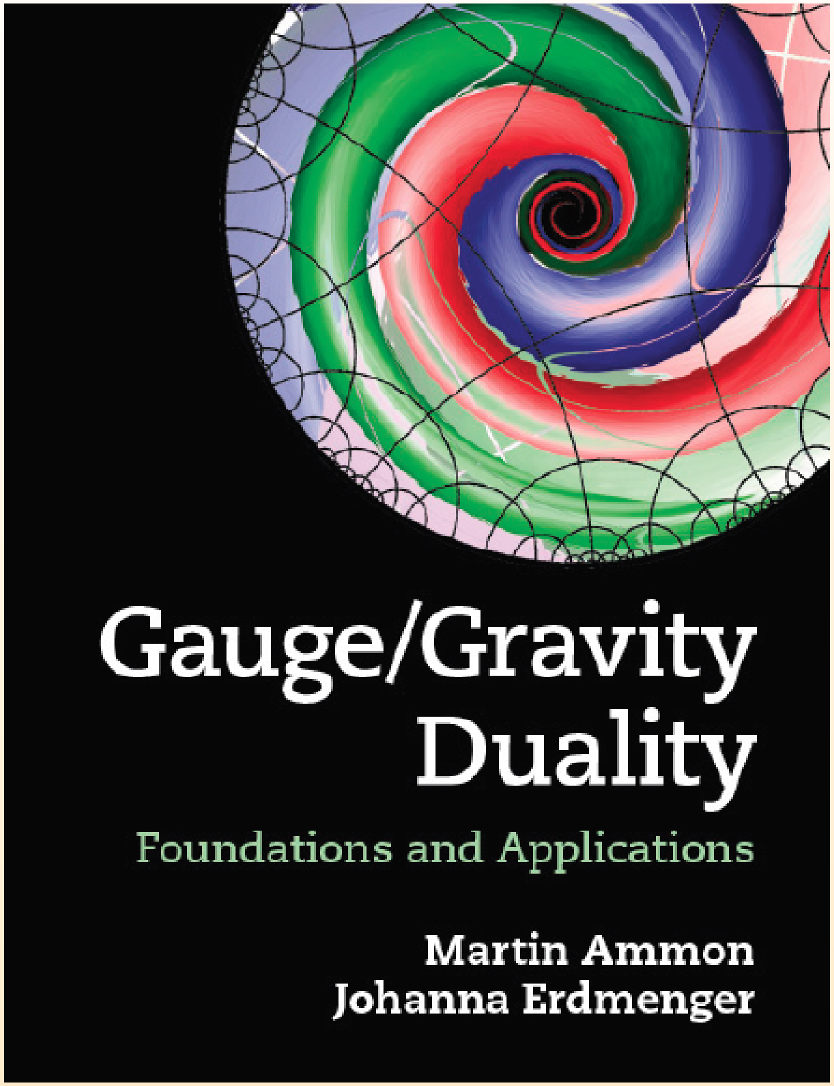
Anti-de Sitter
black hole



Retarded Green's function: $$G^R_{\mathcal{O}_A\mathcal{O}_B} = \left.\frac{\delta\langle\mathcal{O}_A\rangle}{\delta\phi_{B(0)}}\right|_{\delta\phi=0} = \frac{\delta\phi_{A(1)}}{\delta\phi_{B(0)}}$$

subject to infalling boundary condition at horizon

Book on gauge/gravity duality



Entanglement entropy: Quantum mechanics

Density matrix $\rho = \sum_n |\Psi_n\rangle\langle\Psi_n|$

Von Neumann entropy $S_{\mathrm{v}N} = -\mathrm{Tr}(\rho \ln \rho)$

Maximised when ρ diagonal with equal entries,
vanishes for pure states where $\rho^2 = \rho$

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Consider product Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Reduced density matrix

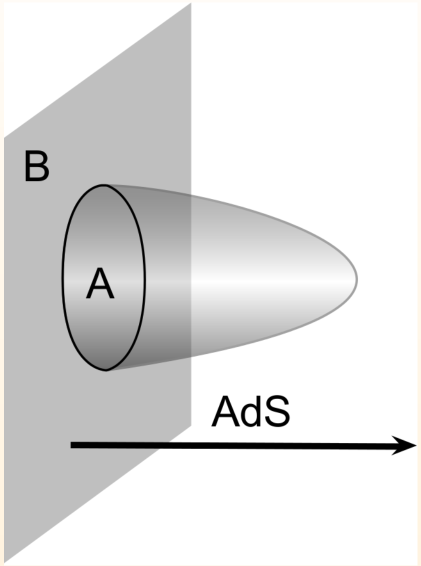
$$\rho_A = \mathrm{Tr}_B \rho_{\mathrm{tot}}$$

Entanglement entropy

$$S_A = -\mathrm{Tr}_A \rho_A \ln \rho_A$$

Analogy to black hole entropy
('Lost information' hidden in B)

Entanglement entropy: Gauge/gravity duality



Ryu-Takayanagi 2006:

$$S_A = \frac{\mathrm{Area}\gamma_A}{4G_N}$$

γ_A: Minimal area bulk surface with $\partial A = \partial\gamma_A$

Satisfies strong subadditivity

$$S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$$

Entangelement entropy: Examples

Conformal field theory in 1+1 dimensions (Cardy, Calabrese):

$$S = \frac{c}{3} \ln(\ell \Lambda)$$

Reproduced by Ryu-Takayanagi result

$\Lambda \propto 1/\epsilon$, ϵ boundary cut-off in radial direction

$c = 3L/(2G_3)$

Finite temperature (at small ℓ):

$$S(\ell) = \frac{c}{3} \ln\left(\frac{1}{\pi \epsilon T} \sinh(2\pi \ell T)\right)$$

J.E., Miekley 1709.07016

Analytic expression in closed form for strip region:

$$S_{EE} = \frac{L^{d-1}\left(\tilde{\ell}/\epsilon\right)^{d-2}}{2(d-2)G_N} + \frac{\sqrt{\pi}L^{d-1}}{4(d-1)G_N}\frac{\tilde{\ell}^{d-2}}{z_\star^{d-2}}\sum_{\Delta m=0}^{\frac{2(d-1)}{\chi}-1}\frac{(1/2)_{\Delta m}}{\Delta m!}\frac{\Gamma\left(\frac{d}{\chi}a^{\mathrm{EE}}_{-1/2}\right)}{\Gamma\left(\frac{d}{\chi}a^{\mathrm{EE}}_{0}\right)}\left(\frac{z_\star}{z_h}\right)^{\Delta m d} \qquad (5.5\mathrm{b})$$

$$\times_{\frac{3d-2}{\chi}+1}F_{\frac{3d-2}{\chi}}\left(1, a^{\mathrm{EE}}_{-\frac{1}{2}}, \ldots, a^{\mathrm{EE}}_{\frac{d}{\chi}-\frac{3}{2}}, b^{\mathrm{EE}}_{\frac{1}{2}}, \ldots, b^{\mathrm{EE}}_{\frac{2(d-1)}{\chi}-\frac{1}{2}}; a^{\mathrm{EE}}_{0}, \ldots, a^{\mathrm{EE}}_{\frac{d}{\chi}-1}, b^{\mathrm{EE}}_{1}, \ldots, b^{\mathrm{EE}}_{\frac{2(d-1)}{\chi}}; \left(\frac{z_\star}{z_h}\right)^{\frac{2(d-1)d}{\chi}}\right)$$

z_*: Turning point of minimal surface

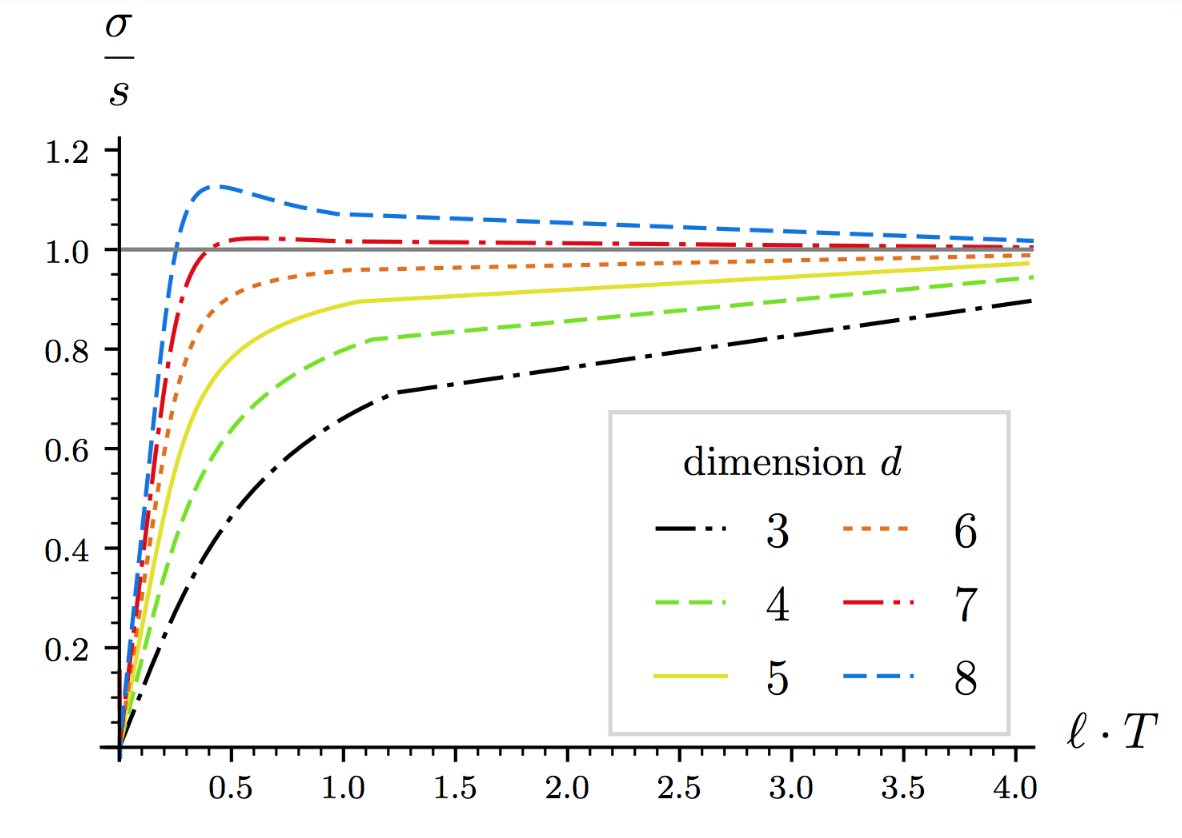
Given implicity in terms of strip width ℓ

Holographic entanglement entropy: Arbitrary dimensions

J.E., Miekley 1709.07016

Entanglement density

$$\sigma = \frac{S(T) - S(T=0)}{vol(A)}$$



Non-monotonic behaviour signals violation of area theorem

Holographic entanglement entropy: Finite temperature

Hubeny, Rangamani, Takayanagi 0705.0016

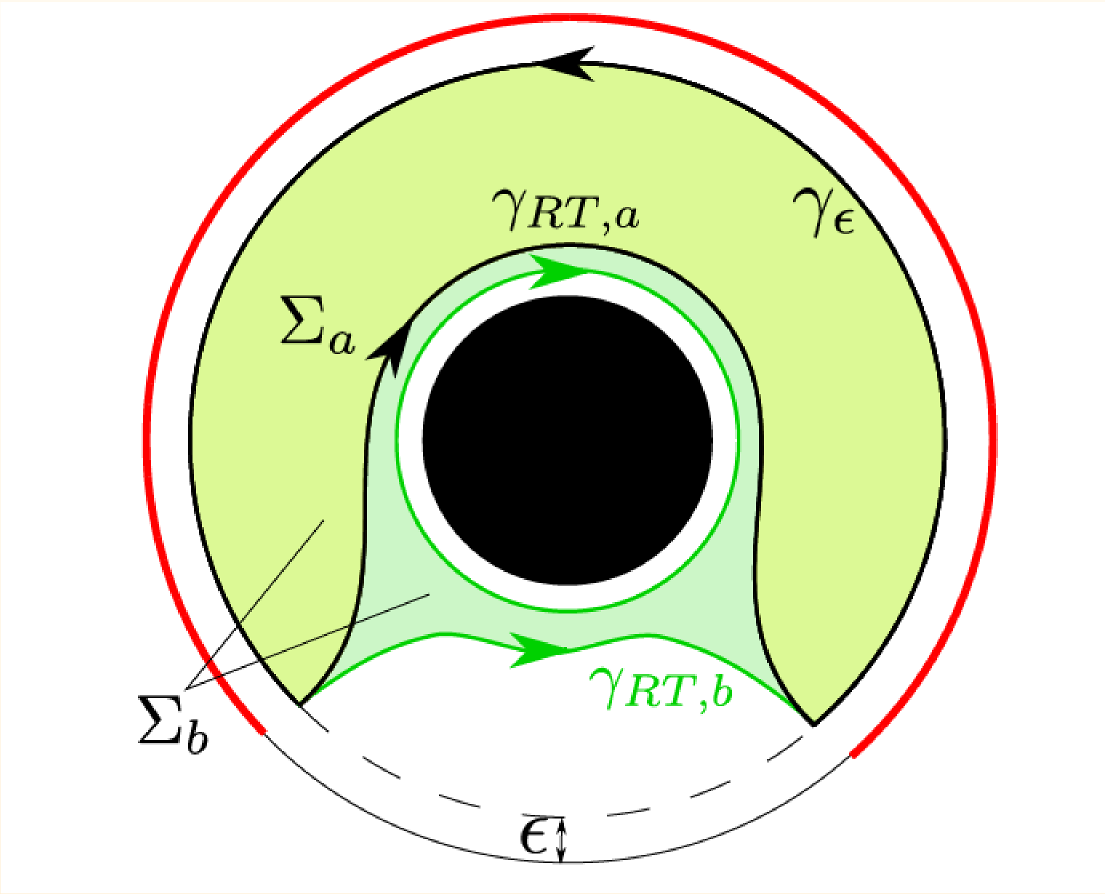
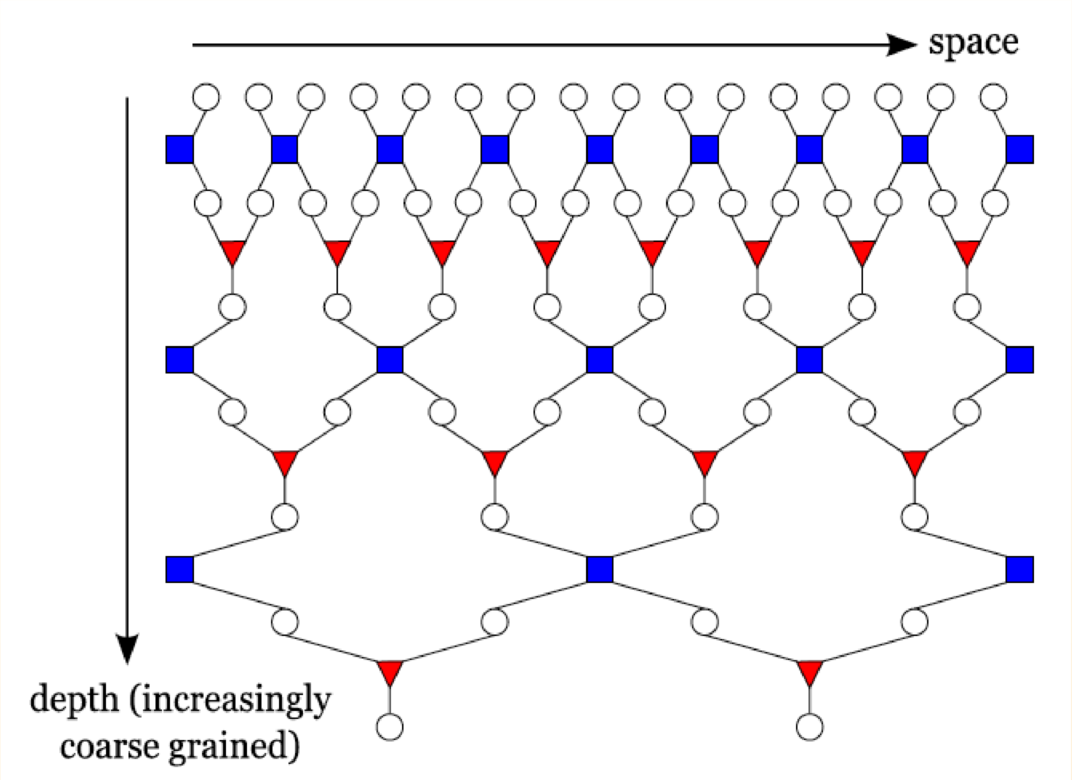


figure by Raimond Abt

Entanglement entropy: Tensor networks



MERA networks:
CFT ground states

Implement RG idea

Networks defined on discretizations of hyperbolic space

cf. AdS/CFT: Extra dimension corresponds to RG scale

MERA Network:
Entanglement entropy bounded from above by Ryu-Takayanagi formula
(Swingle 0905.1317)

Susskind, Stanford

A fixed spatial volume is associated to each tensor

Subnetwork connecting RT surface to boundary provides map $\mathcal{H}_{\mathrm{RT}} \to \mathcal{H}_A$

RT surface:
Smallest Hilbert space for any cut through the network bounded by ∂A

Entanglement entropy: Tensor networks

Susskind, Stanford

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Natural realization of complexity = volume conjecture
Number of tensors measures complexity

Entanglement entropy: Tensor networks

Quantum error-correcting codes satisfy discretized Ryu-Takayanagi formula

(Pastawski, Yoshida, Harlow, Preskill 1503.06237)

Building block: Tensor with maximal entanglement along any bipartition

$\Rightarrow$ Isometry from the bulk Hilbert space to the boundary Hilbert space

Random tensor network:

Observables obtained by averaging over tensor network states
built from random tensors living on a fixed graph

Random tensor networks may be mapped to an associated Ising model
Hayden et al 1601.01694

Average value of second Renyi entropy related to partition function

$$\overline{\mathrm{Tr}(\rho_A^2)} \sim Z_A$$

New paper (complexity): Abt, J.E., Hinrichsen, Melby-Thompson, Meyer, Northe, Reyes

Entanglement entropy: Random tensor networks

Numerical simulation of entanglement entropy in black hole background

Map to associated Ising model

H. Hinrichsen, Würzburg University, 1710.01327

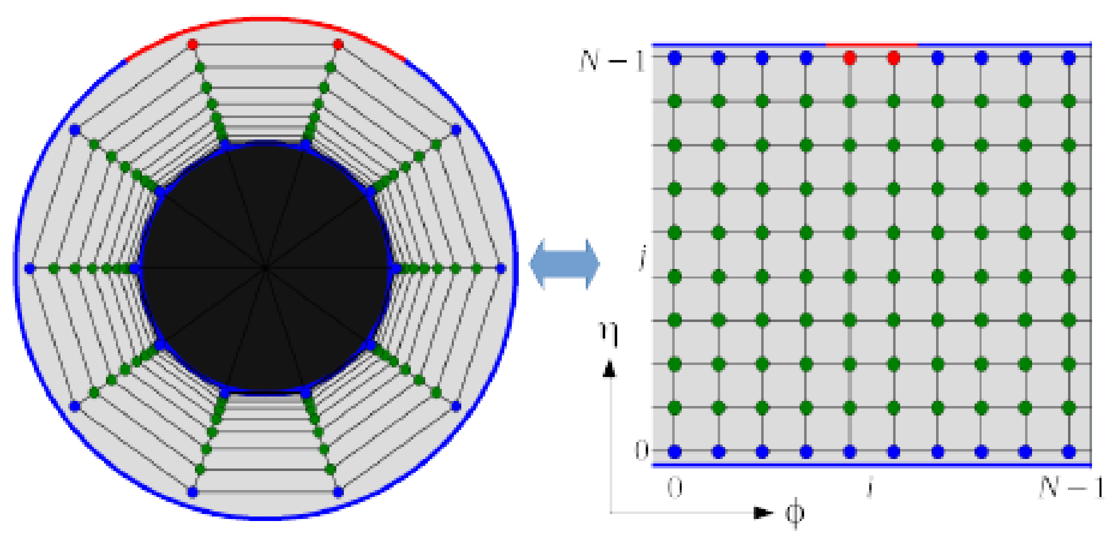
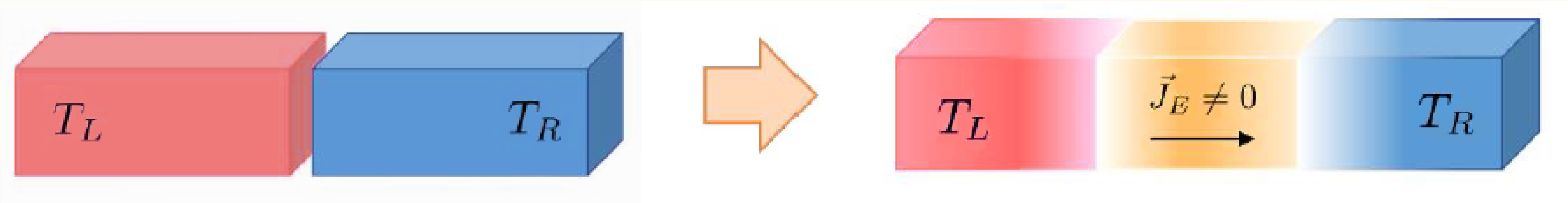
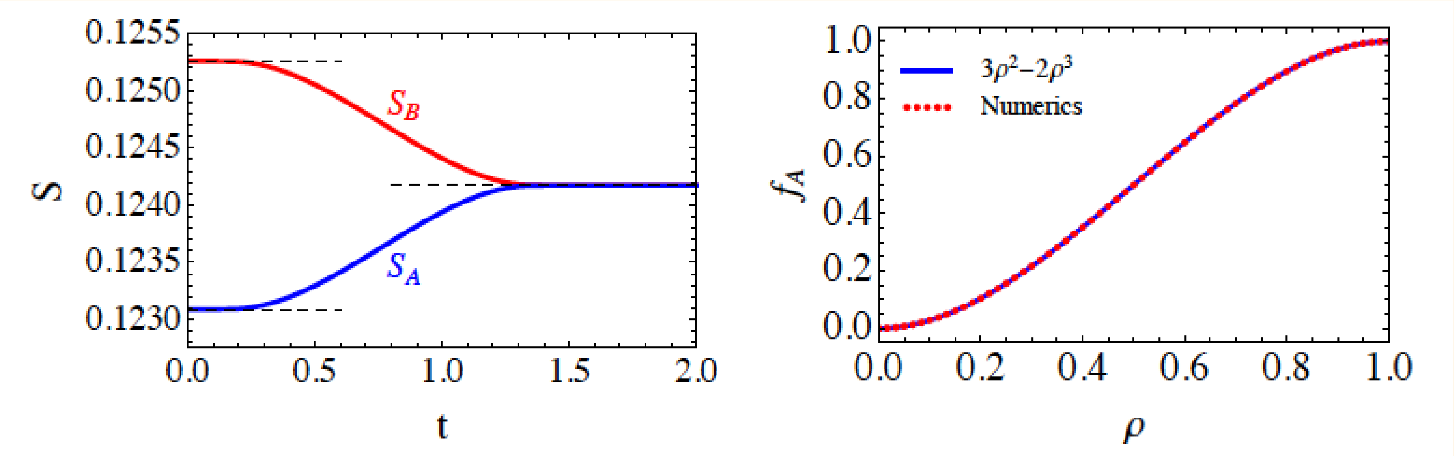


FIG. 6. BTZ black hole (left) with radial coordinate $\arctan(r/L)$ mapped to conformal coordinates ϕ, η (right) where an Ising model on a square lattice is embedded. The top and the bottom row of spins are fixed according to the respective boundary conditions (red=↑, blue=↓) while the green spins are allowed to fluctuate.

Gauge/gravity duality: Time evolution of entanglement entropy

J.E., Flory, Fernandez, Megias, Straub, Witkowski 1705.04696





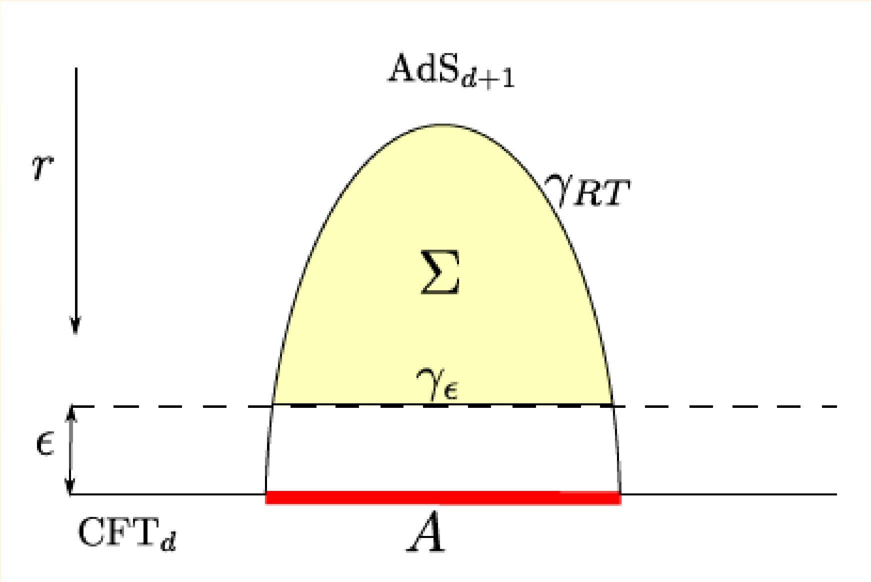
Very large temperature differences: $\Delta S(t) \propto v_E s_{\text{eq}} A\, t + \ldots$

Complexity:

Susskind: Complexity = Volume and Complexity = Action proposals

Abt, J.E., Hinrichsen, Melby-Thompson, Meyer, Northe, Reyes:
Modify volume proposal to

$$\mathcal{C}(\mathcal{A}) = -\frac{1}{2}\int_{\Sigma} d^{d+1}x R d\sigma$$



Complexity

For $d = 2$, evaluate complexity using Gauss-Bonnet theorem
see talk by I. Reyes

For black hole:

$$\mathcal{C} = \frac{x}{\epsilon} - \pi$$

$$\Delta\mathcal{C} = 2\pi$$

Complexity

For $d = 2$, evaluate complexity using Gauss-Bonnet theorem
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For black hole:

$$\mathcal{C} = \frac{x}{\epsilon} - \pi$$

$$\Delta\mathcal{C} = 2\pi$$

This is reproduced using random tensor networks

Complexity: Random tensor networks

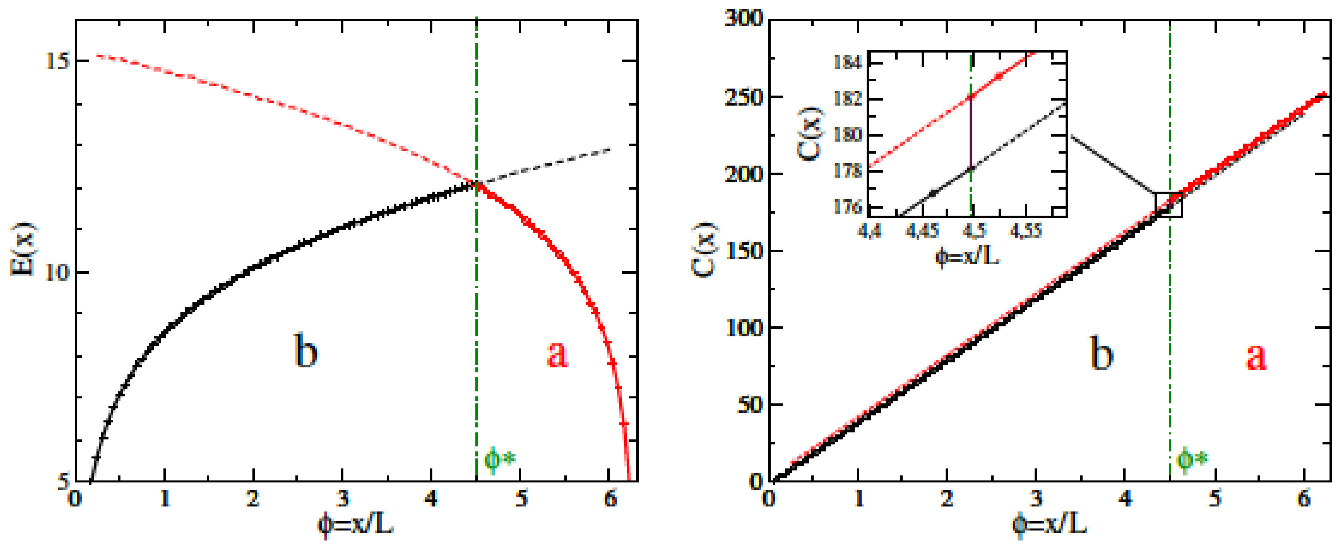
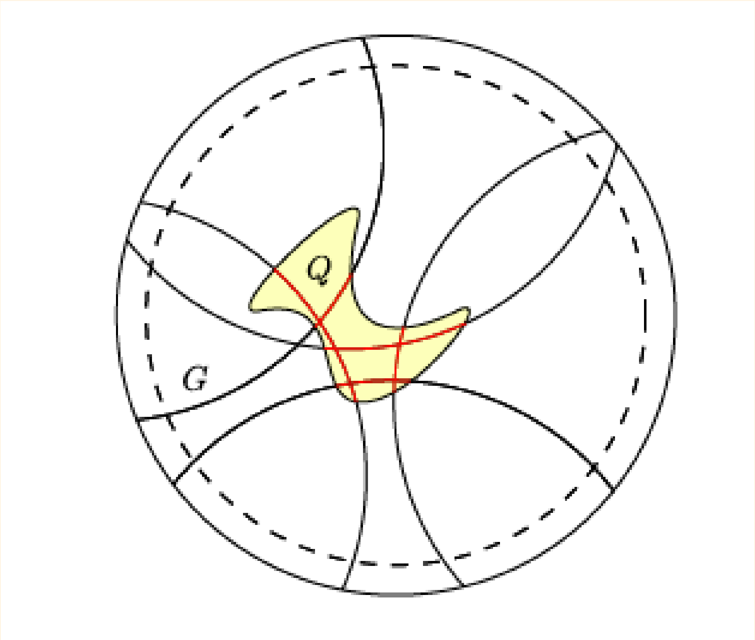


FIG. 8. Numerical results on a lattice with 200×200 sites for a BTZ black hole with mass $M = 0.1$. Left: Numerically measured entanglement of the two solutions as functions of the subregion size. As can be seen, the lines cross precisely at the theoretically expected transition point, marked by the vertical green dashed line. Right: Corresponding complexity, reproducing the linear law. The inset shows a magnification where the discontinuous jump occurs.

Kinematic space: Czech et al

Kinematic space of an asymptotically AdS_3 spacetime:
Space of its oriented, boundary-anchored geodesics

Use this approach to relate bulk volumes to field-theory entanglement entropy
Volume from integration over geodesics that intersect it



Fisher information metric

Use of geometry widespread in information theory

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Fisher metric in information theory: Metric on space of probability distributions

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Fisher metric in information theory: Metric on space of probability distributions

Probability distribution $p(x,\vec{\theta})$, x a stochastic variable, $\vec{\theta}$ a set of n external parameters

Spectrum $\gamma(x,\vec{\theta}) \equiv -\ln p(x,\theta)$

Fisher metric

$$g_{\mu\nu}(\vec{\theta}) = \int dx\, p(x,\vec{\theta}) \frac{\partial\gamma(x,\theta)}{\partial\theta^\mu}\frac{\partial\gamma(x,\theta)}{\partial\theta^\nu} = \langle \partial_\mu\gamma\partial_\nu\gamma\rangle$$

For Gaussian distribution (saddle point approximation)

$$p(x_1,\ldots,x_n) = \frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left(-\sum_{i=1}^{n}\frac{(x_i-\bar{x}_i)^2}{2\sigma^2}\right)$$

Fisher metric gives Anti-de Sitter space:

$$ds^2 = \frac{1}{\sigma^2}\left(d\bar{x}_i d\bar{x}^i + 2n d\sigma^2\right)$$

Fisher information metric

Fisher information metric

Question: Understanding the dynamics governing this metric

Fisher information metric

Question: Understanding the dynamics governing this metric

- In AdS/CFT, gravity action and dynamics obtained from saddle-point approximation of string theory
- Information theory may help to establish the gravity dynamics more generally
- Further insight using tensor networks?

Banerjee, J.E., Sarkar 1701.02319

Fidelity susceptibility F

Fisher metric

$$G_{mn} = \frac{\partial^2 F}{\partial \lambda^m \partial \lambda^n}$$

Couplings λ_m are dual to to deformations of the AdS metric

Proposal: $F = vol(\lambda) - vol(0)$ (finite expression)

For λ a metric deformation the result matches

Conclusion and outlook

- New relations between quantum theory and gravity
- Tensor networks may contribute to a further understanding of gauge/gravity duality
- Conversely,
  gauge/gravity duality provides predictions for strongly coupled systems