Differential Equations and Dispersion Relations for Feynman Amplitudes

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Based on collaboration with *Lorenzo Tancredi*

- Veltman's largest time equation
- Imaginary parts and cut graphs – examples
- Imaginary parts and discontinuities
- Imaginary parts/cut graphs and differential equations
- (Dispersion relations)
- Differential equations for the imaginary part only [?]
- using Dispersion relations for merging inhomogeneous terms in the differential equations

- The largest time equation (M. Veltman, 1963) .

An elementary derivation of the Cutkosky (1960) rule :

to start with, definition and properties of the (scalar) Feynman propagator $\Delta(x)$ in configuration and then momentum space

$$\Delta(x) = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + m^2 - i\epsilon}\ \mathrm{e}^{ipx}\ ,$$

with $px = \vec{p} \cdot \vec{x} - p_0 x_0$ (written in $1 + 3 = 4$ dimensions, but of immediate extension to $1 + (d - 1)$ space dimensions).

Properties of the propagator

$$\Delta(x) = \Delta(-x) = \int \frac{d^4p}{(2\pi)^4} \frac{-i}{p^2 + m^2 - i\epsilon} \, \mathrm{e}^{ipx} \; ,$$

$$\Delta(x) = \theta(x_0)\Delta^+(x) + \theta(-x_0)\Delta^-(x) \; ,$$

$$\Delta^+(x) = \int \frac{d^4p}{(2\pi)^4}(2\pi)\theta(+p_0) \; \delta(p^2 + m^2)\mathrm{e}^{ipx} \; ,$$

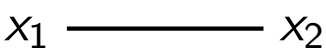
$$\Delta^-(x) = \int \frac{d^4p}{(2\pi)^4}(2\pi)\theta(-p_0) \; \delta(p^2 + m^2)\mathrm{e}^{ipx} \; ,$$

$$\Delta^+(-x) = \Delta^-(x) \; , \qquad\qquad (\; \Delta^+(x) \;)^* = \Delta^-(x) \; ,$$

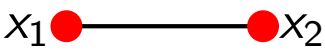
$$\Delta^*(x) = \Delta^*(-x) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 + m^2 + i\epsilon} \, \mathrm{e}^{ipx} \; ,$$

$$\Delta^*(x) = \theta(x_0)\Delta^-(x) + \theta(-x_0)\Delta^+(x) \; .$$

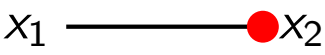
Graphical representations



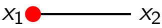
$$= \Delta(x_2 - x_1) = \Delta(x_1 - x_2)$$



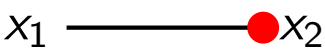
$$= \Delta^*(x_2 - x_1) = \Delta^*(x_1 - x_2)$$



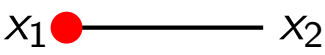
$$= \Delta^+(x_2 - x_1) = \Delta^-(x_1 - x_2)$$



$$= \Delta^-(x_2 - x_1) = \Delta^+(x_1 - x_2)$$

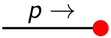


and

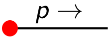


are usually called cut propagator lines.

In the momentum representation,



$$= 2\pi\theta(p_0)\delta(p^2 + m^2) \ ,$$



$$= 2\pi\theta(-p_0)\delta(p^2 + m^2) \ ,$$

i.e. a cut propagator line represents a positive energy flow from the uncircled to the red circled vertex;
in a propagator line without circles or with two circles

 or  ,

the energy can flow in both directions.

A Feynman graph in configuration space is represented by N vertex points $x_1, x_2, .., x_N$, suitably joined by propagator lines.
The corresponding Feynman graph amplitude $F(x_i)$ (scalar case only, and omitting coupling constants) is the product of a factor i for each vertex and a (scalar) propagator $\Delta(x_i - x_j) = \Delta(x_j - x_i)$ for each line joining any two points x_i, x_j.
as a consequence, in particular

$$F(-x_i) = F(x_i) \; .$$

To be more precise, the above $F(x_i)$ should also be integrated on the internal points.
For simplicity, finally, all the propagators will be given a same mass m, but the arguments which follows apply also to the case of different masses.

preparing Veltman's derivation:
Given an N vertices graph, consider the sum of the 2^N related amplitudes, obtained by taking the original graph and putting a circle on each of the N vertices in all possible ways;
associate to each graph with circles an amplitude defined, according to the previous graphical representation, with the rules:

- a factor i for each original vertex, $(-i)$ for each circled vertex;
- $\Delta(x_i - x_j) = \Delta(x_j - x_i)$ for each line joining two uncircled vertices (x_i, x_j);
- $\Delta^+(x_i - x_j) = \Delta^-(x_j - x_i)$ for each line joining a circled x_i to an uncircled x_j;
  or, which is the same
- $\Delta^-(x_i - x_j) = \Delta^+(x_j - x_i)$ for each line joining an uncircled x_i to a circled x_j;
- $\Delta^*(x_i - x_j) = \Delta^*(x_j - x_i)$ for each line joining two circled vertices (x_i, x_j).

The *circling* operation is related to complex conjugation; for any of the *circled* graphs, the corresponding graph with reversed circling is its complex conjugate;
in particular, if the amplitude of the original Feynman graph is $F(x)$, (where x stands for some set of x_i) the amplitude of the graph with N circles is $F^*(x_i)$, *i. e.*

$$F(x_i) + F^*(x_i) = 2\mathtt{Re}F(x_i) \ .$$

It is to be recalled, however, that $\mathtt{Re}F(x_i)$ is usually called the *imaginary part* of the Feyman amplitude ....

Further: if $\tilde{F}(p)$ is the Fourier transform of $F(x)$

$$\tilde{F}(p) = \int dx\ F(x)\mathrm{e}^{-ipx}$$

recalling $F(-x) = F(x)$, $F^*(-x) = F^*(x)$, one has

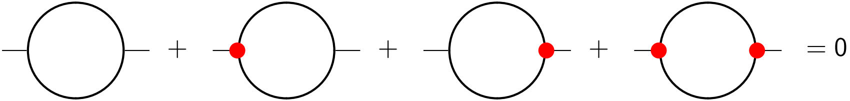
$$\left(\tilde{F}(p)\right)^* = \int dx\ F^*(x)\mathrm{e}^{ipx} = \int dx\ F^*(-x)\mathrm{e}^{-ipx} = \int dx\ F^*(x)\mathrm{e}^{-ipx}$$

so that

$$\mathtt{Re}\tilde{F}(p) = \int dx\ \mathtt{Re}F(x)\mathrm{e}^{-ipx}\ ,$$

$$\mathtt{Im}\tilde{F}(p) = \int dx\ \mathtt{Im}F(x)\mathrm{e}^{-ipx}\ ,$$

The largest time equation tells us that the sum of all the 2^N *circled* amplitudes defined above, including the original amplitude $F(x_i)$, vanishes.
The proof can best be followed in a first explicit example, the bubble with 2 vertices, for which the largest time equation graphically means

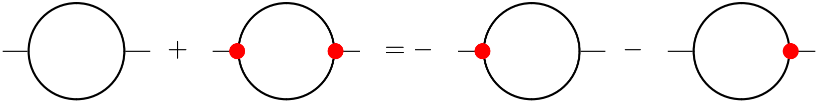


Proof: let x_m^0 be the largest of all the x_i^0 times;
take as a first graph one of the 2^{N-1} graphs in which the vertex m is uncircled and the others $(N-1)$ vertices are in any of the possible circled configuration;
pick up as a second graph the corresponding graph where the vertex m is circled, and ALL the other vertices are in the same configuration as the first graph;
if the first graph has a factor $\Delta(x_m - x_i)$, corresponding to a line from an uncircled point x_i to x_m, in the second graph the factor corresponding to that line is $\Delta^+(x_m - x_i)$; but as $x_0^m > x_0^i$, $\Delta(x_m - x_i) = \Delta^+(x_m - x_i)$, that factor is the same;

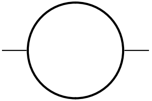
if the first graph has a factor $\Delta^{+}(x_i - x_m) = \Delta^{-}(x_m - x_i)$, corresponding to a line from a circled point x_i to x_m, the second graph has a factor $\Delta^{*}(x_i - x_m) = \Delta^{*}(x_m - x_i)$, which is again equal to $\Delta^{-}(x_m - x_i)$ because $x_0^m > x_0^i$;
the argument applies to all the lines with a vertex in m ;
the only difference in the two graphs is the change of the factor i associated to the vertex into $-i$;
so that the sum of the two graphs vanishes.

The same applies to all the 2^{N-1} graphs in which the vertex m is uncircled;
the sum of the 2^N graphs is equal to a sum of 2^{N-1} pairs of graphs, each pair vanishing;
the sum of the 2^N graphs vanishes.

Considering again the 1-loop bubble, the largest time equation gives

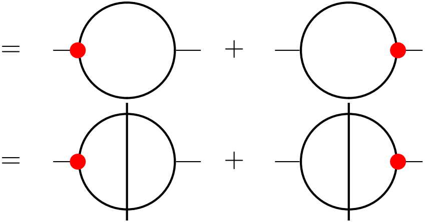


By using, from now on, only the momentum representation, define

$$\frac{1}{i}A(p) = \; p$$  ;

the largest time equation then becomes

$$2\ \mathtt{Im}A(p) =$$



where the *cut lines* are joined to give a *cut graph*.

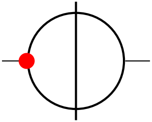
One has in general (different masses, $M > m$)

$$A(p) = -i \int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1 - i\epsilon} \frac{1}{D_2 - i\epsilon} \ .$$

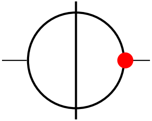
with

$$D_1 = k^2 + m^2 = -k_0^2 + \vec{k}^2 + m^2 \ ,$$
$$D_2 = (p-k)^2 + M^2 = -(p_0 - k_0)^2 + (\vec{p} - \vec{k})^2 + M^2 \ ,$$

so that

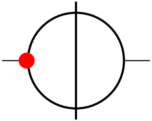


$$= \int \frac{d^d k}{(2\pi)^{d-2}} \theta(-k_0)\delta(D_1)\theta(-(p_0 - k_0))\delta(D_2)$$



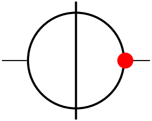
$$= \int \frac{d^d k}{(2\pi)^{d-2}} \theta(k_0)\delta(D_1)\theta(p_0 - k_0)\delta(D_2)$$

$\delta(D_1) = \delta(-k_0^2 + \vec{k}^2 + m^2)$ implies $|k_0| > m$;
$\delta(D_2) = \delta(-(p_0 - k_0)^2 + (\vec{p} - \vec{k})^2 + M^2)$ implies $|p_0 - k_0| > M$; for timelike $p = (p_0, 0)$, $u = p_0^2$



$$= \int \frac{d^d k}{(2\pi)^{d-2}} \theta(-k_0)\delta(D_1)\theta(-(p_0 - k_0))\delta(D_2)$$

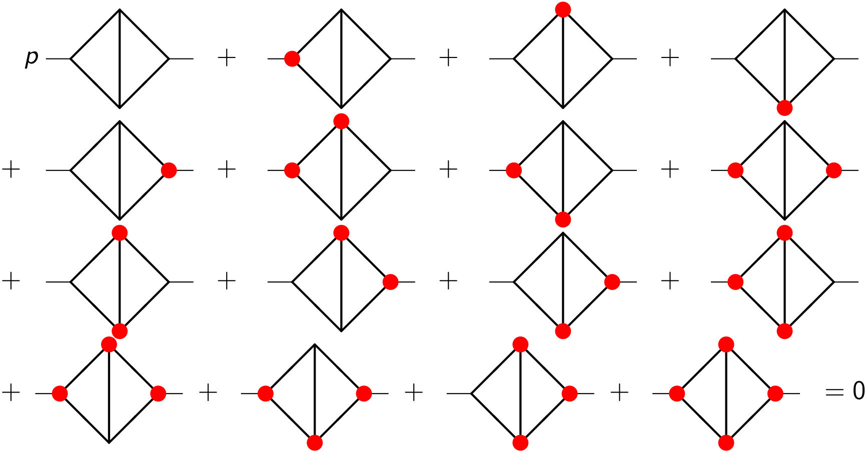
vanishes unless $p_0 < -(M + m)$, $u > (M + m)^2$



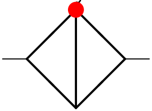
$$= \int \frac{d^d k}{(2\pi)^{d-2}} \theta(k_0)\delta(D_1)\theta(p_0 - k_0)\delta(D_2)$$

vanishes unless $p_0 > (M + m)$, $u > (M + m)^2$

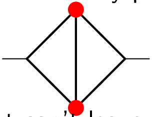
Another example, the (massive) kite; the largest time equation reads



But many of the (cut) graphs vanish:

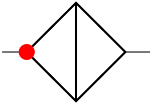


$= 0$; indeed, all the three cut lines meeting in the upper vertices carry positive energy, violating momentum (energy) conservation;



$=0$, positive energy flows to the upper and lower vertex but can't leave them;

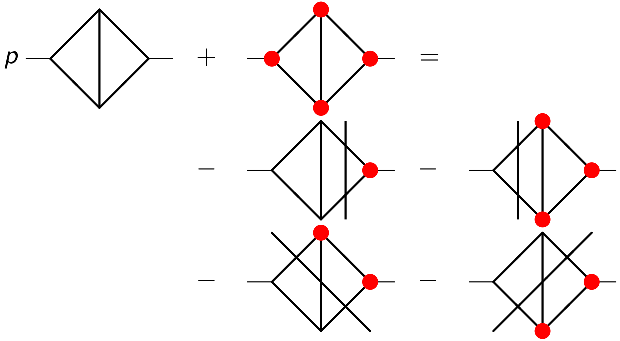
for timelike $p = (p_0, 0)$, with $p_0 > 0$,



$= 0$;

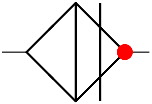
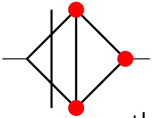
the energy flows from the incoming external line and the two internal propagator lines to the left vertex and can't leave anymore...
etc.

in conclusion, for timelike $p = (p_0, 0)$, with $p_0 > 0$, (and in the equal mass case)

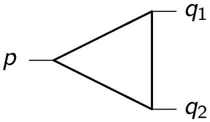


$p_0 > 2m$

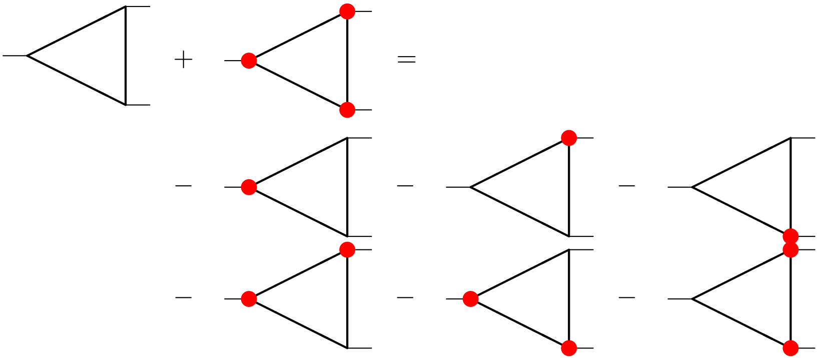
$p_0 > 3m$

 and  are the c.c. of each other, so their sum is real (as it should be..)

One more example: the 1-loop vertex



The largest time equation is immediately written



But the meaning of *imaginary part* and of the cut graphs requires more care than for the self-mass case.
Quite in general, if a Feynman amplitude $F(u)$ depends on a single Mandelstam variable only, say u, one has

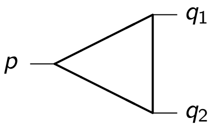
$$F(u^*) = F^*(u)$$

(the constant parameters entering in the definition of $F(u)$ are only the masses, which are real).
The discontinuity across a cut along the real axis of u is then given by

$$F(u+i\epsilon) - F(u-i\epsilon) = F(u+i\epsilon) - F^*(u+i\epsilon) = 2i\mathtt{Im}F(u+i\epsilon)$$

i.e. the discontinuity is twice the imaginary part in the $\epsilon \to 0$ limit.
That is the case for the self-mass graphs.

But the case of the vertex is different, because a vertex amplitude

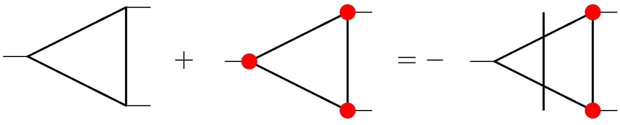


depends on equal footing on three Mandelstam variables, $(-p^2), (-q_1^2), (-q_2)^2$, and discontinuities in several variables at the same time can occur.

Correspondingly, many cut graphs (or even all of them) can give non vanishing contributions for a same kinematical configuration of the Mandelstam variables.

(A practical criterion might be to start with all the three variables in the Euclidean region, where the amplitude is real, continue judiciously the amplitude in one of the variables, then continue again in another variable, crossing the fingers, possibly, and so on...)

A simple case corresponds to timelike $p = (p_0, 0)$, with $p_0 > 0$, and, say, $(-q_1^2) = (-q_2^2) = m^2$; the largest time equation then becomes



where the cut graph does not vanish if $p_0 > 2m$.

Back to the 1-loop bubble; in $d = 2$ dimensions (for simplicity!), for timelike $p = (p_0, 0)$, but with two different masses $M > m$,

$$D_1 = k^2 + m^2, \qquad\qquad D_2 = (p-k)^2 + M^2,$$

$$\delta(D_1)\delta(D_2) = \delta(k_0^2 - k_z^2 - m^2)\delta(p_0^2 - 2p_0 k_0 + m^2 - M^2)$$

and the largest time equation reads

$$\begin{aligned} 2\ \mathtt{Im}A(p) = &\int dk_0 dk_z\ \theta(-k_0)\delta(D_1)\theta(-(p_0-k_0))\delta(D_2) \\ &+ \int dk_0 dk_z\ \theta(k_0)\delta(D_1)\theta(p_0-k_0)\delta(D_2) \end{aligned}$$

An explicit (simple) calculation gives

$$\begin{aligned} \mathtt{Im}A(p) = &\frac{1}{4}\theta(+p_0 - (M+m))\frac{1}{\sqrt{(p_0^2-(M+m)^2)(p_0^2-(M-m)^2)}} \\ &+ \frac{1}{4}\theta(-p_0 - (M+m))\frac{1}{\sqrt{(p_0^2-(M+m)^2)(p_0^2-(M-m)^2)}} \end{aligned}$$

It can be of interest to compare the previous result with the direct, explicit calculation of the whole amplitude; recalling

$$\frac{1}{x - i\epsilon} = \frac{x + i\epsilon}{x^2 + \epsilon^2} = \frac{\mathcal{P}}{x} + i\pi\delta(x)$$

where $\mathcal{P}$ stands for the principal value, in $d = 2$ dimensions

$$\begin{aligned} A(p) &= -i \int \frac{dk_0 dk_z}{(2\pi)^2} \left[\frac{1}{D_1 - i\epsilon} \frac{1}{D_2 - i\epsilon} \right] \\ &= \int \frac{dk_0 dk_z}{(2\pi)^2} \left[\pi \frac{\mathcal{P}}{D_1} \delta(D_2) + \pi\delta(D_1) \frac{\mathcal{P}}{D_2} + i \left(\pi^2 \delta(D_1)\delta(D_2) - \frac{\mathcal{P}}{D_1} \frac{\mathcal{P}}{D_2} \right) \right] \end{aligned}$$

The product of the two δ's gives in fact 4 terms

$$\begin{aligned}\delta(D_1)\delta(D_2) &= \big(\theta(k_0)+\theta(-k_0)\big)\delta(D_1)\big(\theta(p_0-k_0)+\theta(-(p_0-k_0))\big)\delta(D_2)\\ &= \theta(k_0)\delta(D_1)\theta(p_0-k_0)\delta(D_2)\\ &+ \theta(-k_0)\delta(D_1)\theta(p_0-k_0)\delta(D_2)\\ &+ \theta(k_0)\delta(D_1)\theta(-(p_0-k_0))\delta(D_2)\\ &+ \theta(-k_0)\delta(D_1)\theta(-(p_0-k_0))\delta(D_2)\end{aligned}$$

For $p_0 > (M+m)$ the term with the product of the two δ's gives

$$\begin{aligned}\int \frac{dk_0 dk_z}{(2\pi)^2}\pi^2\delta(D_1)\delta(D_2) &= \frac{1}{4}\int dk_0 dk_z\ \theta(k_0)\delta(D_1)\theta(p_0-k_0)\delta(D_2)\\ &= \frac{1}{8}\frac{1}{\sqrt{(p_0^2-(M+m)^2)(p_0^2-(M-m)^2)}}\end{aligned}$$

The previous value of $\mathtt{Im}A(p)$ is recovered when considering also the contributions of the product of the two principal values.

For completeness, let us list the values of the integrals with two δ's and the various specifications of the signs of the roots, considering, for simplicity, only the case of timelike p with $p_0 > 0$

$$\int dk_0 dk_z \; \theta(k_0)\delta(D_1)\theta(p_0 - k_0)\delta(D_2) = \frac{1}{2}\frac{\theta(p_0 - (M+m))}{\sqrt{(p_0^2 - (M+m)^2)(p_0^2 - (M-m)^2)}}$$

$$\int dk_0 dk_z \; \theta(-k_0)\delta(D_1)\theta(p_0 - k_0)\delta(D_2) = \frac{1}{2}\frac{\theta(p_0)\theta(M - m - p_0)}{\sqrt{((M+m)^2 - p_0^2)((M-m)^2 - p_0^2)}}$$

$$\int dk_0 dk_z \; \theta(k_0)\delta(D_1)\theta(-(p_0 - k_0))\delta(D_2) = 0$$

$$\int dk_0 dk_z \; \theta(-k_0)\delta(D_1)\theta(-(p_0 - k_0))\delta(D_2) = 0$$

For $0 < p_0 < M + m$ the integral $\int d^2k\delta(D_1)\delta(D_2)$ does not vanish, but $\mathtt{Im}A(p) = 0$;
in this case, the contribution of the δ's is compensated, again, by the contribution of the principal values.

Finally, for spacelike $p = (0, p_z)$ (with $p_0 = 0$),

$$\int dk_0 dk_z \ \delta(D_1)\delta(D_2) = 2\frac{1}{2}\frac{1}{\sqrt{(p_z^2 + (M+m)^2)(p_z^2 + (M-m)^2)}}$$

where the factor 2 reminds the 2 contributions from positive and negative k_0.

Note that in all cases, if the Mandelstam variable $u = p_0^2 - p_z^2$ is used, the value of all the specifications of the cuts is always proportional to the inverse of the square root $\sqrt{(u-(M+m)^2)(u-(M-m)^2)}$, proper of the two-body phase space.

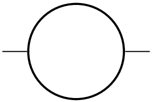
Imaginary parts and differential equations.

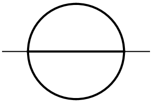
Take some Feynman amplitude $A(u)$ which satisfies a given differential equation; the equation will have, in general, a homogeneous part, involving only $A(u)$, and an inhomogenous part, involving *simpler* amplitudes, corresponding to graphs in which some of the propagators appearing in $A(u)$ are missing, supposedly known.
In the simplest cases, for instance the `Bubble`, the inhomogeneous terms are just real tadpoles, and the differential equation for the imaginary part of $A(u)$ is just the homogeneous part of the complete equation.
More in general, also the inhomogeneous terms can develop an imaginary part, expected to be somewhat simpler than the full amplitude, so that the resulting equation is somewhat (marginally?) simpler than the original equation.

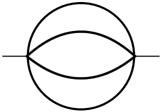
As $\mathtt{Re}A(u)$, $\mathtt{Im}A(u)$ satisfy separately the real and imaginary parts of the equation, the equation for $\mathtt{Im}A(u)$ is presumably simpler than the original equation.

If:
all the inhomogeneous terms are tadpoles, so that the differential equation for the imaginary part of the amplitude is the homogeneous part of the equation,
and one is able to evaluate directly (from the graph, say by using the largest time equation) the imaginary part,
one can then use that calculation for obtaining a first solution of the homogeneous equation – as a first step to a more complete discussion of the complete equation.

The first obvious examples are `Bubble`, `Sunrise` and `Banana`







Let us start from $\texttt{Bub}(u)$, the $\texttt{Bubble}$ amplitude, considering only the $d = 2$ limit for simplicity;
the homogeneous part of the equation for $\texttt{Bub(u)}$ is

$$\frac{d}{du}\texttt{Bub}(u) = -\frac{1}{2}\left(\frac{1}{u-(M+m)^2} + \frac{1}{u-(M-m)^2}\right)\texttt{Bub}(u)$$

with solution, up to a multiplicative constant, irrelevant here, is

$$\texttt{Bub}(u) = \frac{1}{\sqrt{(u-(M+m)^2)(u-(M-m)^2)}}$$

(Let us recall that, when considering only the real solutions of the previous equation, the multiplicative constant must be specified separately for each of the u intervals with end points $(-\infty, (M-m)^2, (M+m)^2, \infty)$.

The result is obviously in agreement with the already seen imaginary part of $A(p)$, obtained, according to the largest time equation, by considering the cut graph with the *proper* signs of the energy solutions of the δ-function conditions.
But also the other choices of the signs, whose meaning is not related to the Feynman graph amplitude, give the same result – and therefore satisfy also the equation.
To clarify this point, recall that the equation for the original Feynman amplitude was obtained by repeated use of the IBP's
(Integration by Parts Identities, K.G.Chetyrkin,F.V.Tkachov,1981)
for the integral of the product of two propagators in the loop variable $d^d k$ from $-\infty$ to $+\infty$, with the integration contours fixed by the usual $m^2 \to m^2 - i\epsilon$ Feynman prescription.

An essential feature of those IBP's is the absence of end point contributions (the integrands of the Feynman graphs can be considered as vanishing at infinity as a consequence of the properties of the continuous dimensional integration).
But end point contributions are absent also when the integration contour is a closed loop (not crossing a discontinuity cut of the integrand), symbolically

$$\oint_{C_1} dz \, \frac{df}{dz} = \oint_{C_2} dz \, \frac{df}{dz} = 0 \; ,$$

where $f(z)$ is some suitable function and C_1, C_2 two different contours, not crossing a discontinuity of $f(z)$.
If $D = k^2 + m^2 = -(k_0^2 - \vec{k}^2 - m^2)$, the corresponding Feynman propagator is $1/(D - i\epsilon)$, with two poles at, say, $k_0 = \pm\sqrt{\vec{k}^2 + m^2}$, and in the Feynman amplitude the integration contour of k_0 runs along the real axis from $-\infty$ to $+\infty$, passing below and above the negative and positive values of k_0 at the two poles.

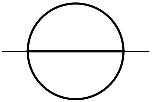
Replacing the Feynman propagator $1/(D - i\epsilon)$ by, say, $\theta(k_0)\delta(D)$ amounts then to keep the factor $1/D$ in the integrand, replacing however the previous *standard* integration path of k_0 with a (small) circle around $k_0 = \sqrt{\vec{k}^2 + m^2}$;
replacing $1/(D - i\epsilon)$ by $\delta(D) = (\theta(-k_0) + \theta(k_0))\delta(D)$, amounts to keep again $1/D$ in the integrand, while integrating around two (small) circles around both zeroes, *etc.*;
the procedure can be repeated for any other propagator as well.
As the integrand does not change, and end point contributions are always absent, the IBP's for the amplitudes for the modified contours are exactly the same IBP's for the original Feynman amplitude;
in particular, original Feynman amplitude and amplitudes with modified contours obey, *formally*, the same differential equations.

Why *formally* ?
The IBP's can generate, among many othe terms, a numerator consisting of polynomials in the scalar products of the various occurring vectors. Assume that there is a propagator $1/(D - i\epsilon)$ in the integrand of the original Feynman amplitude, and that a factor D is generated by the IBP's in the numerator; as

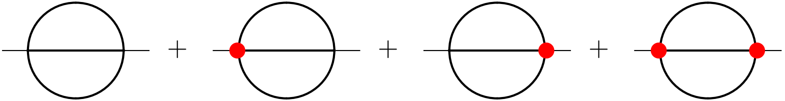
$$\frac{1}{D - i\epsilon}\ D = 1$$

in the resulting term the propagator $1/(D - i\epsilon)$ is missing, so that the resulting term belongs to a *subtopology*, ending up, in the equation, to an inhomogeneous term.
In the modified amplitude with a cut propagator (no matter how cut) $D(1/D) = 1$ implies that the singularities associated to $D = 0$ are canceled out, so that the integrations along the contours around those singularities give a vanishing contribution (even simpler: $D\delta(D) = 0$); the inhomogeneos term associated with the factor D in the numerator is therefore absent.

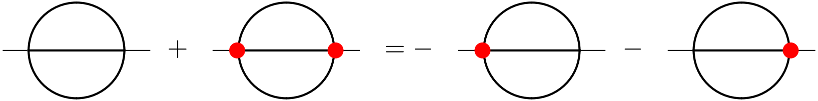
In the case of the `Bubble`, the double cut $\delta(D_1)\delta(D_2)$ implies that any inhomogeneous term corresponding to the absence of one of the propagators is absent; as the two propagators are both cut (so called *maximal cut*), there is no inhomogeneous term.
One recovers, in that way, the homogeneous equation for the imaginary part.
As the equation is a first order equation, with just one solution, there is no surprise that all cut amplitudes, being all solutions of a same homogeneous equation, are equal (more exactly, proportional).

Now the Sunrise 

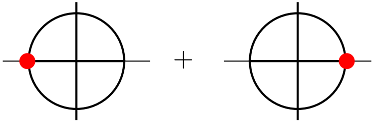
The largest time equation gives (same structure as the `Bubble`)

 $= 0$

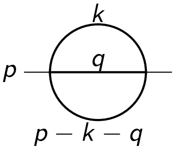
so that



or

$2\mathtt{ImSun}(u) =$ 

Some formulas for the equal mass case



$$D_1 = k^2 + m^2, \qquad D_2 = q^2 + m^2, \qquad D_3 = (p-k-q)^2 + m^2$$

$u = W^2 = (-p^2)$, in $d = 2$ dimensions (for simplicity; and neglecting overall normalization)

$$\mathrm{Sun}(u) = -i \int \frac{d^2k \; d^2q}{(D_1 - i\epsilon)(D_2 - i\epsilon)(D_3 - i\epsilon)}$$

The homogeneous (second order) differential equation for $\texttt{Sun}(u)$ is

$$\left\{ \frac{d^2}{du^2} + \left[\frac{1}{u} + \frac{1}{u - m^2} + \frac{1}{u - 9m^2} \right] \frac{d}{du} + \frac{1}{m^2} \left[-\frac{1}{3u} + \frac{1}{4(u - m^2)} + \frac{1}{12(u - 9m^2)} \right] \right\} \texttt{Sun}(u) = 0$$

The triple cut of the largest time equation (which is also a maximal cut) corresponds also to the (physical) three particle phase space at energy $W > 3m, u = W^2 > 9m^2$, which is known to be, long since (Dalitz-Fabri plot, 1953-1954).
Calling $I_0(u)$ that phase space, for $W > 3m$ one has (again up to a normalization factor)

$$I_0(u) = \int_{4m^2}^{(W-m)^2} \frac{db}{\sqrt{R_4(u, b)}} \ ,$$

where $R_4(u, b)$ is the fourth order polynomial in b

$$R_4(u, b) = b(b - 4m^2)((W - m)^2 - b)((W + m)^2 - b)$$

One can check that $I_0(u)$ is indeed a solution, by a kind of suitable integration by parts identities, in which the vanishing of $R_4(u,b)$ at the end-points of the integration interval plays an essential role; (one might, equally well, consider the integral in b of the same integrand along a closed contour including $4m^2$ and $(W-m)^2$, the end points of the integration).
Therefore, also all the integrals in b of the same integrand between any other two other zeroes of $R_4(u,b)$ (or any other contour including those points) are solutions of the equation.
Standard considerations for contour integrals along a closed path show then that only two of them are independent, say the previous $I_0(u)$ and, for instance

$$J_0(u) = \int_0^{4m^2} \frac{db}{\sqrt{-R_4(u,b)}} \ .$$

Once the integral representation of the two independent solutions of the homogeneous equation are obtained, one can switch to special function mathematics, which gives

$$I_0(u) = \frac{2}{\sqrt{(W+3m)(W-m)^3}}\; K\left(\frac{(W-3m)(W+m)^3}{(W+3m)(W-m)^3}\right) \; ,$$

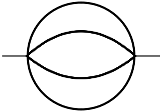
$$J_0(u) = \frac{2}{\sqrt{(W+3m)(W-m)^3}}\; K\left(1 - \frac{(W-3m)(W+m)^3}{(W+3m)(W-m)^3}\right) \; ,$$

where $K(x)$ is a complete elliptic integral.

One can then use the Euler's formulas for obtaining the solution of the complete inhomogeneous equation, as well as for obtaining the next terms in the $(d-2)$ expansion etc., as repeated integrations of rational fractions times the homogeneous solutions in $d=2$ dimensions, $I_0(u)$ and $J_0(u)$.

As $I_0(u)$ and $J_0(u)$ are complete elliptic integrals, it is natural to call, somewhat loosely, (generalized) Elliptic Polylogarithms all the (new) integrals appearing when following the Euler's method, by analogy with the (generalized) Polylogarithms appearing when the solutions of the differential equations can be expressed as repeated integrals of rational functions only.

More rigorous, unambiguous definitions of Elliptic Polylogarithms exist, linking them to the general theory and formalism of elliptic functions.

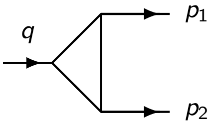
A few words on the (equal mass) Banana graph 

(A.Primo, L.Tancredi 2017)
In this case, the homogeneous equation is a third order equation, at first sight of impossible solution.
But, as an obvious extension of the sunrise case, it is known that the imaginary part of its amplitude is the maximally cut graph, which is also the 4-body phase;
the 4-body phase space can be easily written by "joining" two phase spaces of two particles each.
Following this path one obtains, in a relatively simple way, a double integral representation for the phase space, whose integrand is the (inverse) square root of a sixth order polynomial in two integration variables, say (a, b).

By suitably changing the integration region (in highly non-trivial ways) one obtains three integrals corresponding to three independent solutions; which, in turn, are found to be products of two complete elliptic integrals of suitable arguments.

Message: once more a bottom up approach, relying on physical, mathematically simple considerations, can provide a first step in the mathematical treatment of a (new) problem.

Consider now the vertex (or triangle loop), in continuous d dimensions:

$\mathrm{Tri}(d;s) =$ 

$$= \int \frac{\mathfrak{D}^{\,d} k}{(k^2+m^2)((k-p_1)^2+m^2)((k-p_1-p_2)^2+m^2)}$$

where $\mathfrak{D}^{\,d} k$ refers to the d-dimensional integration and normalization factors, in the kinematical configuration $q = p_1 + p_2$, $p_1^2 = p_2^2 = 0$ and $-q^2 = s$ (q^2 is positive when q is spacelike). The differential equation is

$$\frac{d}{ds}\mathrm{Tri}(d;s) = -\frac{1}{s}\mathrm{Tri}(d;s) + \frac{(d-2)}{8m^4}\left(\frac{1}{s-4m^2} - \frac{1}{s}\right)\mathrm{Tad}(d;m)$$
$$+ \frac{(d-3)}{4\,m^2}\left(\frac{1}{s-4m^2} - \frac{1}{s}\right)\mathtt{Bub}(d;s)\ ,$$

where $\mathrm{Tad}(d;m)$ is a tadpole, and $\mathtt{Bub}(d;s)$ is the equal mass `Bubble`.

The equation is very simple, especially its homogeneous part (which has solution $1/s$).
On the contrary, the equation for the imaginary part

$$\frac{d}{ds}\mathtt{ImTri}(d;s) = -\frac{1}{s}\mathtt{ImTri}(d;s) + \frac{(d-3)}{4\,m^2}\left(\frac{1}{s-4m^2} - \frac{1}{s}\right)\mathtt{ImBub}(d;s)$$

is not particularly simpler: it is *not* a homogeneous equation.
To get a homogeneous equation, one should look not at the imaginary part, but to the maximally cut amplitude, which in this case would be

$$\mathtt{Cut}_3(s) = \int \mathfrak{D}^{\,d}k\;\delta(k^2+m^2)\delta((k-p_1)^2+m^2)\delta((k-p_1-p_2)^2+m^2)$$

(... but it vanishes for real values of the vectors).

The dispersion relations

.

Any Feynman amplitude $\mathtt{A}(u)$ satisfies a dispersion relation of the form

$$\mathtt{A}(u) = \frac{1}{\pi}\int_{u_0}^{\infty} \frac{dv}{v-u-i\epsilon}\mathtt{ImA}(v)$$

where u_0 is a threshold and $\mathtt{ImA}(u)$ the imaginary part (better, when the amplitude depends, besides u, on other Mandelstam variables: the discontinuity in u) of $\mathtt{A}(u)$ discussed at length in the previous slides.

Almost for free, one can write a *subtracted* differential equation, such as

$$\mathtt{A}(u) = \mathtt{A}(0) + u\frac{1}{\pi}\int_{u_0}^{\infty} \frac{dv}{v(v-u-i\epsilon)}\mathtt{ImA}(v)$$

which can be sometimes useful to fix boundary conditions or convergence problems.

A first (practical) use of the dispersion relations:
evaluate $\mathtt{ImA}(v)$ first, up to the required order in $(d-2)$ or $(d-4)$, then the complete amplitude by means of the dispersion relation.
That implies one more integration, but of *standard* type in the repeated integration framework of the "usual" Generalized Polylogarithms formalism, with one more *standard* rational factor $1/(v-u-i\epsilon)$, where u is just one more parameter.

The procedure (obtain the imaginary part first, say solving the differential equation, then the whole amplitude with the dispersion relation) has some advantages, but somewhat limited, as the equation for the imaginary part is not in general much simpler or easier to solve than the complete equation.

A second (and better) practical use of the dispersion relations:
write the inhomogeneous terms in dispersive form.
Let's go back to the triangle amplitude; the inhomogeneous term contain the Bubble amplitude $\mathtt{Bub}(d;s)$;

$$\frac{d}{ds}\mathrm{Tri}(d;s) = -\frac{1}{s}\mathrm{Tri}(d;s) + \frac{(d-2)}{8m^4}\left(\frac{1}{s-4m^2} - \frac{1}{s}\right)\mathrm{Tad}(d;m) + \frac{(d-3)}{4\,m^2}\left(\frac{1}{s-4m^2} - \frac{1}{s}\right)\mathtt{Bub}(d;s)\ ,$$

writing $\mathtt{Bub}(d;s)$ in dispersive form, the equation becomes

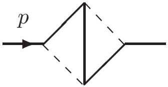
$$\frac{d}{ds}\mathrm{Tri}(d;s) = -\frac{1}{s}\mathrm{Tri}(d;s) + \frac{(d-2)}{8m^4}\left(\frac{1}{s-4m^2} - \frac{1}{s}\right)\mathrm{Tad}(d;m) + \frac{(d-3)}{4\,m^2}\frac{1}{\pi}\int_{u_0}^{\infty} dv\ \mathtt{ImBub}(d,v)\left(\frac{1}{s-4m^2} - \frac{1}{s}\right)\frac{(-1)}{s-v+i\epsilon}$$

As a first step, the equation can be solved by fully ignoring the integration in v and the actual value of $\texttt{ImBub}(d, v)$, but treating the new factor $1/(s - v + i\epsilon)$ on the same footing as the factors $1/s$, $1/(s - 4m^2)$ already appearing in the equation, *i.e.* considering the quantity v a kind of new constant or parameters.

Only as a second step one has to consider the actual value of $\texttt{ImBub}(d, v)$, and worry about the integration in v.

Another example, the QED 2-loop self-mass kite (Sabry, 1962)

$$\mathcal{I}(n_1, n_2, n_3, n_4, n_5) =$$



$$= \int \mathfrak{D}^d k \, \mathfrak{D}^d l \, \frac{1}{D_1^{n_1} D_2^{n_2} D_3^{n_3} D_4^{n_4} D_5^{n_5}}$$

In the d continuous dimensional regularization, the problem involves a total of 8 Master Integrals, among them the scalar amplitude with all the 5 propagators at the first power, say $f_8(d, u)$, and the scalar (equal mass) sunrise, called here $f_6(d, u)$.
In QED one is interested in the $d \to 4$ limit; in that limit, $f_6(d, u)$ develops a (trivial) double pole, so it can be convenient (even if somewhat confusing ...) to rescale all the amplitudes by suitable powers of $(d - 4)$, so that the rescaled set of equations has a finite limit at $d = 4$, and, as a further "simplification", (which might however increase the confusion) to use also the Tarasov-Lee dimensional shift from $d = 4$ to $d = 2$.

In terms of the already rescaled amplitudes, the equation for $f_8(d,u)$ reads

$$\frac{d}{du}f_8(d;u) = (d-4)\left(\frac{1}{u-m^2} - \frac{1}{2u}\right) f_8(d;u) + \frac{(d-4)^3}{24}\left(\frac{1}{m^2} - \frac{8}{u-m^2}\right) f_6(d;u) + \ldots$$

where the dots stand for a few inhomogeneous terms which turn out to be expressible with ordinary Polylogarithms.

The homogeneous equation for $f_8(d;u)$ is simple, the non trvial part is in the inhomogeneous part $f_6(d;u)$, the *elliptic* sunrise.

The first non trivial order is $(d-4)^3$:

$$\frac{d}{du} f_8^{(3)} u) = \left(\frac{1}{m^2} - \frac{8}{u - m^2} \right) f_6^{(0)}(u)$$
$$+ \frac{1}{u - m^2} \left(\frac{\pi^2}{96} - \frac{1}{16} G(0, m^2, u) \right) + \frac{1}{8u} G(m^2, m^2, u)$$

where $G(0, m^2, u)$, $G(m^2, m^2, u)$ are ordinary Polylogarithms, while $f_6^{(0)}(u)$ is the (scalar) amplitude of the sunrise in $d = 2$ dimensions, satisfying the dispersion relation

$$f_6^{(0)}(u) = \int_{9m^2}^{\infty} \frac{dt}{t - u - i\epsilon} I_0(t).$$

The integration of the differential equation becomes trivial and gives

$$f_8^{(3)} u) = \frac{1}{8} G(0, m^2, m^2, u) - \frac{1}{16} G(m^2, 0, m^2, u) - \frac{\pi^2}{96} G(m^2, u)$$
$$- \frac{1}{24} \int_{9m^2}^{\infty} dt \; I_0(t) \left(\frac{1}{m^2} - \frac{8}{t - m^2} \right) G(t, u) \; ,$$

where the dispersive "kernel" of the sunrise is completely factorised;

this feature remains valid at higher orders in the $(d-4)$, with $I_0(t)$ replaced by the sunrise imaginary part at that order.

Concluding remarks.

•The cut graph amplitudes offer significant information concerning the structure of the original graphs;

•in best cases they are closely related to simple and wellknown physical quantities, such as a many-particle phase space, which can be used as a first hint in the study of the differential equations for the concerned amplitudes;

•the dispersion relation representation is a kind of universal tool for "merging" the amplitude of a "subgraph" appearing in inhomogeneous terms, regardless of its peculiar analytic properties, within the solution of the complete equation.

In this approach:
the contributions of each of the subgraphs is “factorised” as

$$\int dv\ K(v)\ \frac{-1}{u - v + i\epsilon}\ ,$$

where u is the variable of the differential equation, v is proper of the subgraph and $K(v)$ is the imaginary part of the subgraph;
the dispersive factor $1/(u - v)$ fits naturally in the by now established frame of Generalised Polylogarithms, introducing only one more parameters (or “letter”) v, so that:
• the equation in u can be solved, without worrying about the actual value of $K(v)$,
• the problem of the integral in v can be tackled once the equation in u is solved.
The procedure: does not require the explicit knowledge of the dispersive kernel,
can be used even for amplitudes whose analytic structure is still to be discovered.