



KMPB Conference: Elliptic Integrals,  
Elliptic Functions & Modular Forms  
in Quantum Field Theory

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**Twisted elliptic multiple zeta values  
and non-planar superstring one-loop amplitudes**

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Oliver Schlotterer (AEI Potsdam & Perimeter Institute)

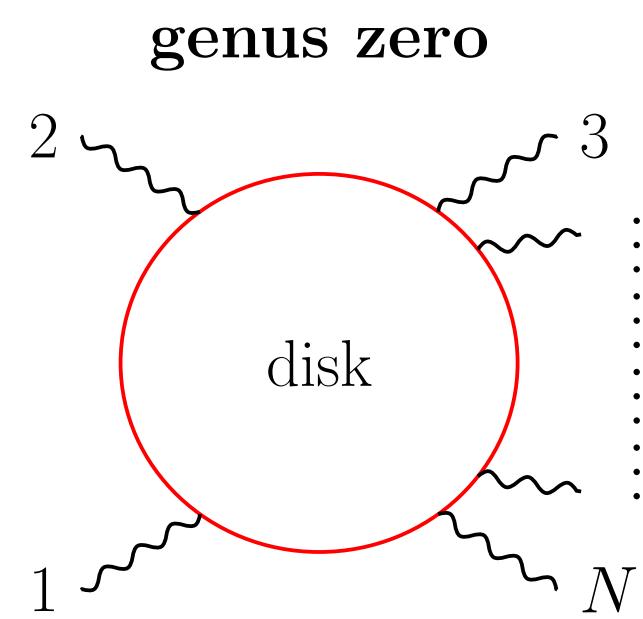
based on arXiv:1412.5535, 1507.02254 & 1704.03449

with J. Brödel, C. Mafra, N. Matthes, G. Richter



**Motivation:** Open-string scattering amplitudes @ tree level and one loop

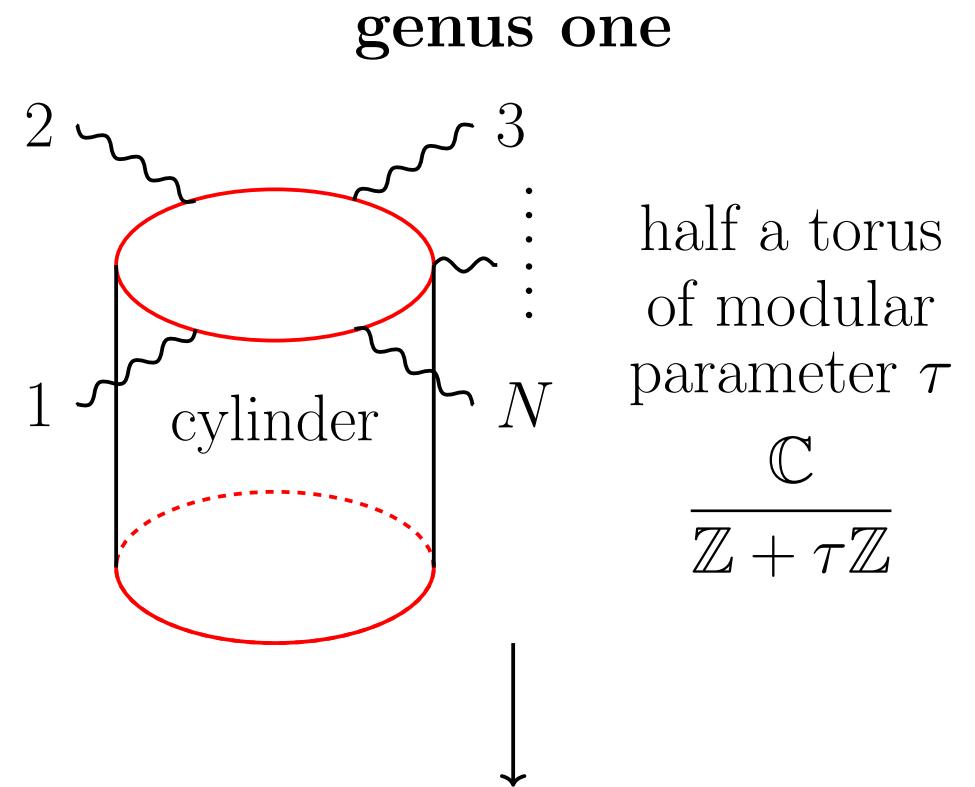
⇒ iterated integrals over the worldsheet boundaries



peel off  
YM trees

$$\int \prod_{0 \leq z_i \leq z_{i+1} \leq 1} \frac{dz_j}{z_j - n_j} = \text{MZVs}$$

[earlier work]



half a torus  
of modular  
parameter  $\tau$

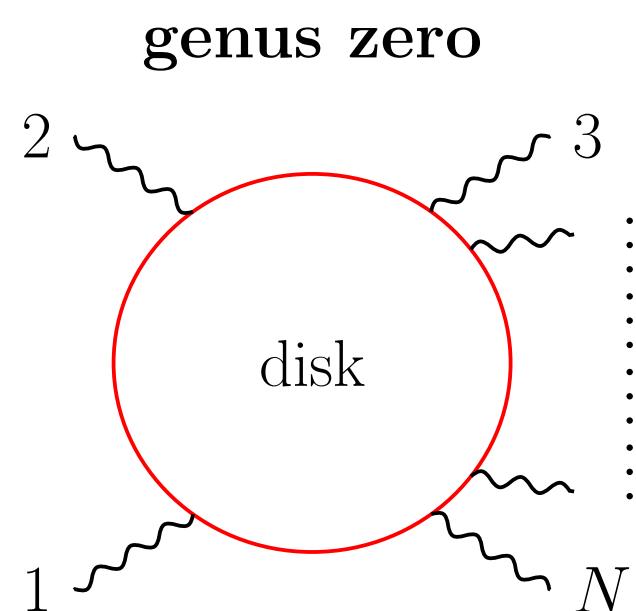
$$\frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}$$

$$\int \prod_{0 \leq z_i \leq z_{i+1} \leq 1} f^{(n_j)}(z_j) dz_j = \text{eMZVs}$$

[this talk]

**Motivation:** Open-string scattering amplitudes @ tree level and one loop

⇒ iterated integrals over the worldsheet boundaries

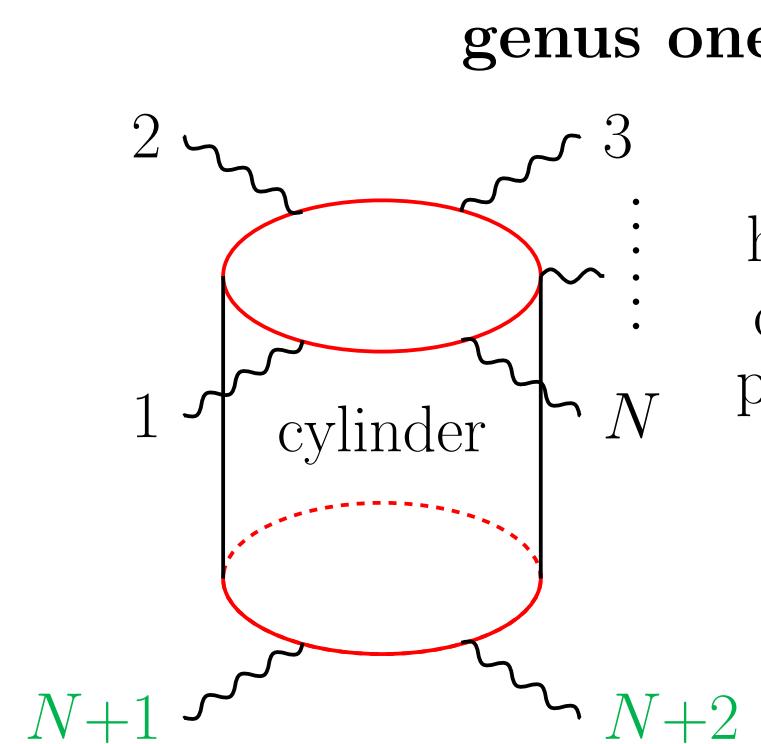


peel off  
YM trees

expand  
in  $\alpha'$

$$\int_{0 \leq z_i \leq z_{i+1} \leq 1} \prod \frac{dz_j}{z_j - n_j} = \text{MZVs}$$

[earlier work]



half a torus  
of modular  
parameter  $\tau$

$$\frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}$$

$$\int_{0 \leq z_i \leq z_{i+1} \leq 1} \prod f^{(n_j)}(z_j - \frac{\tau}{2}) dz_j = \text{twisted eMZVs}$$

[this talk]

# Outline

I. Definition of eMZVs and  $f^{(n)}$

[Brown, Levin, Enriquez, ...]

II. Planar one-loop string amplitudes

[Brödel, Mafra, Matthes, OS 1412.5535]

III. eMZV relations & indecomposables

[Brödel, Matthes, OS 1507.02254]

IV. Twisted eMZVs & non-planar amplitudes

[Brödel, Matthes, Richter, OS 1704.03449]

## I. 1 Iterated integrals: genus-zero versus elliptic

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Recall that multiple polylogarithms [Goncharov]

$$G(a_1, a_2, \dots, a_r; z) \equiv \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_r; t), \quad G(; z) \equiv 1$$

specialize to multiple zeta values (MZVs) upon  $z \rightarrow 1$  and  $a_j \rightarrow \{0, 1\}$ :

$$\zeta(a_1, a_2, \dots, a_r) \equiv (-1)^{\sum_{j=1}^r a_j} \int_{\substack{0 \leq z_i \leq z_{i+1} \leq 1}} \frac{dz_1}{z_1 - a_1} \frac{dz_2}{z_2 - a_2} \dots \frac{dz_r}{z_r - a_r}$$

Both are said to have weight  $r \equiv$  number of integrations.

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Both are said to have weight  $r \equiv$  number of integrations.

Can recursively bypass clashes  $a_j = z$  via partial fraction:

$$G(a, 0, z; z) = G(0, 0, a; z) - G(0, a, a; z) - \zeta_2 G(a; z)$$

resting on:  $\frac{1}{(z - a)(z - b)} = \frac{1}{a - b} \left( \frac{1}{z - a} - \frac{1}{z - b} \right)$

## I. 1 Iterated integrals: genus-zero versus elliptic

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Elliptic iterated integrals with suitable  $f^{(n)}$ ,  $n \in \mathbb{N}$  [Brown, Levin]

$$\Gamma \left( \frac{n_1}{a_1} \frac{n_2}{a_2} \dots \frac{n_r}{a_r}; z \right) \equiv \int_0^z dt \ f^{(n_1)}(t - a_1) \ \Gamma \left( \frac{n_2}{a_2} \dots \frac{n_r}{a_r}; t \right) , \quad \Gamma(; z) \equiv 1$$

specialize to elliptic multiple zeta values (eMZVs) upon  $z \rightarrow 1$  and  $a_j \rightarrow 0$ :

$$\omega(n_1, n_2, \dots, n_r) \equiv \int_{0 \leq z_i \leq z_{i+1} \leq 1} f^{(n_1)}(z_1) dz_1 \ f^{(n_2)}(z_2) dz_2 \ \dots \ f^{(n_r)}(z_r) dz_r .$$

Both are said to have length  $r$  and weight  $\sum_{j=1}^r n_j$ . [Enriquez 1301.3042]

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$$\Gamma \left( \begin{smallmatrix} n_1 & n_2 & \dots & n_r \\ a_1 & a_2 & \dots & a_r \end{smallmatrix} ; z \right) \equiv \int_0^z dt \ f^{(n_1)}(t - a_1) \ \Gamma \left( \begin{smallmatrix} n_2 & n_3 & \dots & n_r \\ a_2 & a_3 & \dots & a_r \end{smallmatrix} ; t \right) , \quad \Gamma(; z) \equiv 1$$

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Both are said to have length  $r$  and weight  $\sum_{j=1}^r n_j$ . [Enriquez 1301.3042]

Can recursively bypass clashes  $a_j = z$  via Fay relations ( $C_{p,q} \in \mathbb{Z}$ ):

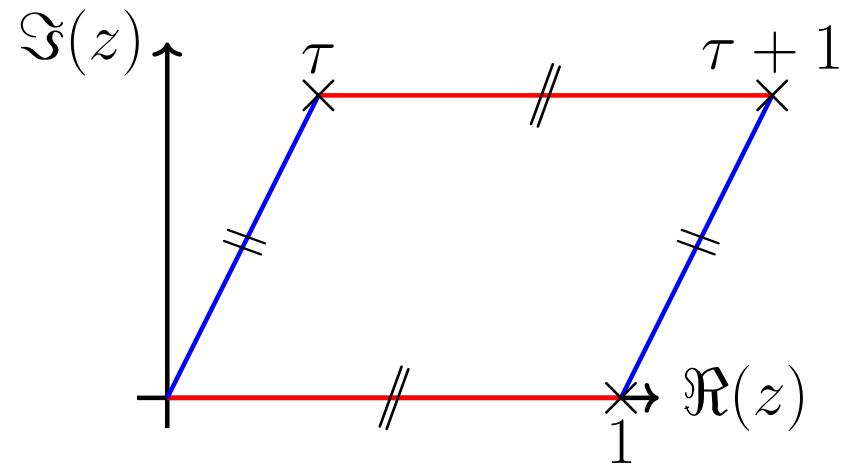
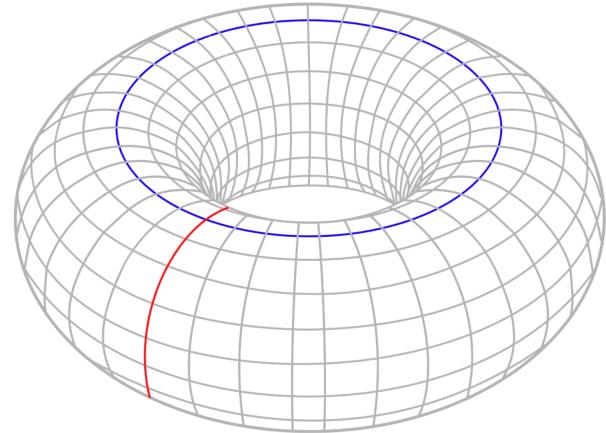
$$\Gamma \left( \begin{smallmatrix} 1 & 1 \\ z & 0 \end{smallmatrix} ; z \right) = 2\Gamma \left( \begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix} ; z \right) + \Gamma \left( \begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix} ; z \right) - 2\Gamma \left( \begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix} ; z \right) + \zeta_2$$

from  $f^{(m)}(z-a)f^{(n)}(z-b) = \sum_{p+q=m+n} C_{p,q} f^{(p)}(z-a)f^{(q)}(a-b) + \sum_{\substack{(m,a) \\ (n,b)}} \uparrow$

# I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{dz}{z}$ with Fay relation

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Parametrization of elliptic curve  $\equiv$  torus



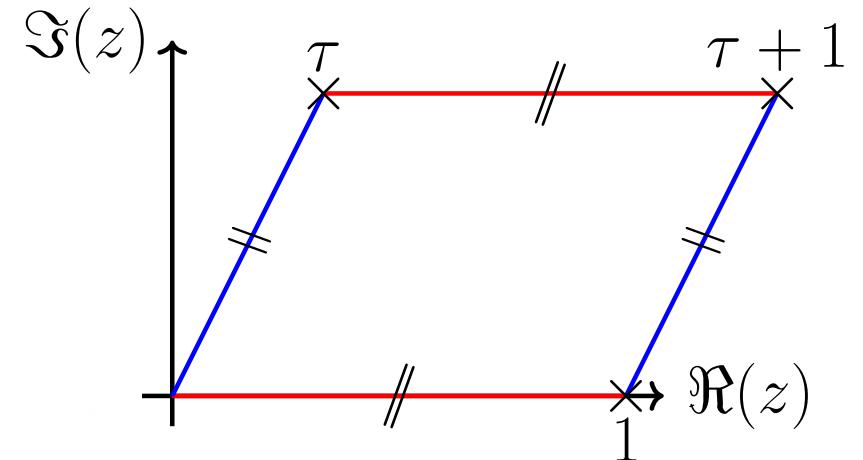
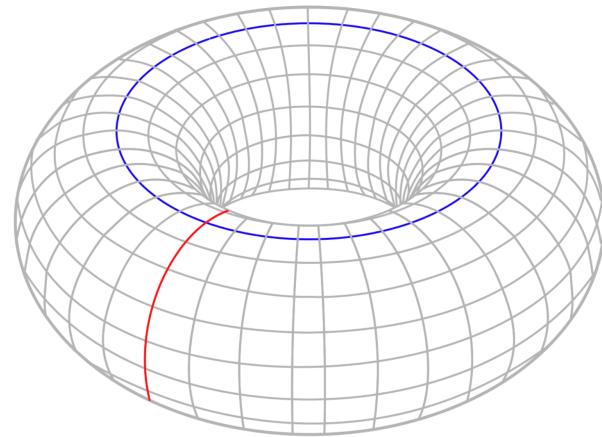
Jacobi  $\theta$ -function takes role of the identity map on the torus

$$\theta(z, \tau) \equiv \sin(\pi z) \prod_{n=1}^{\infty} (1 - e^{2\pi i(n\tau+z)})(1 - e^{2\pi i(n\tau-z)})$$

uplift  $\frac{1}{z} \rightarrow \partial_z \ln \theta(z, \tau) + 2\pi i \frac{\Im(z)}{\Im(\tau)} \equiv f^{(1)}(z, \tau)$

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uplift  $\frac{1}{z} \rightarrow \partial_z \ln \theta(z, \tau) + \underbrace{2\pi i \frac{S(z)}{S(\tau)}}_{\text{sacrifice holomorphicity for } f^{(1)}(z, \tau) = f^{(1)}(z+\tau, \tau)}$  poles like both  $\frac{1}{z}$  and  $\frac{1}{1-z}$

$$f^{(1)}(z, \tau) = f^{(1)}(z+1, \tau)$$

## I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{dz}{z}$ with Fay relation

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To go beyond

$$\text{uplift } \frac{1}{z} \rightarrow f^{(1)}(z, \tau) \equiv \partial_z \ln \theta(z, \tau) + 2\pi i \frac{\Im(z)}{\Im(\tau)},$$

check partial fraction  $\Rightarrow$  new function  $f^{(2)}$  without pole

$$\begin{array}{ccc} \frac{1}{(z-a)(z-b)} + \text{cyc}(z, a, b) & = & 0 \\ \updownarrow & & \\ f^{(1)}(z-a)f^{(1)}(z-b) + \text{cyc}(z, a, b) & = & f^{(2)}(z-a) + \text{cyc}(z, a, b) \end{array}$$

- similar identity for  $f^{(1)}(z-a)f^{(2)}(z-b)$  yields new function  $f^{(3)}$ , etc.
- formally adjoin  $f^{(0)} \equiv 1$
- drop the second argument  $\tau$  of  $\theta$  and  $f^{(n)}$  here and henceforth

## I. 2 $f^{(n)}$ – doubly periodic extension of $\frac{dz}{z}$ with Fay relation

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Generating fct.: non-holomorphic version of Kronecker-Eisenstein series

$$\Omega(z, \alpha) \equiv \exp\left(2\pi i \alpha \frac{\Im(z)}{\Im(\tau)}\right) \frac{\theta'(0)\theta(z + \alpha)}{\theta(z)\theta(\alpha)} = \sum_{n=0}^{\infty} \alpha^{n-1} f^{(n)}(z)$$

[Kronecker, Brown, Levin]

- doubly-periodic  $\Omega(z, \alpha) = \Omega(z + \tau, \alpha) = \Omega(z + 1, \alpha)$  thanks to  $\exp(\dots)$
- e.g.  $f^{(2)}(z) \equiv \frac{1}{2} \left\{ \left( \partial_z \ln \theta(z) + 2\pi i \frac{\Im(z)}{\Im(\tau)} \right)^2 + \partial_z^2 \ln \theta(z) - \frac{\theta'''(0)}{3\theta'(0)} \right\}$

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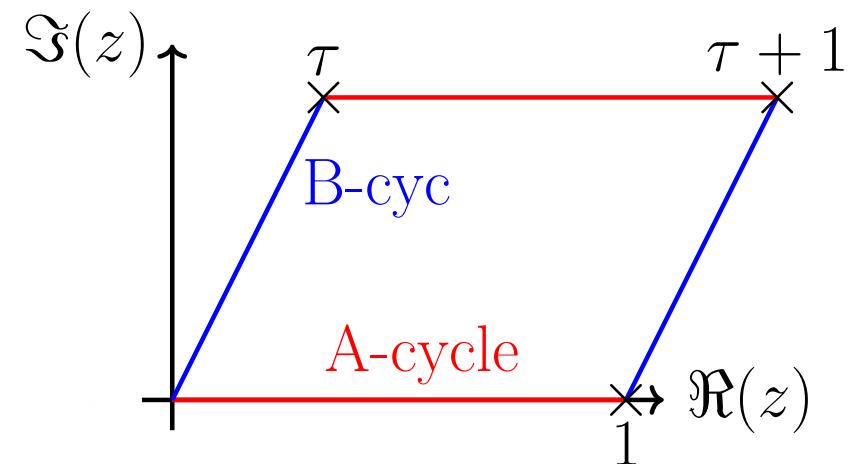
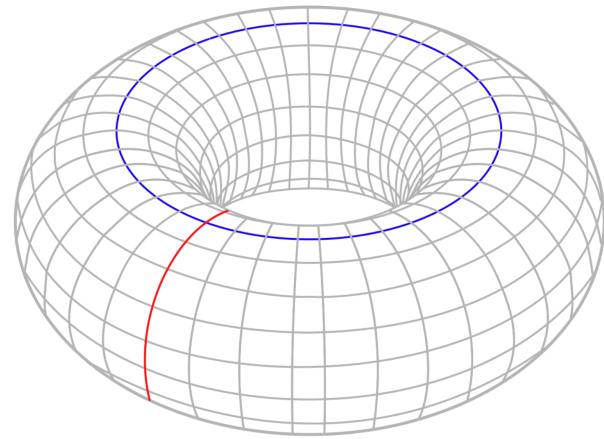
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- $\text{Res}_{z=0} \Omega(z, \alpha) = 1 \Rightarrow$  no other poles than  $f^{(1)}(z) \sim \frac{1}{z}$
- partial fraction generalizes to Fay relation:

$$\Omega(z_1, \alpha_1) \Omega(z_2, \alpha_2) = \Omega(z_1, \alpha_1 + \alpha_2) \Omega(z_2 - z_1, \alpha_2) + (1 \leftrightarrow 2)$$

**Disclaimer:** Only discuss A-elliptic MZVs in this talk

with integration along the A-cycle  $(0, 1)$



The modular S-transformation  $\tau \rightarrow -\frac{1}{\tau}$  maps A-elliptic to

B-elliptic MZVs – iterated integrals over the B-cycle from 0 to  $\tau$

[Enriquez 1301.3042]

→ harder to evaluate and interesting for the closed string

[see Federico Zerbini's talk]

## II. Planar one-loop superstring amplitude

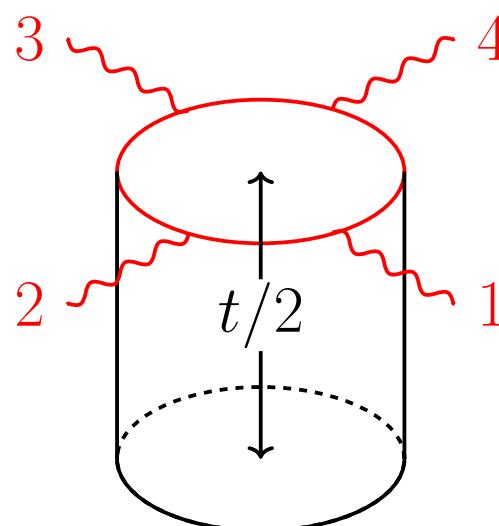
Focus on planar cylinder diagram at four-points:

$$A_{\text{string}}^{\text{1-loop}}(1, 2, 3, 4) = s_{12}s_{23}A_{\text{YM}}^{\text{tree}}(1, 2, 3, 4) \int_0^\infty dt \ I_{1234}(s_{ij}, \tau = it)$$

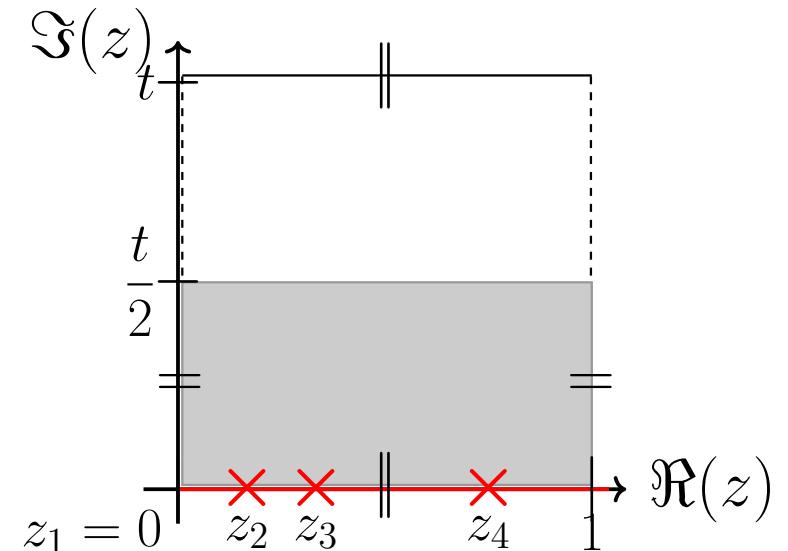
$$I_{1234}(s_{ij}, \tau) = \int_0^1 dz_4 \int_0^{z_4} dz_3 \int_0^{z_3} dz_2 \ \exp \left( \sum_{i < j}^4 s_{ij} P(z_i - z_j, \tau) \right) \Big|_{z_1=0}$$

[Brink, Green, Schwarz 1982]

with dimensionless  $s_{ij} \equiv \alpha'(k_i + k_j)^2$  and worldsheet propagator  $\partial P = f^{(1)}$ .



parametrized as  
→



Analytic  $\alpha'$ -dependence from expanding the exponentials

$$I_{1234}(s_{ij}, \tau) = \int_0^1 dz_4 \int_0^{z_4} dz_3 \int_0^{z_3} dz_2 \prod_{i < j}^4 \sum_{n_{ij}=0}^{\infty} \frac{[s_{ij} P(z_i - z_j, \tau)]^{n_{ij}}}{n_{ij}!} \Big|_{z_1=0}$$

and integrating  $P(z_i - z_j) = \int_{z_j}^{z_i} dx f^{(1)}(x - z_j)$  order by order in  $\alpha'$ .

Each monomial in  $s_{ij}$  is accompanied by eMZVs, e.g.

$$s_{ij}^0 \leftrightarrow \int_0^1 dz_4 f^{(0)}(z_4) \int_0^{z_4} dz_3 f^{(0)}(z_3) \int_0^{z_3} dz_2 f^{(0)}(z_2) = \omega(0, 0, 0)$$

after formally inserting  $f^{(0)} = 1$

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after formally inserting  $f^{(0)} = 1$  as well as

$$\left. \begin{array}{c} s_{12} \\ s_{13} \end{array} \right\} \leftrightarrow \int_0^1 dz_4 f^{(0)} \int_0^{z_4} dz_3 f^{(0)} \int_0^{z_3} dz_2 f^{(0)} \left\{ \begin{array}{l} \int_0^{z_2} dx f^{(1)}(x) \\ \int_0^{z_3} dx f^{(1)}(x) \end{array} \right.$$

$$\implies s_{12} \leftrightarrow \omega(1, 0, 0, 0), \quad s_{13} \leftrightarrow \underbrace{\omega(1, 0, 0, 0)}_{\text{from } 0 \leq x \leq z_2} + \underbrace{\omega(0, 1, 0, 0)}_{\text{from } z_2 \leq x \leq z_3}$$

At higher order ...

$$\begin{aligned}
 s_{12}s_{23} &\leftrightarrow \int_0^1 dz_4 \int_0^{z_4} dz_3 \int_0^{z_3} dz_2 \left( \int_{z_3}^{z_2} dx f^{(1)}(x - z_3) \right) \left( \int_0^{z_2} dy f^{(1)}(y) \right) \\
 &= - \int_0^1 dz_4 \int_0^{z_4} dz_3 \Gamma \left( \begin{smallmatrix} 1 & 0 & 1 \\ z_3 & 0 & 0 \end{smallmatrix}; z_3 \right)
 \end{aligned}$$

... need Fay relations

$$\Gamma \left( \begin{smallmatrix} 1 & 0 & 1 \\ z_3 & 0 & 0 \end{smallmatrix}; z_3 \right) = 2 \Gamma \left( \begin{smallmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \end{smallmatrix}; z_3 \right) + \Gamma \left( \begin{smallmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix}; z_3 \right) - 2 \Gamma \left( \begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{smallmatrix}; z_3 \right) + \zeta_2 \Gamma \left( \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}; z_3 \right) .$$

Above example then integrates to

$$\begin{aligned}
 s_{12}s_{23} &\leftrightarrow -2\omega(2, 0, 0, 0, 0) - \omega(0, 2, 0, 0, 0) + 2\omega(1, 1, 0, 0, 0) - \zeta_2 \omega(0, 0, 0) \\
 &= -\omega(1, 0, 0, 0, 1)
 \end{aligned}$$

after using eMZV relations at length five.

After using momentum conservation for  $s_{ij}$ , first orders simplify to

$$\begin{aligned} I_{1234}(s_{ij}) &= \omega(0,0,0) + 2\omega(0,1,0,0)s_{13} + 2\omega(0,1,1,0,0)(s_{12}^2 + s_{23}^2) \\ &\quad - 2\omega(0,1,0,1,0)s_{12}s_{23} + \beta_5(s_{12}^3 + 2s_{12}^2s_{23} + 2s_{12}s_{23}^2 + s_{23}^3) - \beta_{2,3}s_{12}s_{23}s_{13} + \dots \end{aligned}$$

with shorthands

[Brödel, Mafra, Matthes, OS 1412.5535]

$$\begin{aligned} \beta_5 &= \frac{4}{3} [\omega(0,0,1,0,0,2) + \omega(0,1,1,0,1,0) - \omega(2,0,1,0,0,0) - \zeta_2 \omega(0,1,0,0)] \\ \beta_{2,3} &= \frac{\zeta_3}{12} + \frac{8\zeta_2}{3} \omega(0,1,0,0) - \frac{5}{18} \omega(0,3,0,0) . \end{aligned}$$

Choice of indecomposable eMZVs requires guidance,

in particular at higher  $\alpha'$ -order  $\leftrightarrow$  weight  $\leftrightarrow$  length ...

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**also:**  $n$ -point amplitude naturally compatible with eMZV language !

### III. 1 eMZV relations and indecomposables

---

The following MZVs are believed to be indecomposable over  $\mathbb{Q}$ :

weight $w$	0	1	2	3	4	5	6	7	8	9	10	11	12
indec. MZVs	1	$\emptyset$	$\zeta_2$	$\zeta_3$	$\emptyset$	$\zeta_5$	$\emptyset$	$\zeta_7$	$\zeta_{3,5}$	$\zeta_9$	$\zeta_{3,7}$	$\zeta_{11}, \zeta_{3,3,5}$	$\zeta_{3,9}, \zeta_{1,1,4,6}$

MZVs satisfy shuffle and stuffle relations

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MZVs satisfy shuffle and stuffle relations, and eMZVs obey

- shuffle  $\omega(n_1, \dots, n_r) \omega(k_1, \dots, k_s) = \omega((n_1, \dots, n_r) \sqcup (k_1, \dots, k_s))$
- reflection:  $\omega(n_1, n_2, \dots, n_r) = (-1)^{n_1+n_2+\dots+n_r} \omega(n_r, \dots, n_2, n_1)$
- Fay rel's  $\omega(n_1, n_2, \dots) \leftrightarrow \omega(n_1+j, n_2-j, \dots)$  such as  $\omega(0, 5) = \omega(2, 3)$

all of which preserve the weight  $\sum_{j=1}^r n_j$  of  $f^{(n_j)}$  integrands.

→ Which eMZVs remain indecomposable w.r.t.  $\mathbb{Q}[\text{MZV}]$ ?

However,  $f^{(0)}$  @ zero weight  $\Rightarrow \exists \infty$  eMZVs  $\omega(n, \underbrace{0, \dots, 0}_{\text{any number}})$  @ weight  $n$

$\Rightarrow$  organize relations by length  $r$ :

$$\underline{\text{length } r = 1: \text{only constant eMZVs:}} \quad \omega(n) = \begin{cases} -2\zeta_n & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases}$$

length  $r = 2$ : shuffle and reflection reduce even-weight eMZVs to  $r = 1$ :

$$\omega(n_1, n_2) \Big|_{n_1+n_2 \text{ even}} = \begin{cases} 2\zeta_{n_1}\zeta_{n_2} & : n_1, n_2 \text{ both even} \\ 0 & : n_1, n_2 \text{ both odd} \end{cases}$$

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$$\underline{\text{length } r = 1: \text{only constant eMZVs:}} \quad \omega(n) = \begin{cases} -2\zeta_n & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases}$$

length  $r = 2$ : shuffle and reflection reduce even-weight eMZVs to  $r = 1$ :

$$\omega(n_1, n_2) \Big|_{n_1+n_2 \text{ even}} = \begin{cases} 2\zeta_{n_1}\zeta_{n_2} & : n_1, n_2 \text{ both even} \\ 0 & : n_1, n_2 \text{ both odd} \end{cases}$$

odd-weight  $\omega(n_1, n_2)$  depend on  $\tau$  and can be reduced to  $\omega(0, 2p - 1)$ :

$$\omega(n_1, n_2) \Big|_{n_1+n_2 \text{ odd}} = (-1)^{n_1} \omega(0, n_1 + n_2) + 2\delta_{n_1, 1} \zeta_{n_2} \omega(0, 1) - 2\delta_{n_2, 1} \zeta_{n_1} \omega(0, 1)$$

$$+ 2 \left\{ \sum_{p=1}^{\lceil \frac{1}{2}(n_2-3) \rceil} \binom{n_1 + n_2 - 2p - 2}{n_1 - 1} \zeta_{n_1+n_2-2p-1} \omega(0, 2p+1) - (n_1 \leftrightarrow n_2) \right\}$$

any length  $r$ : eMZVs with  $(-1)^{\text{length}} = (-1)^{\text{weight}}$  reducible to lower length

length  $r = 3$ :  $\lceil \frac{w}{6} \rceil$  indecomposable  $\omega(n_1, n_2, n_3)$  @ even  $w = \sum_{j=1}^3 n_j$ :

$w$	2	4	6	8	10	12	14
indec.	$\omega(0, 0, 2)$	$\omega(0, 0, 4)$	$\omega(0, 0, 6)$	$\omega(0, 0, 8)$	$\omega(0, 0, 10)$	$\omega(0, 0, 12)$	$\omega(0, 0, 14)$
eMZVs at $r = 3$				$\omega(0, 3, 5)$	$\omega(0, 3, 7)$	$\omega(0, 3, 9)$	$\omega(0, 3, 11)$
							$\omega(0, 5, 9)$

at  $w = 8$ , for instance, shuffle, reflection & Fay imply

$$\begin{aligned} \omega(0, 6, 2) = & -\frac{21}{2} \zeta_8 + 2 \omega(0, 3) \omega(0, 5) - 14 \zeta_6 \omega(0, 0, 2) \\ & - 6 \zeta_4 \omega(0, 0, 4) - \frac{9}{2} \omega(0, 0, 8) - \frac{2}{5} \omega(0, 3, 5) \end{aligned}$$

Further eMZV relations available @ <https://tools.aei.mpg.de/emzv>

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eMZVs at				$\omega(0, 3, 5)$	$\omega(0, 3, 7)$	$\omega(0, 3, 9)$	$\omega(0, 3, 11)$
$r = 3$							$\omega(0, 5, 9)$

length  $r = 4$ : conjecture  $\left\lfloor \frac{1}{2} + \frac{1}{48}(w+5)^2 \right\rfloor$  indecomp's @ odd  $w = \sum_{j=1}^4 n_j$ :

$w$	1	3	5	7	9
indec.	$\omega(0, 0, 1, 0)$	$\omega(0, 0, 0, 3)$	$\omega(0, 0, 0, 5)$	$\omega(0, 0, 0, 7)$	$\omega(0, 0, 0, 9), \omega(0, 0, 4, 5)$
eMZVs at			$\omega(0, 0, 2, 3)$	$\omega(0, 0, 2, 5)$	$\omega(0, 0, 2, 7), \omega(0, 1, 3, 5)$
$r = 4$				$\omega(0, 0, 4, 3)$	

summary @ length  $r \leq 7$ : indecomposable eMZVs @  $(-1)^{\text{weight}} \neq (-1)^{\text{length}}$

$w \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	1		1		1		1		1		1		1		1		1		1		1		1
3		1		1		1		2		2		2		3		3		3		4		4	
4	1		1		2		3		4		5		7		8		10		12		14		16
5		1		2		4		6		9		13		17		23		30		37		47	
6	1		2		4		8		13		22		31		45		?		?		?		?
7		1		4		8		16		29		48		?		?		?		?		?	

- strong evidence that shuffle, reflection & Fay  $\Rightarrow$  **any** eMZV relation
- will derive these numbers from **combinatorial principles**

### III. 2 Direct computation of eMZVs

eMZVs satisfy ODE w.r.t. modular parameter  $q = e^{2\pi i \tau}$ : [Enriquez 1301.3042]

$$-4\pi^2 q \frac{d}{dq} \text{eMZV} \left( \begin{smallmatrix} \text{weight } w \\ \text{length } \ell \end{smallmatrix} \right) \sim G_k(q) \times \text{eMZV} \left( \begin{smallmatrix} \text{weight } w-k+1 \\ \text{length } \ell-1 \end{smallmatrix} \right)$$

recursive in length, generating *Eisenstein series*  $G_k(q)$  in each step:

$$G_k(q) = \begin{cases} 0 & : k \text{ odd} \\ 2\zeta_k + \frac{2(2\pi i)^k}{(k-1)!} \sum_{m,n=1}^{\infty} m^{k-1} q^{mn} & : k \text{ even} \end{cases}$$

Supplement the above ODE in  $q$  by boundary values: [Enriquez 1301.3042]

$$\lim_{q \rightarrow 0} \text{eMZV} \left( \begin{smallmatrix} \text{weight } w \\ \text{length } \ell \end{smallmatrix} \right) \sim (2\pi i)^{\pm k} \times \text{MZVs}(\text{weight } w \mp k)$$

generating series for eMZV degenerates to Drinfeld associator

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recursively get  $q$ -expansion from *truncated Eisenstein series*  $G_k^0$ :

$$G_k^0(q) \sim \begin{cases} -1 & : k = 0 \\ \sum_{m,n=1}^{\infty} m^{k-1} q^{mn} & : k = 2, 4, 6, \dots \end{cases}$$

$\Rightarrow$  eMZVs are  $\mathbb{Q}[\text{MZV}]$  linear combination of iterated Eisenstein integrals

$$\gamma(k_1, k_2, \dots, k_r) \equiv \frac{1}{(4\pi^2)^r} \int_{0 \leq q_i \leq q_{i+1} \leq q} G_{k_1}^0(q_1) \frac{dq_1}{q_1} G_{k_2}^0(q_2) \frac{dq_2}{q_2} \dots G_{k_r}^0(q_r) \frac{dq_r}{q_r}$$

special cases of iterated Shimura integrals

[Manin 2005; Brown 2014]

Proof of linear independence

[Matthes 1601.05743]

$q$ -expansion of Eisenstein integrals easy to compute

$$\int_0^q (q')^N \frac{dq'}{q'} = \frac{q^N}{N}$$

and turns out to resemble MZVs as nested sums

$$\gamma(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \dots, k_r, 0^{p_r-1}) \sim \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{\sigma_{k_1-1}(n_1) \sigma_{k_2-1}(n_2 - n_1) \dots \sigma_{k_r-1}(n_r - n_{r-1}) q^{n_r}}{n_1^{p_1} n_2^{p_2} \dots n_r^{p_r}}$$

$$\zeta(1, 0^{p_1-1}, 1, 0^{p_2-1}, \dots, 1, 0^{p_r-1}) \sim \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{1}{n_1^{p_1} n_2^{p_2} \dots n_r^{p_r}} = \zeta_{p_1, p_2, \dots, p_r}$$

$q$ -expansion of Eisenstein integrals easy to compute

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$$\gamma(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \dots, k_r, 0^{p_r-1}) \sim \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{q^{n_r} \sigma_{k_1-1}(n_1) \sigma_{k_2-1}(n_2 - n_1) \dots \sigma_{k_r-1}(n_r - n_{r-1})}{n_1^{p_1} n_2^{p_2} \dots n_r^{p_r}}$$

$$\zeta(1, 0^{p_1-1}, 1, 0^{p_2-1}, \dots, 1, 0^{p_r-1}) \sim \sum_{0 < n_1 < n_2 < \dots < n_r} \frac{1}{n_1^{p_1} n_2^{p_2} \dots n_r^{p_r}} = \zeta_{p_1, p_2, \dots, p_r}$$

Divisor sums  $\sigma_k(n) \equiv \sum_{d|n} d^k$  obstruct analogues of stuffle-relations

$\Rightarrow$  no relations among  $\gamma(\dots)$  beyond shuffle  $\gamma(W_1)\gamma(W_2) = \gamma(W_1 \sqcup W_2)$

[Matthes 1601.05743]

Length-two example: indecomposable eMZVs  $\omega(0, k)$  @  $k = 3, 5, 7, \dots$

$$\omega(0, k) = k \gamma(k+1) = \text{rational} \times \pi^{k-1} \sum_{m,n=1}^{\infty} \frac{m^{k-1}}{n} q^{mn}$$

### III. 3 Back to counting indecomposable eMZVs

---

Naively: indecomposable eMZVs  $\leftrightarrow$  shuffle-independent Eisenstein int's?

$$\text{eMZV} \left( \begin{array}{c} \text{weight } \sum_j k_j - r \\ \text{length } r+1 \end{array} \right) \xleftrightarrow{?} \frac{\{\gamma(k_1, k_2, \dots, k_r) @ k_j = 0, 2, 4, \dots\}}{\text{shuffle-relations } \gamma(W_1)\gamma(W_2) = \gamma(W_1 \sqcup W_2)}$$

Correct at length two  $\omega(0, k) \leftrightarrow \gamma(k + 1)$ , but  $\exists$  problems at length three:

$$\text{e.g. } \underbrace{\omega(0, 0, 12), \omega(0, 3, 9)}_{2 \times \text{indecomposable}} \leftrightarrow \underbrace{\gamma(0, 14), \gamma(2, 12), \gamma(4, 10), \gamma(6, 8)}_{4 \times \text{shuffle-independent}}$$

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Additional selection rules arise from derivation algebra  $\{\epsilon_0, \epsilon_2, \epsilon_4, \dots\}$

”dual” to Eisenstein series  $G_0, G_2, G_4, \dots$  subject to relations

$$[\epsilon_2, \epsilon_k] = 0 , \quad [\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6] = 0 , \quad \text{etc.}$$

Explains above mismatch “ $2 \times \omega(\dots)$  vs.  $4 \times \gamma(\dots)$ ”

## Generating series of eMZVs: elliptic associator [Enriquez 1301.3042]

$$A(q) \equiv \sum_{r \geq 0} (-1)^r \sum_{n_1, n_2, \dots, n_r \geq 0} \omega(n_1, n_2, \dots, n_r) \text{ad}_x^{n_r}(y) \dots \text{ad}_x^{n_2}(y) \text{ad}_x^{n_1}(y)$$

Derivations  $\epsilon_0, \epsilon_2, \epsilon_4, \dots$  act on non-commutative variables  $x, y$  via

$$\epsilon_{2n}(x) = (\text{ad}_x)^{2n}(y) , \quad \epsilon_0(y) = 0 , \quad [\text{Nakamura, Tsunogai 1995, } \dots]$$

$$\epsilon_{2n}(y) = \sum_{0 \leq j < n} (-1)^j [(\text{ad}_x)^j(y), (\text{ad}_x)^{2n-1-j}(y)] \quad @ n > 0$$

satisfy various commutator relations and enter the associator's ODE

$$q \frac{d}{dq} A(q) = \frac{1}{4\pi^2} \left( \sum_{n=0}^{\infty} (2n-1) G_{2n}(q) \epsilon_{2n} \right) A(q) .$$

- ⇒ more precise version of “ $\epsilon_{2n}$  are dual to  $G_{2n}$ ”
- ⇒ commutator relations among  $\epsilon_{2n}$  impose selection rules on  $\gamma(\dots)$

Hence, more accurate picture is

$$\text{eMZV} \left( \begin{array}{c} \text{weight } \sum_j k_j - r \\ \text{length } r+1 \end{array} \right) \longleftrightarrow \frac{\{\gamma(k_1, k_2, \dots, k_r) @ k_j = 0, 2, 4, 6, \dots\}}{\{\text{shuffle-relations}\} \times \{\epsilon_k\text{-relations}\}}$$

For instance,  $[\epsilon_2, \epsilon_k] = 0$  obstructs non-trivial appearance of  $k_j = 2$  and

$$\omega(0, n) \leftrightarrow \gamma(n+1), \quad \omega(0, 3, 5) \leftrightarrow \gamma(4, 6)$$

$$\omega(0, 0, n) \leftrightarrow \gamma(n+2, 0), \quad \omega(0, 3, 7) \leftrightarrow \gamma(4, 8)$$

$$\left. \begin{array}{l} \omega(0, 3, 9) \\ \omega(0, 5, 7) \end{array} \right\} \leftrightarrow 81 \gamma(10, 4) + 35 \gamma(8, 6)$$

selected by  $[\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6] = 0$

[Brödel, Matthes, OS 1507.02254]

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selected by  $[\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6] = 0$

Cusp forms control  $\epsilon_k$ -relations, e.g. [Brown; Hain, Matsumoto; Pollack]

$$\text{wt. 12 cusp form} \leftrightarrow 0 = [\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6], \quad \text{[Ihara, Takao 1993; Schneps 2006]}$$

$$\text{wt. 16 cusp form} \leftrightarrow 0 = 2[\epsilon_{14}, \epsilon_4] - 7[\epsilon_{12}, \epsilon_6] + 11[\epsilon_{10}, \epsilon_8]$$

At length  $\ell = 4$ , for instance,

$$\text{eMZV}\left(\begin{smallmatrix} \text{weight } \sum_j k_j - 3 \\ \text{length } 4 \end{smallmatrix}\right) \longleftrightarrow \frac{\{\gamma(k_1, k_2, k_3) @ k_j = 0, 2, 4, 6, \dots\}}{\{\text{shuffle-relations}\} \times \{\epsilon_k\text{-relations}\}}$$

indecomposable eMZVs governed by derived relations, e.g.

$$0 = [\epsilon_k, [\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6]] = [\epsilon_k, 2[\epsilon_{14}, \epsilon_4] - 7[\epsilon_{12}, \epsilon_6] + 11[\epsilon_{10}, \epsilon_8]]$$

and irreducible relations controlled by cusp forms

[Pollack 2009]

$$\begin{aligned} 0 &= 80 [\epsilon_{12}, [\epsilon_4, \epsilon_0]] + 16 [\epsilon_4, [\epsilon_{12}, \epsilon_0]] - 250 [\epsilon_{10}, [\epsilon_6, \epsilon_0]] \\ &\quad - 125 [\epsilon_6, [\epsilon_{10}, \epsilon_0]] + 280 [\epsilon_8, [\epsilon_8, \epsilon_0]] - 462 [\epsilon_4, [\epsilon_4, \epsilon_8]] - 1725 [\epsilon_6, [\epsilon_6, \epsilon_4]] \end{aligned}$$

At length  $\ell = 4$ , for instance,

$$\text{eMZV}\left(\begin{array}{c} \text{weight } \sum_j k_j - 3 \\ \text{length } 4 \end{array}\right) \longleftrightarrow \frac{\{\gamma(k_1, k_2, k_3) @ k_j = 0, 2, 4, 6, \dots\}}{\{\text{shuffle-relations}\} \times \{\epsilon_k\text{-relations}\}}$$

indecomposable eMZVs governed by derived relations, e.g.

$$0 = [\epsilon_k, [\epsilon_{10}, \epsilon_4] - 3[\epsilon_8, \epsilon_6]] = [\epsilon_k, 2[\epsilon_{14}, \epsilon_4] - 7[\epsilon_{12}, \epsilon_6] + 11[\epsilon_{10}, \epsilon_8]]$$

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Perfect matching with  
indecomposable eMZVs

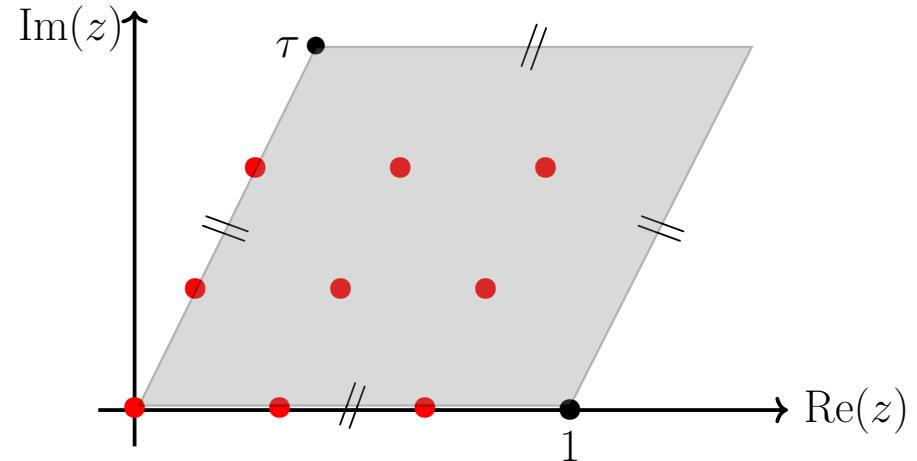
[Brödel, Matthes, OS 1507.02254]

wt	1	2	3	4	5	6	7	8	9	10	11	12	13
$\ell=2$	1		1		1		1		1		1		1
$\ell=3$		1		1		1		2		2		2	
$\ell=4$	1		1		2		3		4		5		7

## IV. Twisted eMZVs and non-planar string amplitudes

**Idea:** shift arguments  $z$  of  $f^{(n)}$

by rational lattice  $\Lambda \subset \mathbb{Q} + \mathbb{Q}\tau$



### untwisted eMZVs

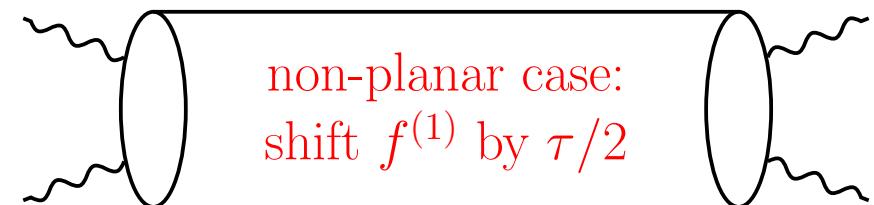
iterated integrals over  $f^{(n)}(z)$



planar one-loop  
string amplitude

### twisted eMZVs

iterated int's over  $f^{(n)}(z - \mathbb{Q} - \mathbb{Q}\tau)$



non-planar case:  
shift  $f^{(1)}$  by  $\tau/2$

$\tau \rightarrow i\infty \Rightarrow$  MZVs &  $\frac{1}{2\pi i}$

$\tau \rightarrow i\infty \Rightarrow$  cyclotomic MZVs &  $\frac{1}{2\pi i}$

$\frac{\partial}{\partial \tau} \Rightarrow G_k = -f^{(k)}(0)$

$\frac{\partial}{\partial \tau} \Rightarrow f^{(k)}(\mathbb{Q} + \mathbb{Q}\tau)$

## IV. 1 Appearance in non-planar open-string amplitudes

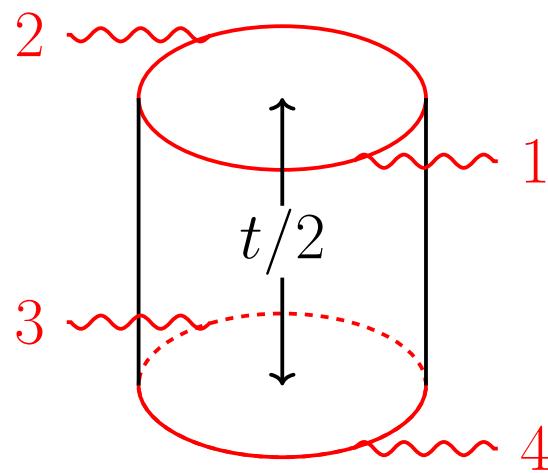
Non-planar cylinder diagram at four-points:

$$A_{\text{string}}^{\text{1-loop}}(1, 2 | 3, 4) = s_{12}s_{23}A_{\text{YM}}^{\text{tree}}(1, 2, 3, 4) \int_0^\infty dt I_{12|34}(s_{ij}, \tau = it)$$

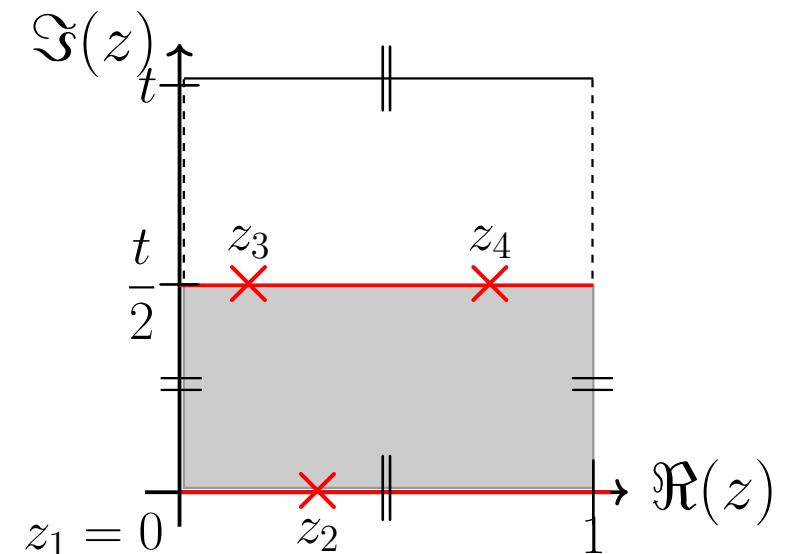
$$I_{12|34}(s_{ij}, \tau) = \int_{\tau/2}^{\tau/2+1} dz_4 \int_{\tau/2}^{\tau/2+1} dz_3 \int_0^1 dz_2 \exp \left( \sum_{i < j}^4 s_{ij} P(z_i - z_j, \tau) \right) \Big|_{z_1=0}$$

[Brink, Green, Schwarz 1982]

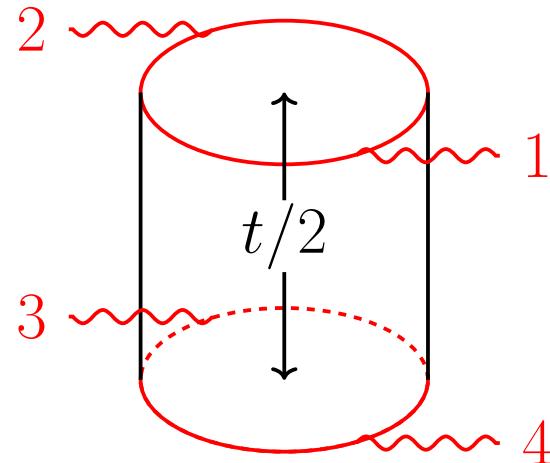
with dimensionless  $s_{ij} \equiv \alpha'(k_i + k_j)^2$  and worldsheet propagator  $\partial P = f^{(1)}$ .



parametrized as



Different iterated integrals for planar & non-planar propagator



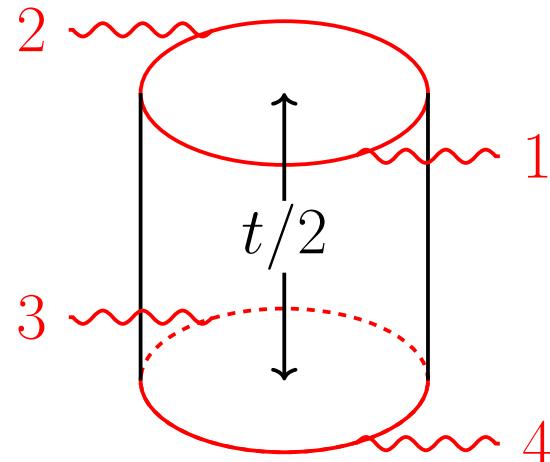
$$P(z_1 - z_2) = \int_0^{z_2} dx f^{(1)}(x)$$

$$P(z_1 - z_3) = \int_0^{z_2} dx f^{(1)}(x - \tau/2) + c(q)$$

Lines up with definition of twisted eMZVs (not necessarily  $b_i \in \{0, \tau/2\}$ )

$$\omega \left( \begin{smallmatrix} n_1, & n_2, & \dots & n_r \\ b_1, & b_2, & \dots & b_r \end{smallmatrix} \right) = \int_{0 \leq z_i \leq z_{i+1} \leq 1} f^{(n_1)}(z_1 - b_1) dz_1 \dots f^{(n_r)}(z_r - b_r) dz_r$$

Different iterated integrals for planar & non-planar propagator



$$P(z_1 - z_2) = \int_0^{z_2} dx \ f^{(1)}(x)$$

$$P(z_1 - z_3) = \int_0^{z_2} dx \ f^{(1)}(x - \tau/2) + c(q)$$

Lines up with definition of twisted eMZVs (teMZVs)

$$\omega \left( \begin{matrix} n_1, n_2, \dots n_r \\ b_1, b_2, \dots b_r \end{matrix} \right) = \int_{\substack{0 \leq z_i \leq z_{i+1} \leq 1}} f^{(n_1)}(z_1 - b_1) dz_1 \cdots f^{(n_r)}(z_r - b_r) dz_r$$

Offset in the non-planar propagator also expressible via teMZVs

$$c(q) = \omega \left( \begin{matrix} 1, 0 \\ 0, 0 \end{matrix} \right) - \omega \left( \begin{matrix} 1, 0 \\ \tau/2, 0 \end{matrix} \right) - \frac{1}{8} \log(q) ,$$

and logarithms can be absorbed as  $I_{12|34}(s_{ij}, \tau) = q^{s_{12}/4} \times (\text{teMZVs})$ .

Order by order expansion  $\Rightarrow$  teMZVs @ twists  $b_i \in \{0, \tau/2\}$ , e.g.

$$I_{12|34}(q) \Big|_{s_{13}s_{23}} = 2\omega \left( \begin{smallmatrix} 1, & 0 \\ \tau/2, & 0 \end{smallmatrix} \right)^2 - 4\omega \left( \begin{smallmatrix} 1, & 1, & 0 \\ \tau/2, & \tau/2, & 0 \end{smallmatrix} \right)$$

However, twisted eMZVs can be eliminated here and @ higher order

$$I_{12|34}(q) \Big|_{s_{13}s_{23}} = -2\omega(0, 0, 2) - \frac{2\zeta_2}{3} \quad \text{untwisted eMZV !}$$

Order by order expansion  $\Rightarrow$  teMZVs @ twists  $b_i \in \{0, \tau/2\}$ , e.g.

$$I_{12|34}(q) \Big|_{s_{13}s_{23}} = 2\omega \left( \begin{smallmatrix} 1, & 0 \\ \tau/2, & 0 \end{smallmatrix} \right)^2 - 4\omega \left( \begin{smallmatrix} 1, & 1, & 0 \\ \tau/2, & \tau/2, & 0 \end{smallmatrix} \right)$$

However, twisted eMZVs can be eliminated here and @ higher order

$$I_{12|34}(q) \Big|_{s_{13}s_{23}} = -2\omega(0, 0, 2) - \frac{2\zeta_2}{3} \quad \text{untwisted eMZV !}$$

**Strategy:** use differential equation of teMZVs with twists  $b_i$

$$-4\pi^2 q \frac{d}{dq} \text{teMZV} \left( \begin{smallmatrix} \text{weight } w \\ \text{length } \ell \end{smallmatrix} \right) \sim f^{(k)}(b_i, q) \times \text{teMZV} \left( \begin{smallmatrix} \text{weight } w-k+1 \\ \text{length } \ell-1 \end{smallmatrix} \right)$$

where  $f^{(k)}(b_i, q)$  generalize Eisenstein series  $G_k(q) = -f^{(k)}(0, q)$  in

$$-4\pi^2 q \frac{d}{dq} \text{eMZV} \left( \begin{smallmatrix} \text{weight } w \\ \text{length } \ell \end{smallmatrix} \right) \sim G_k(q) \times \text{eMZV} \left( \begin{smallmatrix} \text{weight } w-k+1 \\ \text{length } \ell-1 \end{smallmatrix} \right)$$

$\longrightarrow$  modular forms for congruence subgroups of  $\text{SL}_2(\mathbb{Z})$ .

Order by order expansion  $\Rightarrow$  teMZVs @ twists  $b_i \in \{0, \tau/2\}$ , e.g.

$$I_{12|34}(q) \Big|_{s_{13}s_{23}} = 2\omega \left( \begin{smallmatrix} 1, & 0 \\ \tau/2, & 0 \end{smallmatrix} \right)^2 - 4\omega \left( \begin{smallmatrix} 1, & 1, & 0 \\ \tau/2, & \tau/2, & 0 \end{smallmatrix} \right)$$

However, twisted eMZVs can be eliminated here and @ higher order

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where  $f^{(k)}(b_i, q)$  generalize Eisenstein series  $G_k(q) = -f^{(k)}(0, q)$ .

**Moral:** write teMZVs as iterated  $q$ -integral  $\int_{q_i} f^{(k_1)}(b_1, q_1) f^{(k_2)}(b_2, q_2) \dots$   
 $\implies$  expose relations over  $\mathbb{Q}$  such as the example above !

After systematically exploiting relations among teMZVs:

$$I_{12|34}(q) = q^{s_{12}/4} \left\{ 1 + s_{12}^2 \left( \frac{7\zeta_2}{6} + 2\omega(0,0,2) \right) - 2s_{13}s_{23} \left( \frac{\zeta_2}{3} + \omega(0,0,2) \right) \right. \\ \left. - 4\zeta_2\omega(0,1,0,0)s_{12}^3 + s_{12}s_{13}s_{23} \left( \frac{5}{3}\omega(0,3,0,0) + 4\zeta_2\omega(0,1,0,0) - \frac{1}{2}\zeta_3 \right) + \dots \right\}$$

Reduction to untwisted eMZVs should persist to all orders in  $\alpha'$ :

- generic  $\omega \left( \begin{smallmatrix} 1, & \cdots \\ \tau/2, & \dots \end{smallmatrix} \right)$  involve half-odd integer powers  $q^{\mathbb{N}+1/2}$
- $q^{s_{12}/4} \times q^{\mathbb{N}+1/2} \Rightarrow$  unphysical closed-string poles upon  $\int_0^1 \frac{dq}{q}$

After systematically exploiting relations among teMZVs:

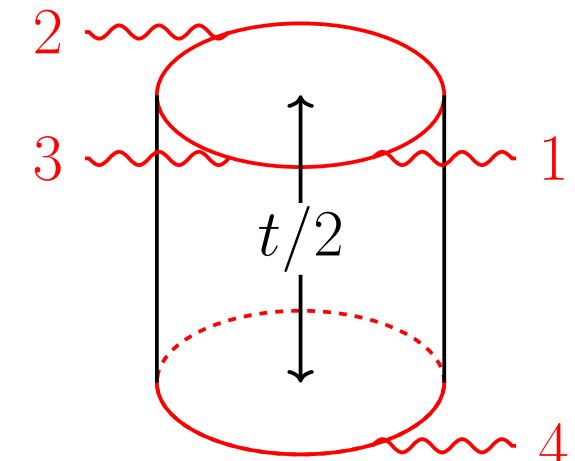
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Reduction to untwisted eMZVs should persist to all orders in  $\alpha'$ :

- generic  $\omega \left( \frac{1}{\tau/2}, \dots \right)$  involve half-odd integer powers  $q^{\mathbb{N}+1/2}$
- $q^{s_{12}/4} \times q^{\mathbb{N}+1/2} \Rightarrow$  unphysical closed-string poles upon  $\int_0^1 \frac{dq}{q}$

Same strategy applies to “3+1” non-planar integral

$$I_{123|4}(q) = \frac{1}{2} + (s_{12}^2 + s_{12}s_{23} + s_{23}^2) \left( \frac{7\zeta_2}{12} + \omega(0,0,2) \right) \\ - s_{12}s_{23}s_{13} \left( 2\zeta_2\omega(0,1,0,0) - \frac{5}{6}\omega(0,3,0,0) + \frac{\zeta_3}{4} \right) + \dots$$

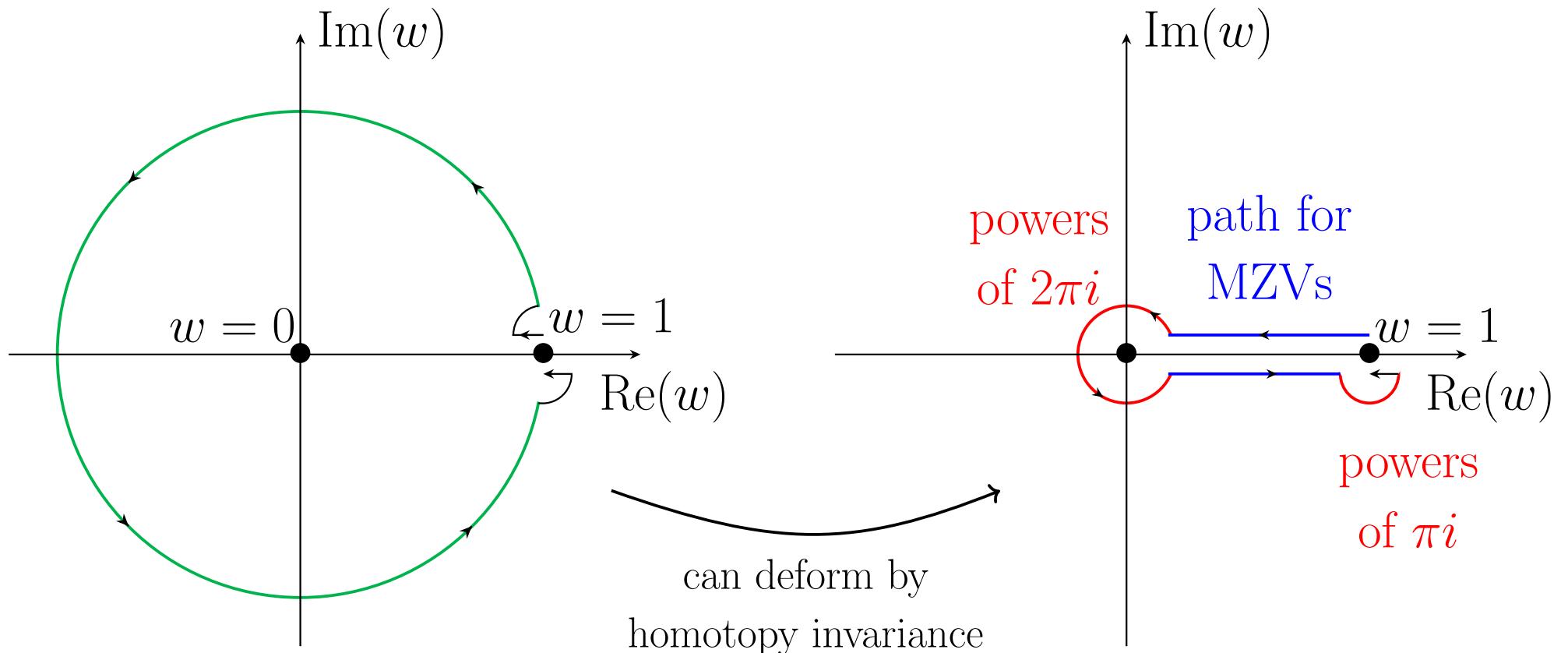


## IV. 2 Constant terms and cyclotomic MZVs

To study  $\tau \rightarrow i\infty$  degeneration of teMZVs, change variables to

$$w = e^{2\pi iz}, \quad dz = \frac{1}{2\pi i} \frac{dw}{w}$$

Integration domain  $z \in (0, 1)$  is mapped to the unit circle



What is the  $\tau \rightarrow i\infty$  degeneration of the  $f^{(k)}(z - b, \tau)$  in the integrand?

$$w = e^{2\pi iz}, \quad dz = \frac{1}{2\pi i} \frac{dw}{w}$$

- most of the  $f^{(k \neq 1)}$  degenerate to multiple of  $\frac{dw}{w}$ , ( $r, s \in \mathbb{Q}$ )

$$\lim_{\tau \rightarrow i\infty} f^{(k)}(z - s - r\tau, \tau) dz = -\frac{dw}{w} \sum_{m=0}^k \frac{B_m r^{k-m} (-2\pi i)^{k-1}}{m! (k-m)!}, \quad k \neq 1$$

- exceptional case: singular form  $f^{(1)}(z, \tau) \sim \frac{1}{z}$ :

$$\lim_{\tau \rightarrow i\infty} f^{(1)}(z - r\tau, \tau) dz = \begin{cases} \left(\frac{1}{2} - r\right) \frac{dw}{w} & : r \neq 0 \\ -\frac{1}{2} \frac{dw}{w} + \frac{dw}{w-1} & : r = 0 \end{cases} \quad \text{alphabet of MZVs}$$

$$\lim_{\tau \rightarrow i\infty} f^{(1)}(z - s, \tau) dz = -\frac{1}{2} \frac{dw}{w} + \frac{dw}{w - e^{2\pi i s}} \Rightarrow \text{cyclotomic MZVs}$$

Examples of degenerations as  $\tau \rightarrow i\infty$ :

- twist  $b = \tau/2$  as seen in non-planar string amplitudes

$$\omega \left( \begin{matrix} n, & 0 \\ \tau/2, & 0 \end{matrix} \right) \rightarrow \begin{cases} \frac{2^{n-1} - 1}{2^{n-1}} \zeta_n & : n \text{ even} \\ 0 & : n \text{ odd} \end{cases}, \quad \omega \left( \begin{matrix} 0, & 1, & 0, & 2 \\ 0, & 0, & 0, & \tau/2 \end{matrix} \right) \rightarrow \frac{\zeta_3}{8}$$

- twist  $b \in \mathbb{Q} \Rightarrow$  cyclotomic MZVs

$$\omega \left( \begin{matrix} 2, & 0, & 1 \\ 1/2, & 0, & 1/2 \end{matrix} \right) \rightarrow \frac{i\pi^3}{24} - \zeta_2 \log(2)$$

$$\omega \left( \begin{matrix} 1, & 1 \\ 1/3, & 2/3 \end{matrix} \right) \rightarrow i\pi \left( \zeta \left( \frac{1}{e^{4\pi i/3}} \right) - \zeta \left( \frac{1}{e^{2\pi i/3}} \right) \right) - 3\zeta_2$$

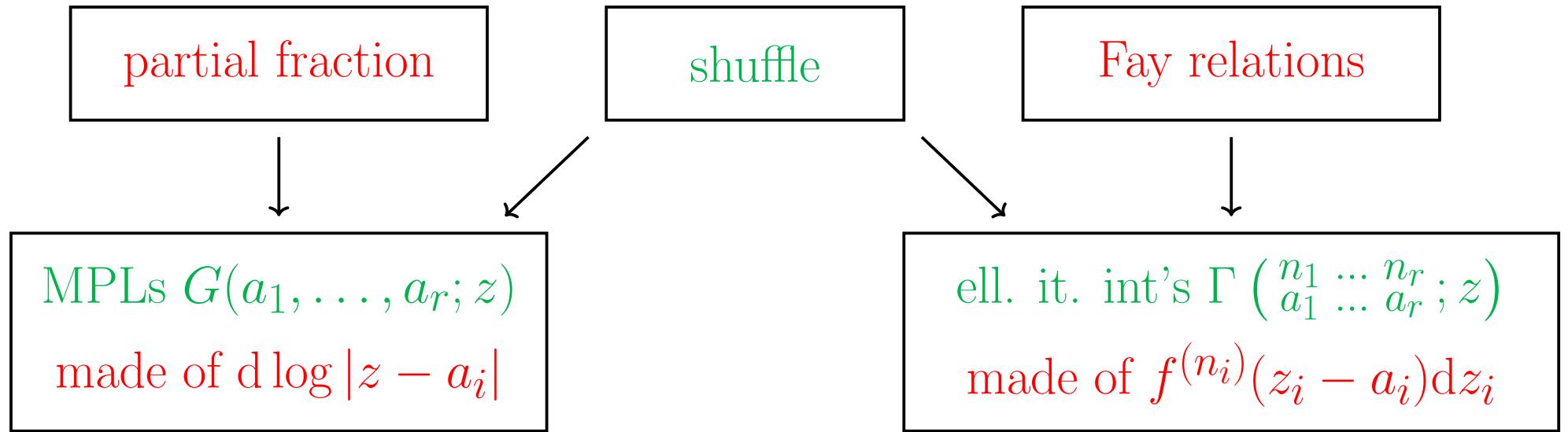
where  $(n_1, n_2, \dots, n_r \geq 1 \text{ and } n_r \geq 2 \text{ if } \sigma_r = 1)$

$$\zeta \left( \begin{matrix} n_1, & n_2, & \dots, & n_r \\ \sigma_1, & \sigma_2, & \dots, & \sigma_r \end{matrix} \right) = \sum_{0 < k_1 < k_2 < \dots < k_r} \frac{\sigma_1^{k_1} \sigma_2^{k_2} \dots \sigma_r^{k_r}}{k_1^{n_1} k_2^{n_2} \dots k_r^{n_r}}$$

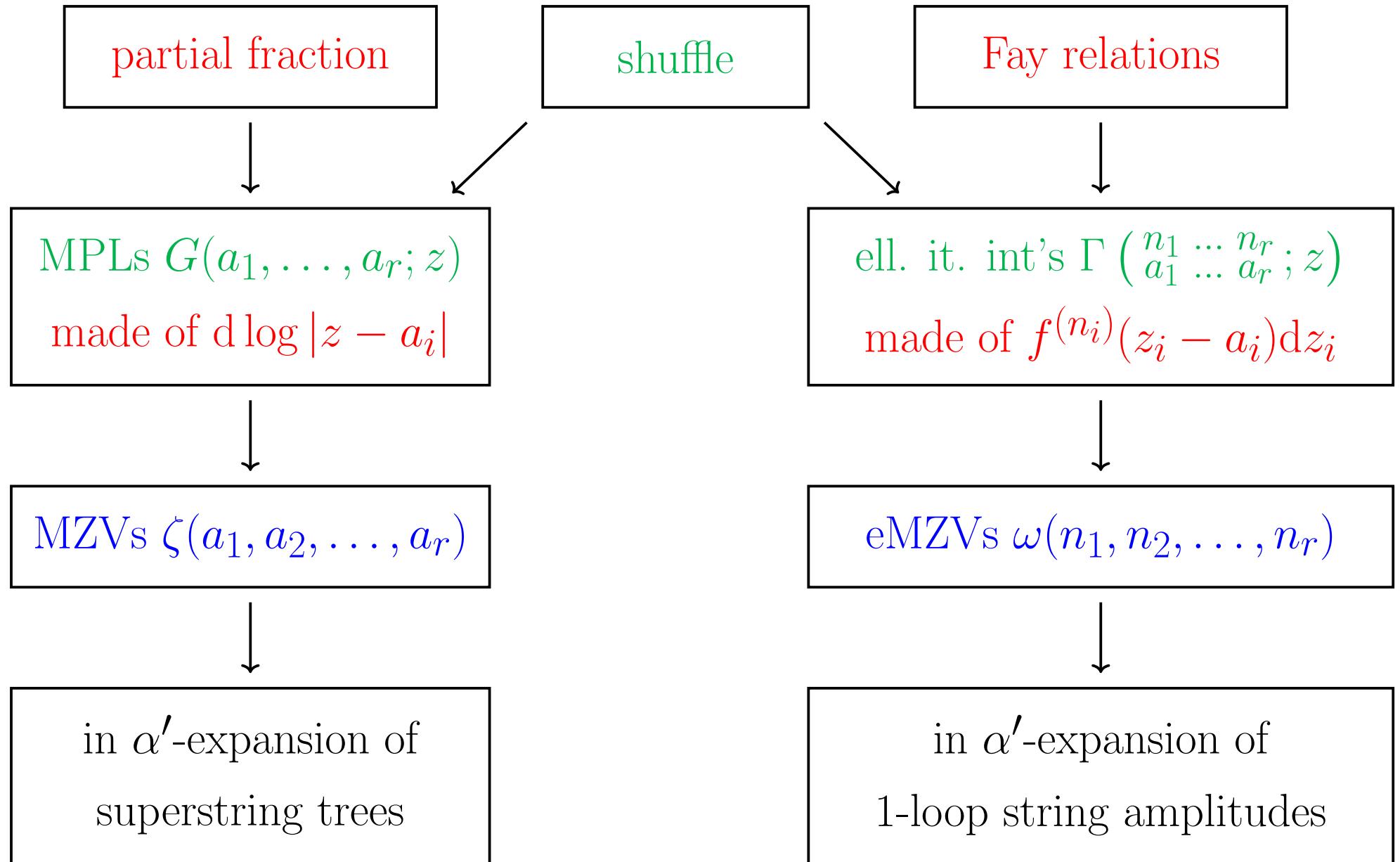
## Further directions

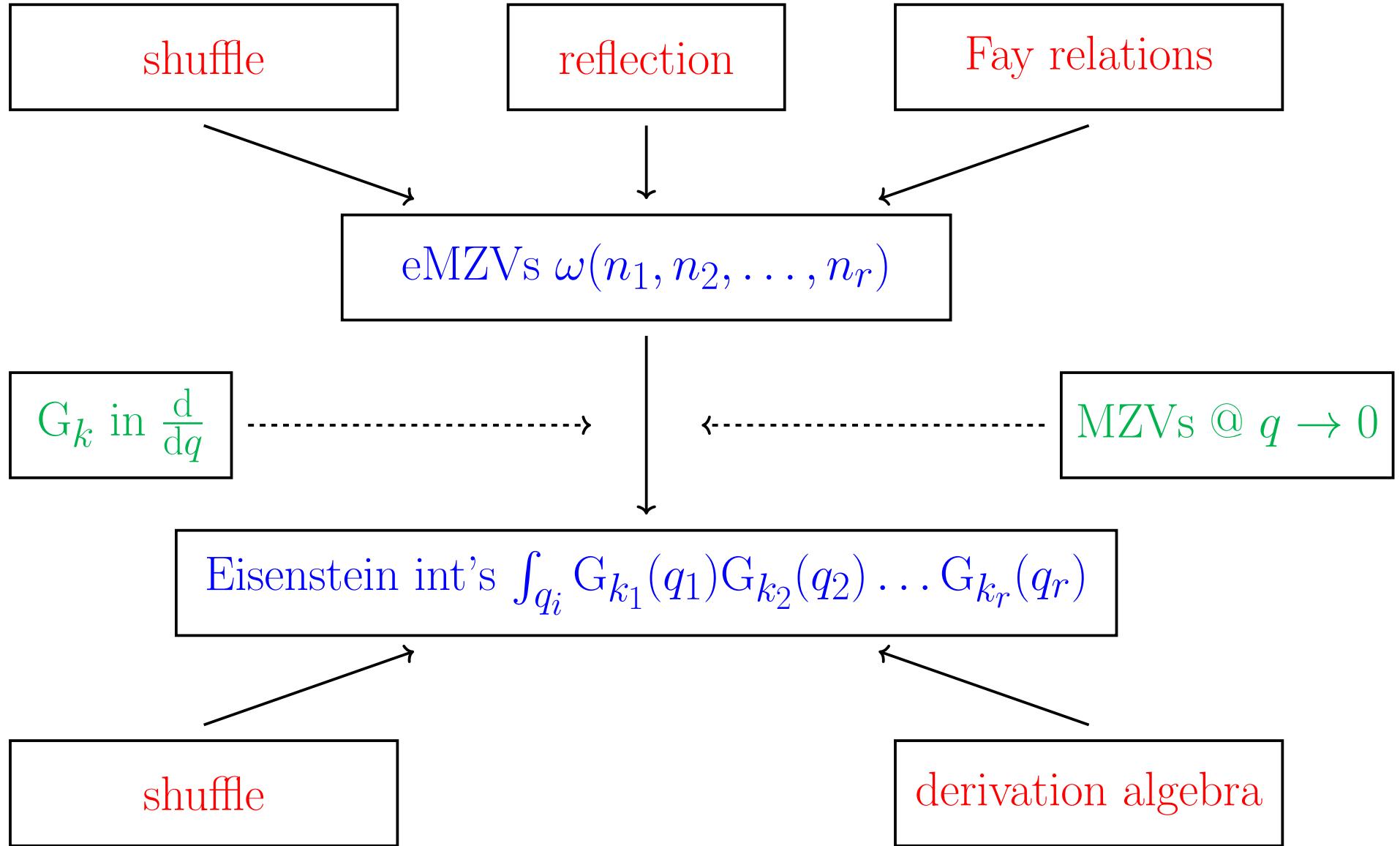
- construct one-loop string amplitude from elliptic associator  $A(q)$   
in analogy to tree-level amplitudes from the Drinfeld associator  
[Brödel, OS, Stieberger, Terasoma 1304.7304]
- identify single-valued eMZVs and connect with closed strings  
[Federico's & Pierre's talks; Basu, D'Hoker, Green, Gürdagan,  
Kleinschmidt, Russo, Vanhove, Verschinin, Zerbini]
- derivation algebra for twisted eMZVs & counting irreducibles  
[Calaque, Gonzalez: to appear]
- higher-genus counterparts of MZVs along with twisted versions  
[closed string: D'Hoker, Green, Pioline, Russo]

## Summary

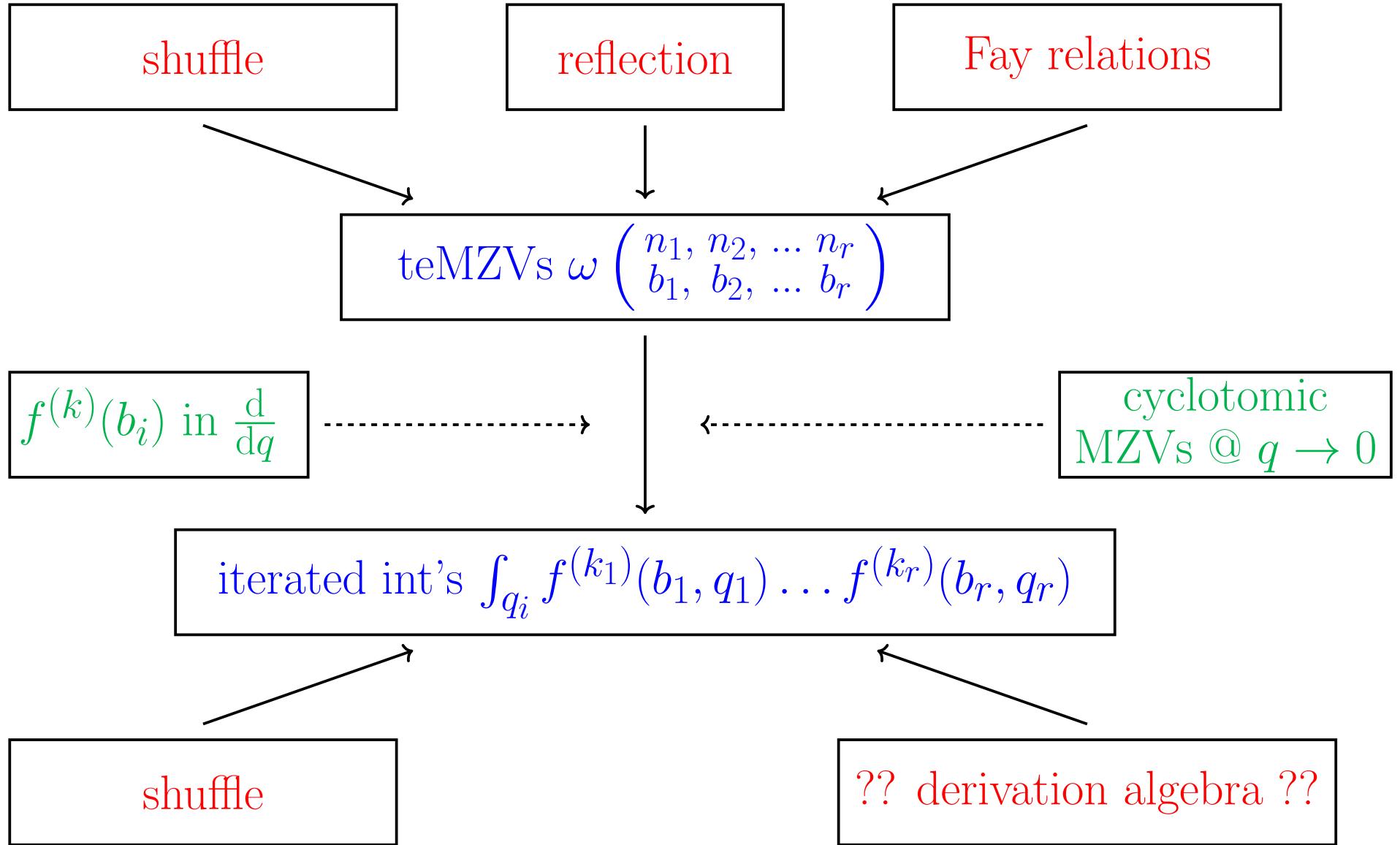


# Summary





Various eMZV relations available at <https://tools.aei.mpg.de/emzv>



Open problem to set up a data mine for twisted eMZVs.