Feynman integrals and iterated integrals of modular forms

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Mathematics intro:	**Periodic functions and periods**
Physics intro:	**Precision calculations**
Part I:	**Multiple Polylogarithms**
Part II:	**Elliptic generalisations**

Periodic functions and periods

Periodic functions

Let us consider a non-constant meromorphic function f of a complex variable z.

A period ω of the function f is a constant such that for all z:

$$f(z+\omega) \quad = \quad f(z)$$

The set of all periods of f forms a lattice, which is either

- trivial (i.e. the lattice consists of $\omega = 0$ only),
- a simple lattice, $\Lambda = \{n\omega \mid n \in \mathbb{Z}\}$,
- a double lattice, $\Lambda = \{n_1\omega_1 + n_2\omega_2 \mid n_1, n_2 \in \mathbb{Z}\}$.

Examples of periodic functions

- Singly periodic function: Exponential function

$$\exp(z).$$

$\exp(z)$ is periodic with peridod $\omega = 2\pi i$.

- Doubly periodic function: Weierstrass's $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2} \right), \qquad \Lambda = \{n_1\omega_1 + n_2\omega_2 | n_1, n_2 \in \mathbb{Z}\},$$

$$\mathrm{Im}(\omega_2/\omega_1) \neq 0.$$

$\wp(z)$ is periodic with periods ω_1 and ω_2.

Inverse functions

The corresponding inverse functions are in general multivalued functions.

- For the exponential function $x = \exp(z)$ the inverse function is the logarithm

$$z = \ln(x).$$

- For Weierstrass's elliptic function $x = \wp(z)$ the inverse function is an elliptic integral

$$z = \int\limits_x^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}, \qquad g_2 = 60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}.$$

Periods as integrals over algebraic functions

In both examples the periods can be expressed as integrals involving only algebraic functions.

- Period of the exponential function:

$$2\pi i = 2i\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}.$$

- Periods of Weierstrass's $\wp$-function: Assume that g_2 and g_3 are two given algebraic numbers. Then

$$\omega_1 = 2\int_{t_1}^{t_2}\frac{dt}{\sqrt{4t^3-g_2t-g_3}}, \qquad \omega_2 = 2\int_{t_3}^{t_2}\frac{dt}{\sqrt{4t^3-g_2t-g_3}},$$

where t_1, t_2 and t_3 are the roots of the cubic equation $4t^3-g_2t-g_3=0$.

Numerical periods

Kontsevich and Zagier suggested the following generalisation:

A numerical period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Remarks:

- One can replace “rational” with “algebraic”.
- The set of all periods is countable.
- Example: $\ln 2$ is a numerical period.

$$\ln 2 \quad = \quad \int_1^2 \frac{dt}{t}.$$

Precision calculations

Precision calculations

Due to the smallness of all coupling constants g, we may compute an observable at high energies reliable in perturbation theory,

$$\sigma = \left(\frac{g}{4\pi}\right)^4 \sigma_{LO} + \left(\frac{g}{4\pi}\right)^6 \sigma_{NLO} + \left(\frac{g}{4\pi}\right)^8 \sigma_{NNLO} + \ldots$$

Cross section related to the square of the scattering amplitude: $\sigma \sim |\mathcal{A}|^2$.

Perturbative expansion of the amplitude:

$$\mathcal{A} = g^2 \mathcal{A}^{(0)} + g^4 \mathcal{A}^{(1)} + g^6 \mathcal{A}^{(2)} + \ldots,$$

where $\mathcal{A}^{(l)}$ contains l loops.

Loop amplitudes

The computation of the tree amplitude $\mathcal{A}^{(0)}$ poses no conceptional problem.

For loop amplitudes we have to calculate Feynman integrals.

Let us write

$$\mathcal{A}^{(l)} = \sum_j c_j I_j,$$

c_j : coefficients, computation tree-like,

I_j : Feynman integrals

We may take the set of Feynman integrals $\{I_1, I_2, ...\}$ to consist of scalar integrals.

(Tarasov, '96, '97)

Feynman integrals

A Feynman graph with n external lines, r internal lines and l loops corresponds (up to prefactors) in D space-time dimensions to the family of Feynman integrals, indexed by the powers of the propagators ν_j

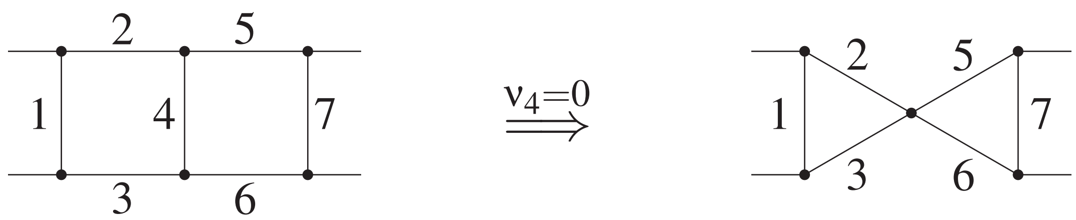
$$I_{\nu_1\nu_2\dots\nu_r} = \int \prod_{s=1}^{l} \frac{d^D k_s}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{r} \frac{1}{(-q_j^2+m_j^2)^{\nu_j}}$$

The momenta flowing through the internal lines can be expressed through the independent loop momenta k_1, ..., k_l and the external momenta p_1, ..., p_n as

$$q_i = \sum_{j=1}^{l} \lambda_{ij} k_j + \sum_{j=1}^{n} \sigma_{ij} p_j, \qquad \lambda_{ij}, \sigma_{ij} \in \{-1,0,1\}.$$

Pinching of propagators

If for some exponent we have $\nu_j = 0$, the corresponding propagator is absent and the topology simplifies:



Integration by parts

Within dimensional regularisation we have for any loop momentum k_i and $v \in \{p_1, ..., p_n, k_1, ..., k_l\}$

$$\int \prod_{s=1}^{l} \frac{d^D k_s}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_i^\mu} v^\mu \prod_{j=1}^{r} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}} = 0.$$

Working out the derivatives leads to relations among integrals with different sets of indices $(\nu_1, ..., \nu_r)$.

This allows us to express most of the integrals in terms of a few master integrals.

Tkachov '81, Chetyrkin '81

Laporta's algorithm

Expressing all integrals in terms of the master integrals requires to solve a rather large linear system of equations.

This system has a block-triangular structure, originating from subtopologies.

Order the integrals by complexity (more propagators $\Rightarrow$ more difficult)

Solve the system bottom-up, re-using the results for the already solved sectors.

Laporta '01

Differential equations

Let t be an external invariant (e.g. $t = (p_i + p_j)^2$) or an internal mass. Let $I_i \in \{I_1, ..., I_N\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial t} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the derivative as a linear combination of the master integrals.

$$\frac{\partial}{\partial t} I_i = \sum_{j=1}^{N} a_{ij} I_j$$

(Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99)

Differential equations

Let us formalise this:

$\vec{I} = (I_1, ..., I_N)$, set of master integrals,

$\vec{x} = (x_1, ..., x_n)$, set of kinematic variables the master integrals depend on.

We obtain a system of differential equations of Fuchsian type

$$d\vec{I} = A\vec{I},$$

where A is a matrix-valued one-form

$$A = \sum_{i=1}^{n} A_i dx_i.$$

The matrix-valued one-form A satisfies the integrability condition

$$dA - A \wedge A = 0.$$

Computation of Feynman integrals reduced to solving differential equations!

Part I

Multiple Polylogarithms

Feynman integrals and periods

Laurent expansion in $\varepsilon = (4 - D)/2$:

$$I_G = \sum_{j=-2l}^{\infty} c_j \varepsilon^j.$$

Question: What can be said about the coefficients c_j ?

Theorem: For rational input data in the euclidean region the coefficients c_j of the Laurent expansion are numerical periods.

(Bogner, S.W., '07)

Next question: Which periods ?

One-loop amplitudes

All one-loop amplitudes can be expressed as a sum of algebraic functions of the spinor products and masses times two transcendental functions, whose arguments are again algebraic functions of the spinor products and the masses.

The two transcendental functions are the logarithm and the dilogarithm:

$$\begin{aligned}
\mathrm{Li}_1(x) &= -\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \\
\mathrm{Li}_2(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n^2}
\end{aligned}$$

Generalisations of the logarithm

Beyond one-loop, at least the following generalisations occur:

Polylogarithms:

$$\text{Li}_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m}$$

Multiple polylogarithms:

$$\text{Li}_{m_1,m_2,...,m_k}(x_1,x_2,...,x_k) = \sum_{n_1>n_2>...>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \ldots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

This is a nested sum:

$$\ldots \sum_{n_j=1}^{n_{j-1}-1} \frac{x_j^{n_j}}{n_j^{m_j}} \sum_{n_{j+1}=1}^{n_j-1} \ldots$$

Multiple polylogarithms

Definition based on nested sums:

$$\text{Li}_{m_1,m_2,\dots,m_k}(x_1,x_2,\dots,x_k) = \sum_{n_1>n_2>\dots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \dots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Definition based on iterated integrals:

$$G(z_1,\dots,z_k;y) = \int_0^y \frac{dt_1}{t_1-z_1} \int_0^{t_1} \frac{dt_2}{t_2-z_2} \dots \int_0^{t_{k-1}} \frac{dt_k}{t_k-z_k}$$

Conversion:

$$\text{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k) = (-1)^k G_{m_1,\dots,m_k}\left(\frac{1}{x_1}, \frac{1}{x_1 x_2}, \dots, \frac{1}{x_1\dots x_k}; 1\right)$$

Short hand notation:

$$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) = G(\underbrace{0,\dots,0}_{m_1-1}, z_1, \dots, z_{k-1}, \underbrace{0\dots,0}_{m_k-1}, z_k; y)$$

The ε-form of the differential equation

If we change the basis of the master integrals $\vec{J} = U\vec{I}$, the differential equation becomes

$$d\vec{J} = A'\vec{J}, \qquad A' = UAU^{-1} - U\,dU^{-1}$$

Suppose one finds a transformation matrix U, such that

$$A' \quad = \quad \varepsilon \sum_j C_j \, d \ln p_j(\vec{x}),$$

where

- ε appears only as prefactor,
- C_j are matrices with constant entries,
- $p_j(\vec{x})$ are polynomials in the external variables,

then the system of differential equations is easily solved in terms of multiple polylogarithms.

Henn '13

Transformation to the ε-form

We may

- perform a rational / algebraic transformation on the kinematic variables

$$(x_1, \ldots, x_n) \quad \to \quad (x'_1, \ldots, x'_n),$$

often done to absorb square roots.

- change the basis of the master integrals

$$\vec{I} \quad \to \quad U\vec{I},$$

where U is rational in the kinematic variables

Henn ’13; Gehrmann, von Manteuffel, Tancredi, Weihs ’14; Argeri et al. ’14; Lee ’14; Meyer ’16; Prausa ’17; Gituliar, Magerya ’17; Lee, Pomeransky ’17;

Numerical evaluations of multiple polylogarithms

Multiple polylogarithms have branch cuts.

Numerical evaluation of multiple polylogarithms $\text{Li}_{m_1,m_2,...,m_k}(x_1,x_2,...,x_k)$ as a function of k complex variables x_1, x_2, ..., x_k:

- Use truncated sum representation within its region of convergence.
- Use integral representation to map arguments into this region.
- Acceleration techniques to speed up the computation.

Implementation in GiNaC, using arbitrary precision arithmetic in C++.

J. Vollinga, S.W. '04

Part II

Elliptic generalisations

Differential equations again

If it is not feasible to compute the integral directly:

Pick one variable t from the set s_{jk} and m_i^2.

1. Find a differential equation for the Feynman integral.

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} I_G(t) \quad = \quad \sum_i q_i(t) I_{G_i}(t)$$

   Inhomogeneous term on the rhs consists of simpler integrals I_{G_i}.

   $p_j(t)$, $q_i(t)$ polynomials in t.

2. Solve the differential equation.

Remark: A single differential equation of order r is equivalent to a system of r first-order differential equations.

Differential equations: The case of multiple polylogarithms

Suppose the differential operator factorises into linear factors:

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j} = \left(a_r(t) \frac{d}{dt} + b_r(t) \right) \dots \left(a_2(t) \frac{d}{dt} + b_2(t) \right) \left(a_1(t) \frac{d}{dt} + b_1(t) \right)$$

Iterated first-order differential equation.

Denote homogeneous solution of the j-th factor by

$$\psi_j(t) = \exp\left(- \int_0^t ds \frac{b_j(s)}{a_j(s)} \right).$$

Full solution given by iterated integrals

$$I_G(t) = C_1 \psi_1(t) + C_2 \psi_1(t) \int_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} + C_3 \psi_1(t) \int_0^t dt_1 \frac{\psi_2(t_1)}{a_1(t_1)\psi_1(t_1)} \int_0^{t_1} dt_2 \frac{\psi_3(t_2)}{a_2(t_2)\psi_2(t_2)} + \dots$$

Multiple polylogarithms are of this form.

Differential equations: Beyond linear factors

Suppose the differential operator

$$\sum_{j=0}^{r} p_j(t) \frac{d^j}{dt^j}$$

does not factor into linear factors.

The next more complicate case:
The differential operator contains one irreducible second-order differential operator

$$a_j(t)\frac{d^2}{dt^2} + b_j(t)\frac{d}{dt} + c_j(t)$$

An example from mathematics: Elliptic integral

The differential operator of the second-order differential equation

$$\left[t\left(1-t^2\right)\frac{d^2}{dt^2}+\left(1-3t^2\right)\frac{d}{dt}-t\right]f(t) \quad = \quad 0$$

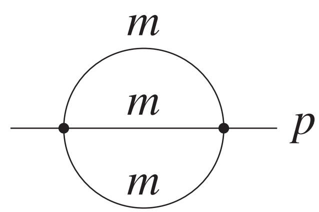
is irreducible.

The solutions of the differential equation are $K(t)$ and $K(\sqrt{1-t^2})$, where $K(t)$ is the complete elliptic integral of the first kind:

$$K(t) \quad = \quad \int\limits_0^1 \frac{dx}{\sqrt{(1-x^2)\,(1-t^2x^2)}}.$$

An example from physics: The two-loop sunrise integral

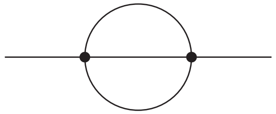
$$S\left(p^2, m\right) \quad =$$



- Two-loop contribution to the self-energy of massive particles.
- Sub-topology for more complicated diagrams.

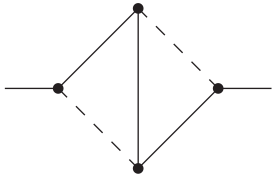
Single scale integrals beyond multiple polylogarithms

Starting from two-loops, there are integrals which cannot be expressed in terms of multiple polylogarithms.



Simplest example: Two-loop sunrise integral with equal masses.

Slightly more complicated: Two-loop kite integral.



Both integrals depend on a single scale t/m^2.

Change variable from t/m^2 to the nome q or the parameter τ with $q = e^{i\pi\tau}$.

Sabry, Broadhurst, Fleischer, Tarasov, Bauberger, Berends, Buza, Böhm, Scharf, Weiglein, Caffo, Czyz, Laporta, Remiddi, Groote, Körner, Pivovarov, Bailey, Borwein, Glasser, Adams, Bogner, Müller-Stach, Schweitzer, S.W, Zayadeh, Bloch, Vanhove, Tancredi, Pozzorini, Gunia, ...

The elliptic curve

How to get the elliptic curve?

- From the Feynman graph polynomial:

$$-x_1 x_2 x_3 t + m^2 (x_1 + x_2 + x_3)(x_1 x_2 + x_2 x_3 + x_3 x_1) = 0$$

- From the maximal cut:

$$y^2 - \left(x - \frac{t}{m^2}\right)\left(x - \frac{t - 4m^2}{m^2}\right)\left(x^2 + 2x + 1 - 4\frac{t}{m^2}\right) = 0$$

Baikov '96; Lee '10; Frellesvig, Papadopoulos, '17; Bosma, Sogaard, Zhang, '17; Harley, Moriello, Schabinger, '17

The periods ψ_1, ψ_2 of the elliptic curve are solutions of the homogeneous differential equation.

Adams, Bogner, S.W., '13; Primo, Tancredi, '16

Set $\quad \tau = \frac{\psi_2}{\psi_1}, \qquad q = e^{i\pi\tau}.$

The elliptic dilogarithm

Recall the definition of the classical polylogarithms:

$$\text{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}.$$

Generalisation, the two sums are coupled through the variable q:

$$\text{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m}q^{jk}.$$

Elliptic dilogarithm:

$$\text{E}_{2;0}(x;y;q) = \frac{1}{i}\left[\frac{1}{2}\text{Li}_2(x) - \frac{1}{2}\text{Li}_2\left(x^{-1}\right) + \text{ELi}_{2;0}(x;y;q) - \text{ELi}_{2;0}\left(x^{-1};y^{-1};q\right)\right].$$

Various definitions of elliptic polylogarithms can be found in the literature

Beilinson '94, Levin '97, Wildeshaus '97, Brown, Levin '11, Bloch, Vanhove '13, Adams, Bogner, S.W. '14, Remiddi, Tancredi '17

Elliptic generalisations

In order to express the sunrise/kite integral to all orders in ε introduce

$$\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q) =$$

$$= \sum_{j_1=1}^{\infty} \cdots \sum_{j_l=1}^{\infty} \sum_{k_1=1}^{\infty} \cdots \sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}} \cdots \frac{x_l^{j_l}}{j_l^{n_l}} \frac{y_1^{k_1}}{k_1^{m_1}} \cdots \frac{y_l^{k_l}}{k_l^{m_l}} \frac{q^{j_1 k_1 + \dots + j_l k_l}}{\prod\limits_{i=1}^{l-1} \left(j_i k_i + \dots + j_l k_l \right)^{o_i}}.$$

Numerical evaluation: G. Passarino '16

The all-order in ε result (ELi-representation)

Taylor expansion of the sunrise integral around $D = 2 - 2\varepsilon$:

$$S = \frac{\psi_1}{\pi} \sum_{j=0}^{\infty} \varepsilon^j E^{(j)}$$

Each term in the ε-series is of the form

$$E^{(j)} \sim \text{linear combination of } \mathrm{ELi}_{n_1,...,n_l;m_1,...,m_l;2o_1,...,2o_{l-1}} \text{ and } \mathrm{Li}_{n_1,...,n_l}$$

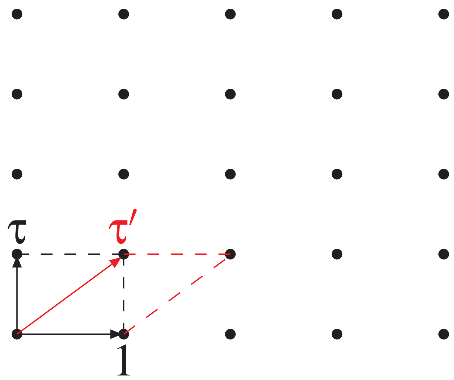
Using dimensional-shift relations this translates to the expansion around $4 - 2\varepsilon$.

$\Rightarrow$ The multiple polylogarithms extended by $\mathrm{ELi}_{n_1,...,n_l;m_1,...,m_l;2o_1,...,2o_{l-1}}$ are the class of functions to express the equal mass sunrise graph to all orders in ε.

Adams, Bogner, S.W., '15

Bases of lattices

The periods ψ_1 and ψ_2 generate a lattice. Any other basis as good as (ψ_2, ψ_1).
Convention: Normalise $(\psi_2, \psi_1) \to (\tau, 1)$ where $\tau = \psi_2/\psi_1$.



Change of basis: $$\begin{pmatrix} \psi_2' \\ \psi_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix},$$

Transformation should be invertible: $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(2, \mathbb{Z}),$$

In terms of τ and τ': $$\tau' = \frac{a\tau + b}{c\tau + d}$$

Modular forms

Denote by $\mathbb{H}$ the complex upper half plane. A meromorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of modular weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

(i) f transforms under Möbius transformations as

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \qquad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

(ii) f is holomorphic on $\mathbb{H}$,

(iii) f is holomorphic at ∞.

Congruence subgroups

Apart from $\mathrm{SL}_2(2,\mathbb{Z})$ we may also look at congruence subgroups, for example

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : a,d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z}) : a,d \equiv 1 \bmod N,\ b,c \equiv 0 \bmod N \right\}$$

Modular forms for congruence subgroups: Require "nice" transformation properties only for subgroup Γ (plus holomorphicity on $\mathbb{H}$ and at the cusps).

Dirichlet character

Let N be a positive integer. A function $\chi : \mathbb{Z} \to \mathbb{C}$ is called a Dirichlet character modulo N, if

(i) $\chi(n) = \chi(n+N) \qquad \forall n \in \mathbb{Z}$,

(ii) $\chi(n) = 0$ if $\gcd(n,N) > 1$ and $\chi(n) \neq 0$ if $\gcd(n,N) = 1$,

(iii) $\chi(nm) = \chi(n)\chi(m) \qquad \forall\, n,m \in \mathbb{Z}$.

The conductor of χ is the smallest positive divisor $d|N$ such that there is a character χ' modulo d with

$$\chi(n) = \chi'(n) \qquad \forall\, n \in \mathbb{Z} \qquad \text{with } \gcd(n,N) = 1.$$

Modular forms with character

We may relax the transformation law:

Let N be a positive integer and let χ be a Dirichlet character modulo N. A function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k for $\Gamma_0(N)$ with character χ if

(i) f is holomorphic on $\mathbb{H}$,

(ii) f is holomorphic at the cusps of $\Gamma_1(N)$,

(iii) $f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

The space of modular forms

- The modular forms for a given congruence subgroup form a vectorspace.
- This vectorspace is finite dimensional.
- It decomposes into a subspace of cusp forms and the Eisenstein subspace.
- We have

$$\mathcal{M}_k(\Gamma_1(N)) = \bigoplus_{\chi} \mathcal{M}_k(N,\chi)$$

  and similar for the subspace of cusp forms and the Eisenstein subspace.
- Basis of Eisenstein subspace $\mathcal{E}_k(N,\chi)$ given in terms of generalised Eisenstein series.

Iterated integrals of modular forms

Iterated integrals of modular forms:

$$I(f_1, f_2, ..., f_n; q) = (2\pi i)^n \int\limits_{\tau_0}^{\tau} d\tau_1 f_1(\tau_1) \int\limits_{\tau_0}^{\tau_1} d\tau_2 f_2(\tau_2) \dots \int\limits_{\tau_0}^{\tau_{n-1}} d\tau_n f_n(\tau_n)$$

Notation:

$$I\left(\{f\}^k; q\right) = I\left(\underbrace{f, f, ..., f}_{k}; q\right)$$

An integral over a modular form is in general not a modular form.
Analogy: An integral over a rational function is in general not a rational function.

Eichler integrals

Let $f(\tau)$ be a modular form of weight $k \geq 2$. Define the Eichler integral by

$$F(\tau) = I(\{1\}^{k-2}, f; q) = (2\pi i)^{k-1} \int_{\tau_0}^{\tau} d\tau_1 \dots \int_{\tau_0}^{\tau_{k-3}} d\tau_{k-2} \int_{\tau_0}^{\tau_{k-2}} d\tau_{k-1} f(\tau_{k-1})$$

Transformation law:

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^{2-k}\left[F(\tau)+P(\tau)\right],$$

where $P(\tau)$ is a polynomial in τ of degree at most $(k-2)$.

The all-order in ε result (iterated integrals)

$$
\begin{aligned}
S &= \frac{\psi_1}{\pi} e^{-\varepsilon I(f_2;q) + 2 \sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta_n \varepsilon^n} \\
&\left\{ \left[\sum_{j=0}^{\infty} \left(\varepsilon^{2j} I\left(\{1, f_4\}^j ; q \right) - \frac{1}{2} \varepsilon^{2j+1} I\left(\{1, f_4\}^j, 1; q \right) \right) \right] \sum_{k=0}^{\infty} \varepsilon^k B^{(k)} \right. \\
&\left. + \sum_{j=0}^{\infty} \varepsilon^j \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor} I\left(\{1, f_4\}^k, 1, f_3, \{f_2\}^{j-2k} ; q \right) \right\}
\end{aligned}
$$

Uniform weight: At order ε^j one has exactly $(j+2)$ integrations.

Alphabet given by modular forms 1, f_2, f_3, f_4.

Adams, S.W., ’17

The letters

Example: The modular form f_3 is given by

$$\begin{aligned} f_3 &= -\frac{1}{24}\left(\frac{\psi_1}{\pi}\right)^3 \frac{t\left(t-m^2\right)\left(t-9m^2\right)}{m^6} \\ &= \frac{3}{i}\left[\mathrm{ELi}_{0;-2}\left(r_3;-1;-q\right)-\mathrm{ELi}_{0;-2}\left(r_3^{-1};-1;-q\right)\right] \\ &= 3\sqrt{3}\frac{\eta\left(2\tau_2\right)^{11}\eta\left(6\tau_2\right)^7}{\eta\left(\tau_2\right)^5\eta\left(4\tau_2\right)^5\eta\left(3\tau_2\right)\eta\left(12\tau_2\right)} \\ &= 3\sqrt{3}\left[E_3\left(\tau_2;\chi_1,\chi_0\right)+2E_3\left(2\tau_2;\chi_1,\chi_0\right)-8E_3\left(4\tau_2;\chi_1,\chi_0\right)\right] \end{aligned}$$

with $\tau_2 = \tau/2$, $r_3 = \exp(2\pi i/3)$, Dedekind's eta function η, Dirichlet characters $\chi_0 = (\frac{1}{n})$, $\chi_1 = (\frac{-3}{n})$ and Eisenstein series E_3.

The ε-form of the differential equation for the sunrise/kite

It is not possible to obtain an ε-form by a rational/algebraic change of variables and/or a rational/algebraic transformation of the basis of master integrals.

However by the (non-algebraic) change of variables from t to τ and by factoring off the (non-algebraic) expression ψ_1/π from the master integrals in the sunrise sector one obtains an ε-form for the kite/sunrise family:

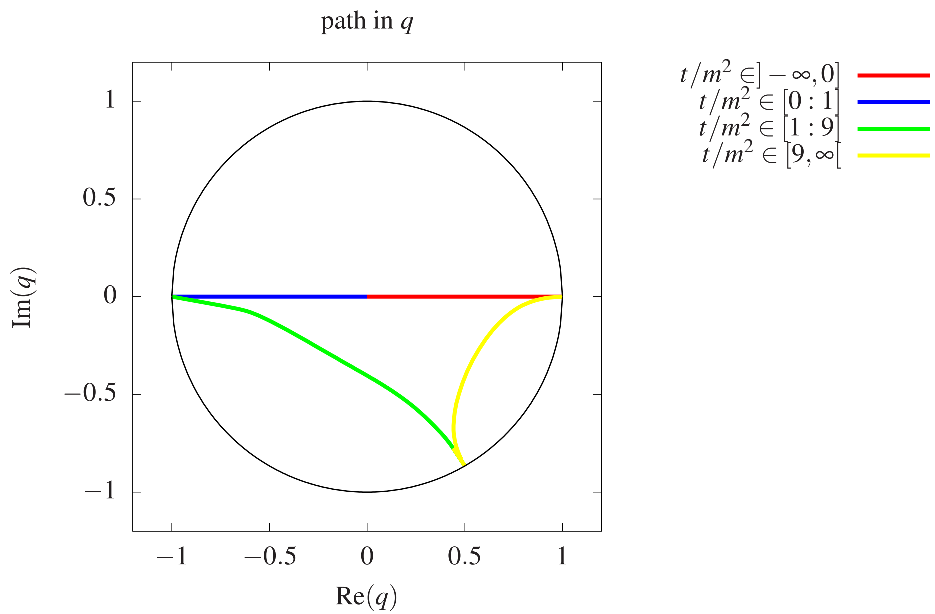
$$\frac{d}{d\tau}\vec{I} = \varepsilon \, A(\tau) \, \vec{I},$$

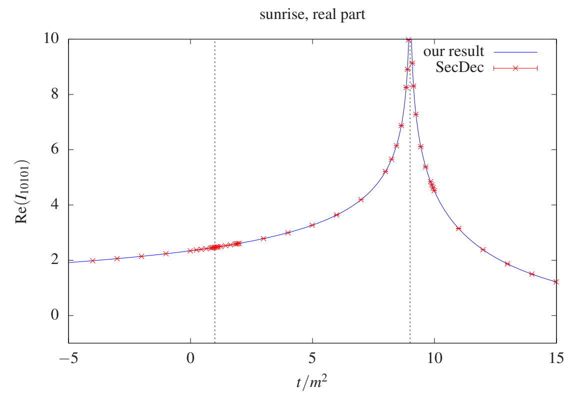
where $A(\tau)$ is an ε-independent 8×8-matrix whose entries are modular forms.

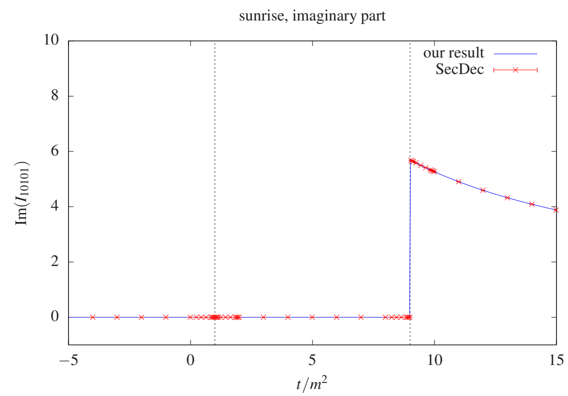
Analytic continuation and numerical evaluations of the kite and sunrise integral

Complete elliptic integrals efficiently computed from arithmetic-geometric mean.

q-series converges for all $t \in \mathbb{R} \backslash \{m^2, 9m^2, \infty\}$.







No need to distinguish the cases $t < 0$, $0 < t < m^2$, $m^2 < t < 9m^2$, $9m^2 < t$!

Bogner, Schweitzer, S.W., ’17

Conclusions

- Differential equations are a powerful tool to compute Feynman integrals.

- If a system can be transformed to an ε-form, a solution in terms of multiple polylogarithm is easily obtained.

- There are system, where within rational transformations at order ε^0 two coupled equations remain.

  Kite/sunrise family:

  - Sum representation in terms of ELi-functions.
  - Iterated integral representation involving modular forms
  - Analytic continuation / numerical evaluation easy.