Genus one superstring amplitudes and modular forms

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Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory
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Outline

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Periods

Definition (Kontsevich-Zagier)

We call **periods** the complex numbers whose real and imaginary part can be written as absolutely convergent integrals of rational function with rational coefficients over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients, and denote the $\mathbb{Q}$-algebra that they generate $\mathcal{P}$.

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- $\overline{\mathbb{Q}} \subset \mathcal{P}$
- $2\pi i = \int_{|z|=1} \frac{dz}{z}$
- $e \notin^? \mathcal{P}$
- $\zeta(k) = \sum_{v \geq 1} \frac{1}{v^k} = \int_{[0,1]^k} \frac{dx_1 \cdots dx_k}{1 - x_1 \cdots x_k}$

Multiple zeta values

Definition

Let $\mathbf{k} \in \mathbb{N}^r$, $k_r \geq 2$. We call **multiple zeta values** (MZVs) the real numbers defined as

$$\zeta(k_1, \dots, k_r) = \sum_{0<v_1<\cdots<v_r} \frac{1}{v_1^{k_1} \cdots v_r^{k_r}}. \tag{1}$$

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- $\zeta(3, 5)$ "irreducible".

Iterated integrals on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$

Let M be a complex manifold, $\omega_1, \ldots, \omega_r$ be smooth 1-forms on M, $\gamma : [0,1] \to M$ be a piecewise smooth path. Write $\gamma^* \omega_i = f_i(t)dt$ for some piecewise smooth function $f_i : [0,1] \to \mathbb{C}$. We define

$$\int_\gamma \omega_1 \cdots \omega_r := \int_{1 \geq t_1 \geq \cdots \geq t_r \geq 0} f_1(t_1) \cdots f_r(t_r)\, dt_1 \cdots dt_r .$$

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Definition

For $\mathbf{k} \in \mathbb{N}^r$ we call (one-variable) **multiple polylogarithm** the holomorphic multi-valued function of $z \in \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ given by

$$\mathrm{Li}_{k_1, \ldots, k_r}(z) = \int_{[0,z]} \underbrace{\omega_0 \cdots \omega_0}_{k_r - 1} \omega_1 \cdots \underbrace{\omega_0 \cdots \omega_0}_{k_1 - 1} \omega_1,$$

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- $\zeta_{\mathrm{sv}}(2k) = 0$
- $\zeta_{\mathrm{sv}}(2k+1) = 2\zeta(2k+1)$
- $\zeta_{\mathrm{sv}}(3,5) = -10\zeta(3)\zeta(5)$
- $\zeta_{\mathrm{sv}}(3,5,3) = 2\zeta(3,5,3) - 2\zeta(3,5)\zeta(3) - 10\zeta(3)^2\zeta(5)$.

The number theory of superstring amplitudes

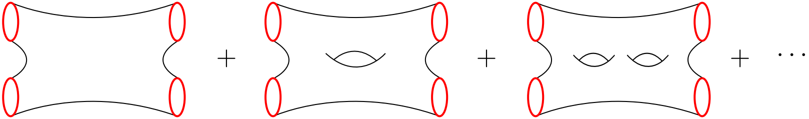

Figure: Four closed strings

$$\mathcal{A}_N = \sum_{g \geq 0} \int_{\overline{\mathcal{M}}_{g,N}} \Omega_{g,N}$$

	Open Strings	Closed Strings
$g = 0, N = 4$ $g = 0, N \geq 5$		
$g = 1$		
$g > 1$		

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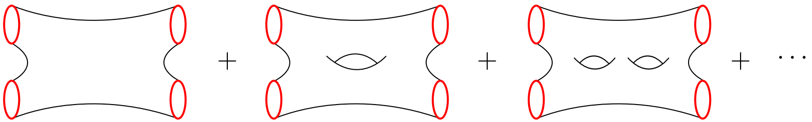

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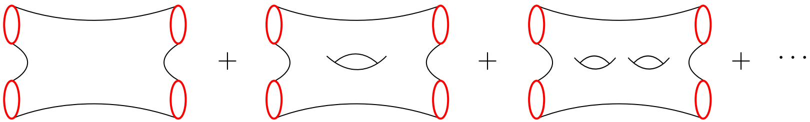

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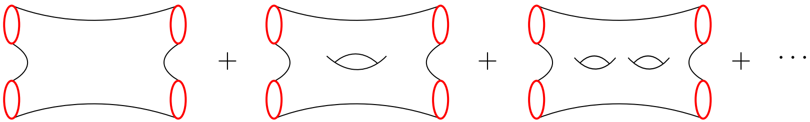

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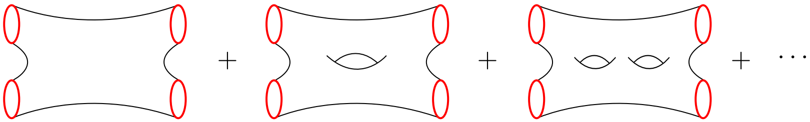

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Broedel, Mafra, Schlotterer, Stieberger, Terasoma, Green, Vanhove...

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Definition

- Graph $\Gamma = (V, E)$, no self-edges, allow multiple edges between pairs of vertices
- Labelling $\xi_1, \ldots \xi_N$ of vertices, for $i < j$ edges oriented from ξ_i to ξ_j
- **Weight** of the graph $l := |E|$
- Incidence matrix $(\Gamma_{i,\alpha})_{\substack{1\leq i\leq N \\ 1\leq\alpha\leq l}}$

Definition (D'Hoker, Green, Gurdogan, Vanhove, 2015)

Let $\tau = \tau_1 + i\tau_2 \in \mathbb{H}$, $\Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z}$. We call **modular graph function**

$$D_\Gamma(\tau) = \left(\frac{\tau_2}{\pi}\right)^l \sum_{\omega_1,\ldots,\omega_l \in \Lambda_\tau^*} \prod_{\alpha=1}^{l} |\omega_\alpha|^{-2} \prod_{i=1}^{N} \delta\Big(\sum_{\beta=1}^{l} \Gamma_{i,\beta}\omega_\beta\Big).$$

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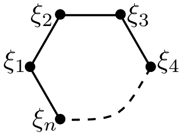
$P_n :=$ 

$D_{P_n}(\tau) = \left(\frac{\tau_2}{\pi}\right)^n \sum_{\omega\in\Lambda_\tau^*} \frac{1}{|\omega_\alpha|^{2n}} = E(n,\tau)$

First properties

- Real analytic
- $D_\Gamma(\gamma\tau) = D_\Gamma(\tau)$ for all $\gamma \in \mathsf{SL}_2(\mathbb{Z})$
- $D_\Gamma(\tau) = d_\Gamma(\pi\tau_2) + O(e^{-\pi\tau_2})$, $d_\Gamma(\pi\tau_2) \in \mathbb{C}[(\pi\tau_2)^{\pm}]$

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Theorem (Zagier, unpublished)

Consider the family of "*banana graphs*" B_n of all graphs with two vertices and n edges.

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Example: $B_3 =$ 

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- $(\Delta - (n-1)n)E(n,\tau) = 0$;
- **D'Hoker, Green, Vanhove:** $(\Delta - s_\Gamma)D_\Gamma(\tau)$ is a polynomial in non-hol. Eis. series and odd Riemann zeta values ($s_\Gamma \in \mathbb{Z}$) for all two-cycle modular graph fcts.

Asymptotic expansion

Definition

Let $\mathbf{k} \in \mathbb{N}^r$, $z_1, \ldots, z_r$ be roots of 1. We call **cyclotomic MZVs**

$$\sum_{0<v_1<\cdots<v_r} \frac{z_1^{v_1} \cdots z_k^{v_k}}{v_1^{k_1} \cdots v_r^{k_r}}$$

and denote $\mathcal{Z}_\infty$ the $\mathbb{Q}$-algebra generated by them.

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and denote $\mathcal{Z}_\infty$ the $\mathbb{Q}$-algebra generated by them.

Theorem (Z. 2015)

$$D_\Gamma(\tau) = \sum_{\mu,\nu \geq 0} d_\Gamma^{(\mu,\nu)}(\pi\tau_2)\, q^\mu\, \overline{q}^\nu,$$

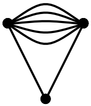
where

$$d_\Gamma^{(\mu,\nu)}(\pi\tau_2) \in \mathcal{Z}_\infty[(\pi\tau_2)^\pm]$$

is a Laurent polynomial of degree $(1-l, l)$ (l is the weight of Γ)

Remark: The proof gives an algorithmic procedure to compute all Laurent polynomials.

Three vertices: single-valued multiple zeta values

Example: $\Gamma =$ 

$$d_\Gamma^{(0,0)}(y) = \frac{62}{10945935}y^7 + \frac{2}{243}\zeta(3)y^4 + \frac{119}{324}\zeta(5)y^2 + \frac{11}{27}\zeta(3)^2 y + \frac{21}{16}\zeta(7)$$
$$+ \frac{46}{3}\frac{\zeta(3)\zeta(5)}{y} + \frac{7115\zeta(9) - 3600\zeta(3)^3}{288y^2} + \frac{1245\zeta(3)\zeta(7) - 150\zeta(5)^2}{16y^3}$$
$$+ \frac{288\zeta(3,5,3) - 288\zeta(3)\zeta(3,5) - 5040\zeta(5)\zeta(3)^2 - 9573\zeta(11)}{128y^4}$$
$$+ \frac{2475\zeta(5)\zeta(7) + 1125\zeta(9)\zeta(3)}{32y^5} - \frac{1575}{32}\frac{\zeta(13)}{y^6}$$

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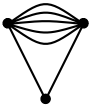
Example: $\Gamma =$ 

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Linear relations

Examples (Zagier, D'Hoker, Green, Vanhove, Kaidi, Basu...)

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- $\mathbf{D}\Big[$  $\Big] - \mathbf{D}\Big[$  $\Big] - \zeta_3 = 0$
- $24\,\mathbf{D}\Big[$  $\Big] - \mathbf{D}\Big[$  $\Big] - 18\,\mathbf{D}\Big[$  $\Big] + 3\,\mathbf{D}\Big[$  $\Big]^2 = 0$

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- **D**[] $-$ **D**[] $- \zeta_3 = 0$
- 24 **D**[] $-$ **D**[] $- 18$ **D**[] $+ 3$ **D**[]^2 $= 0$
- 300 **D**[] $- 40$ **D**[] $+ 120$ **D**[] **D**[] $- 276$ **D**[] $+ 7\zeta_5 = 0$
- 60 **D**[] $-$ **D**[] $+ 10$ **D**[] **D**[] $- 48$ **D**[] $+$ $10\zeta_3$ **D**[] $+ 16\zeta_5 = 0$
- 20 **D**[] $- 10$ **D**[] $- 4$ **D**[] $+ 3\zeta_5 = 0$
- 12 **D**[] $- 30$ **D**[] $+ \zeta_5 = 0$

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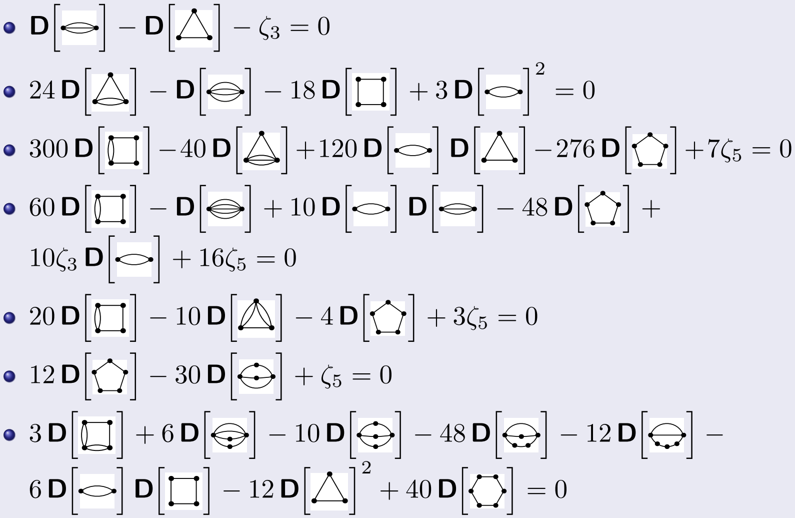

Conjectures and open questions

- **Conjecture 1:** $d_\Gamma^{(\mu,\nu)}(\pi\tau_2) \in \mathcal{Z}^{\mathrm{sv}}[(\pi\tau_2)^{\pm}]$
- **Conjecture 2:** All modular graph functions satisfy "inhomogeneous Laplace equation", whose "complexity" grows with the number of cycles in Γ.
- **Question:** Does $d_\Gamma^{(0,0)}(\pi\tau_2)$ completely determine $D_\Gamma(\tau)$?

Outline

Homotopy invariant iterated integrals on elliptic curves

$$\tau \in \mathbb{H},\ \Lambda_\tau = \tau\mathbb{Z} + \mathbb{Z},\ \mathcal{E}_\tau = \mathbb{C}/\Lambda_\tau,\ \xi = s + r\tau \in \mathcal{E}_\tau^* \ (r, s \in \mathbb{R}),\ \alpha \in \mathcal{E}_\tau^*.$$

$$\Omega(\xi, \alpha, \tau) = e^{2\pi i r\alpha} \frac{\theta'(0,\tau)\theta(\xi+\alpha,\tau)}{\theta(\xi,\tau)\theta(\alpha,\tau)} d\xi =: \sum_{n\geq 0} \omega_n(\xi,\tau)\alpha^{n-1}$$

Homotopy invariant iterated integrals on elliptic curves

$$\tau \in \mathbb{H},\ \Lambda_\tau = \tau\mathbb{Z} + \mathbb{Z},\ \mathcal{E}_\tau = \mathbb{C}/\Lambda_\tau,\ \xi = s + r\tau \in \mathcal{E}_\tau^*\ (r, s \in \mathbb{R}),\ \alpha \in \mathcal{E}_\tau^*.$$

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- **Brown-Levin:** The space of multiple elliptic polylogarithms (i.e. averages of classical MPLs w.r.t. multiplication by q, after Bloch, Zagier, Levin, Beilinson, Racinet, Brown..) coincide with the space of homotopy invariant iterated integrals on $\mathcal{E}_\tau^*$.

Elliptic multiple zeta values: definition and properties

Definition (Enriquez, 2013)

$I^A(n_1,\ldots,n_r;\tau) = \int_{[0,1]} \omega_{n_r} \cdots \omega_{n_1}$ A-elliptic MZVs,

$I^B(n_1,\ldots,n_r;\tau) = \int_{[0,\tau]} \omega_{n_r} \cdots \omega_{n_1}$ B-elliptic MZVs.

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$I^A(\mathbf{n};\tau) = \sum_{j\geq 0} a_j(\mathbf{n})q^j$, where $a_j(\mathbf{n}) \in \mathcal{Z}[(2\pi i)^{\pm}]$,
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- **Enriquez:** elliptic MZVs can be written as certain linear combinations of "iterated primitives" of Eisenstein series $G_k(\tau) = \sum_{\lambda\in\Lambda_\tau^*}\frac{1}{\lambda^k}$ (including $G_0 := -1$ and G_2).

Example: $(2\pi i)^2 I^A(0,1,0,0;\tau) = -3\zeta(3) + 6q + \frac{27}{4}q^2 + \frac{56}{9}q^3 + \cdots$

B-elliptic multiple zeta values

Theorem (Z. 2017)

$$I^B(n_1,\dots,n_r;\tau) = \sum_{s=1-n_1-\cdots-n_r}^{r} \sum_{j\geq 0} b_{s,j}(n_1,\dots,n_r)\tau^s q^j .$$

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Z. (2017): Explicit q-expansion of $I^B(\underbrace{0,\ldots,0}_{r-1},n;\tau)$

Example: $I^B(0,3;\tau) = -\frac{\zeta(3)}{2\pi i\tau} - \frac{6\zeta(4)}{2\pi i\tau^2} + \frac{3\tau}{2\pi i}\int_\tau^{i\infty} z^2 G_4(z)dz$

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Theorem (Z, 2017)

$I^A(\underbrace{0,\dots,0}_{r-1},n;\tau)$ are (components of) vector-valued mod. forms of weight $1-r$ for $\mathrm{SL}_2(\mathbb{Z})$.

Iterated integrals of modular forms

Setting: f modular form of weight k, X, Y formal variables
$\underline{f}(X,Y,\tau) := (2\pi i)^{k-1} f(\tau)(X-\tau Y)^{k-2} d\tau.$

Let $\Theta(X,Y,\tau) = \sum A_f \underline{f}(X,Y,\tau)$, then Brown considered

$$I(\tau,\infty) = 1 + \int_{[\tau,i\infty]} \Theta(X,Y,z) + \int_{[\tau,i\infty]} \Theta(X_1,Y_1,z_1)\,\Theta(X_2,Y_2,z_2) + \dots .$$

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Lemma (Brown, 2014)

For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exists a series $\mathcal{C}_\gamma$ in ∞-many non-commutative variables A_f and ∞-many commutative pairs of variables (X_i, Y_i) s.t.

$$I(\tau, \infty) = \mathcal{C}_\gamma I(\gamma\tau, \infty)|_\gamma.$$

We call the coefficients of $\mathcal{C}_\gamma$ **multiple modular values**.

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Depth one: Eichler integrals of modular forms, $\mathcal{C}_\gamma$ gives the period polynomials.

Outline

Single-valued elliptic multiple zeta values?

- **Levin (1997):** Generating series of **(depth 1) elliptic polylogs**

$$\underline{\Lambda}(\xi,\tau;X,Y)=\sum_{j\in\mathbb{Z}}\sum_{n\geq 1}e^{jX}\Lambda_n(\xi+j\tau)(-Y)^{n-1}$$

with $\Lambda_n(\xi)\sim \mathrm{Li}_n(e^{2\pi i\xi})$.

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Conjecture (D'Hoker, Green, Gurdogan, Vanhove): Modular graph fcts are special values of single-valued multiple elliptic polylogarithms.

Equivariant iterated integrals of modular forms

Lemma (Brown, 2014)

$$\underline{\mathcal{E}}_k(\tau) := \frac{\zeta(k)(k-2)!}{2\pi} \sum_{i+j=k-2} \sum_{\omega \in \Lambda_\tau^*} \frac{\tau_2 (X - \tau Y)^i (X - \overline{\tau} Y)^j}{\omega^{i+1} \overline{\omega}^{j+1}}.$$

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Example: modular graph functions belong to $\mathcal{M}_{0,0}$!

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Theorem (Brown, 2017)

There exists $\mathcal{MI}^E \subset \mathcal{M}$ generated over $\mathcal{Z}^{\mathrm{sv}}$ by certain lin. comb. of real and imaginary parts of iterated integrals of Eisenstein series s.t.

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  where $\mathbb{E}$ = space of Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$.
- The sub-vector space of elements of fixed modular weights and M-degree $\leq m$ is finite dimensional.

A-cycle and B-cycle graph functions

Closed string genus one propagator

$$G_1^{\text{cl}}(\xi,\nu,\tau) = -\frac{1}{4}\log\left|\frac{\theta(\xi-\nu,\tau)}{\eta(\tau)}\right|^2 + \frac{\pi(\xi_2-\nu_2)^2}{2\tau_2}$$

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Four-point closed string amplitude

$$\int_{\mathcal{F}}\int_{(\mathcal{E}_\tau^*)^3}\prod_{1\le i<j\le 4}\exp(s_{i,j}G_1^{\mathrm{cl}}(z_i-z_j,\tau))\,dz_1dz_2dz_3d\tau,$$

where $\mathcal{F}$ is the fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, $\mathcal{E}_\tau^*$ is a the complex torus of modulus τ, $s_{i,j}$ are *Mandelstam variables*.

Expanding w.r.t. Mandelstam variables the inner integral leads to modular graph functions (Green, Vanhove).

A-cycle and B-cycle graph functions

Four-point open string amplitude

$$\int_i^{i\infty} \int_{1 \geq x_1 \geq x_2 \geq x_3 \geq 0} \prod_{1 \leq i < j \leq 4} \exp(s_{i,j} G_1^{op}(x_i - x_j, \tau))\, dx_1 dx_2 dx_3 d\tau$$

Expanding w.r.t. Mandelstam variables the inner integral leads to elliptic MZVs (Broedel, Mafra, Matthes, Schlotterer).

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- We call $B_\Gamma(\tau) = A_\Gamma(-1/\tau)$=**B-cycle graph functions**.

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Expanding w.r.t. Mandelstam variables the inner integral leads to elliptic MZVs (Broedel, Mafra, Matthes, Schlotterer).

"Abelianization" of inner integral

$$\int_{[0,1]^3} \prod_{1\leq i<j\leq 4} \exp(s_{i,j} G_1^{op}(x_i - x_j, \tau))\, dx_1 dx_2 dx_3$$

Expanding w.r.t. Mandelstam variables leads to **A-cycle graph functions** $A_\Gamma(\tau)$ (Broedel, Matthes, Schlotterer, Z., work in progress)

- We call $B_\Gamma(\tau) = A_\Gamma(-1/\tau)$=**B-cycle graph functions**.
- A/B-cycle graph functions can be expressed as A/B-elliptic MZVs.

A-cycle relations

Examples

- **A** [] $-$ **A** [] $= \frac{1}{2}\zeta_3 + 6\zeta_2 I^A(0,1,0,0)$

A-cycle relations

Examples

- **A**[] − **A**[] $= \frac{1}{2}\zeta_3 + 6\zeta_2 I^A(0,1,0,0)$
- **D**[] − **D**[] $= \zeta_3$

A-cycle relations

Examples

- $\mathbf{A}\big[$  $\big] - \mathbf{A}\big[$  $\big] = \frac{1}{2}\zeta_3 + 6\zeta_2 I^A(0,1,0,0)$
- $\mathbf{D}\big[$  $\big] - \mathbf{D}\big[$  $\big] = \zeta_3$
- $\mathbf{A}\big[$  $\big] - 24\,\mathbf{A}\big[$  $\big] + 18\,\mathbf{A}\big[$  $\big] - 3\,\mathbf{A}\big[$  $\big]^2 =$
  $144\zeta_2 I^A(0,0,0,0,2) - 24\zeta_2 I^A(0,0,2) - \frac{31}{2}\zeta_4$

A-cycle relations

Examples

- **A**[] $-$ **A**[] $= \frac{1}{2}\zeta_3 + 6\zeta_2 I^A(0,1,0,0)$
- **D**[] $-$ **D**[] $= \zeta_3$
- **A**[] $-\,24$ **A**[] $+\,18$ **A**[] $-\,3$ **A**[]^2 $=$
  $144\zeta_2 I^A(0,0,0,0,2) - 24\zeta_2 I^A(0,0,2) - \frac{31}{2}\zeta_4$
- 24 **D**[] $-$ **D**[] $-\,18$ **D**[] $+\,3$ **D**[]^2 $= 0$

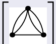
A-cycle relations

Examples

- $\mathbf{A}$[] $-$ $\mathbf{A}$[] $= \frac{1}{2}\zeta_3 + 6\zeta_2 I^A(0,1,0,0)$
- $\mathbf{D}$[] $-$ $\mathbf{D}$[] $= \zeta_3$
- $\mathbf{A}$[] $- 24\,\mathbf{A}$[] $+ 18\,\mathbf{A}$[] $- 3\,\mathbf{A}$[]$^2 =$ $144\zeta_2 I^A(0,0,0,0,2) - 24\zeta_2 I^A(0,0,2) - \frac{31}{2}\zeta_4$
- $24\,\mathbf{D}$[] $-$ $\mathbf{D}$[] $- 18\,\mathbf{D}$[] $+ 3\,\mathbf{D}$[]$^2 = 0$
- $20\,\mathbf{A}$[] $- 10\,\mathbf{A}$[] $- 4\,\mathbf{A}$[] $= \frac{3}{2}\zeta_5 \mod \zeta_2 I^A(\mathbf{n})$

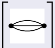
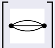
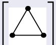
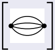
A-cycle relations

Examples

- **A**[] − **A**[] $= \frac{1}{2}\zeta_3 + 6\zeta_2 I^A(0,1,0,0)$
- **D**[] − **D**[] $= \zeta_3$
- **A**[] − 24 **A**[] + 18 **A**[] − 3 **A**[]^2 $=$ $144\zeta_2 I^A(0,0,0,0,2) - 24\zeta_2 I^A(0,0,2) - \frac{31}{2}\zeta_4$
- 24 **D**[] − **D**[] − 18 **D**[] + 3 **D**[]^2 $= 0$
- 20 **A**[] − 10 **A**[] − 4 **A**[] $= \frac{3}{2}\zeta_5 \mod \zeta_2 I^A(\mathbf{n})$
- 20 **D**[] − 10 **D**[] − 4 **D**[] $= 3\zeta_5$

A-cycle relations

Examples

- **A** [] $-$ **A** [] $= \frac{1}{2}\zeta_3 + 6\zeta_2 I^A(0,1,0,0)$
- **D** [] $-$ **D** [] $= \zeta_3$
- **A** [] $- 24$ **A** [] $+ 18$ **A** [] $- 3$ **A** []^2 $=$ $144\zeta_2 I^A(0,0,0,0,2) - 24\zeta_2 I^A(0,0,2) - \frac{31}{2}\zeta_4$
- 24 **D** [] $-$ **D** [] $- 18$ **D** [] $+ 3$ **D** []^2 $= 0$
- 20 **A** [] $- 10$ **A** [] $- 4$ **A** [] $= \frac{3}{2}\zeta_5 \mod \zeta_2 I^A(\mathbf{n})$
- 20 **D** [] $- 10$ **D** [] $- 4$ **D** [] $= 3\zeta_5$

"Rule 1": Relation among A-cycle graph functions + send $\zeta(2k) \to 0$, $\zeta(2k+1) \to 2\zeta(2k+1)$ ⇢ relation among modular graph functions.

From B-cycle to modular graph functions

Let $B_\Gamma(\tau) = \sum_{j\geq 0} b_\Gamma^{(j)}(\tau) q^j$, $\mathsf{b}_\Gamma(\tau) := b_\Gamma^{(0)}(\tau)$, $T := 2\pi i\tau$.

From B-cycle to modular graph functions

Let $B_\Gamma(\tau) = \sum_{j\geq 0} b_\Gamma^{(j)}(\tau) q^j$, $\mathsf{b}_\Gamma(\tau) := b_\Gamma^{(0)}(\tau)$, $T := 2\pi i \tau$.

$\mathbf{b}\Big[$  $\Big]$

$$= \frac{808T^6}{638512875} + \frac{\zeta_2 T^4}{13365} + \frac{\zeta_3 T^3}{9450} + \frac{4\zeta_4 T^2}{1575} + \frac{\zeta_2\zeta_3 T}{378} - \frac{\zeta_5 T}{3780}$$
$$+ \frac{31\zeta_6}{1680} + \frac{\zeta_3\zeta_4}{16T} - \frac{\zeta_2\zeta_5}{144T} + \frac{\zeta_7}{1440T} - \frac{107\zeta_8}{2160T^2} - \frac{35\zeta_2\zeta_7}{96T^3} + \frac{23\zeta_9}{128T^3} + \frac{55\zeta_3\zeta_6}{96T^3}$$
$$- \frac{15\zeta_3\zeta_7}{128T^4} - \frac{525\zeta_{10}}{512T^4} - \frac{45\zeta_3^2\zeta_4}{128T^4} + \frac{15\zeta_2\zeta_3\zeta_5}{32T^4} - \frac{\zeta_5^2}{256T^4} + \frac{5\zeta_2\zeta_{3,5}}{64T^4}$$
$$- \frac{95\zeta_5\zeta_6}{512T^5} + \frac{333\zeta_4\zeta_7}{128T^5} - \frac{5\zeta_3\zeta_8}{48T^5} - \frac{317\zeta_2\zeta_9}{256T^5} + \frac{167\zeta_{11}}{512T^5} - \frac{3435\zeta_{12}}{11056T^6}$$

From B-cycle to modular graph functions

Let $B_\Gamma(\tau) = \sum_{j\geq 0} b_\Gamma^{(j)}(\tau) q^j$, $\mathrm{b}_\Gamma(\tau) := b_\Gamma^{(0)}(\tau)$, $T := 2\pi i \tau$.

b[]

$$= \frac{808T^6}{638512875} + \frac{\zeta_2 T^4}{13365} + \frac{\zeta_3 T^3}{9450} + \frac{4\zeta_4 T^2}{1575} + \frac{\zeta_2\zeta_3 T}{378} - \frac{\zeta_5 T}{3780} + \frac{31\zeta_6}{1680} + \frac{\zeta_3\zeta_4}{16T} - \frac{\zeta_2\zeta_5}{144T} + \frac{\zeta_7}{1440T} - \frac{107\zeta_8}{2160T^2} - \frac{35\zeta_2\zeta_7}{96T^3} + \frac{23\zeta_9}{128T^3} + \frac{55\zeta_3\zeta_6}{96T^3} - \frac{15\zeta_3\zeta_7}{128T^4} - \frac{525\zeta_{10}}{512T^4} - \frac{45\zeta_3^2\zeta_4}{128T^4} + \frac{15\zeta_2\zeta_3\zeta_5}{32T^4} - \frac{\zeta_5^2}{256T^4} + \frac{5\zeta_2\zeta_{3,5}}{64T^4} - \frac{95\zeta_5\zeta_6}{512T^5} + \frac{333\zeta_4\zeta_7}{128T^5} - \frac{5\zeta_3\zeta_8}{48T^5} - \frac{317\zeta_2\zeta_9}{256T^5} + \frac{167\zeta_{11}}{512T^5} - \frac{3435\zeta_{12}}{11056T^6}$$

d[]

$$= \frac{808y^6}{638512875} + \frac{\zeta_3 y^3}{9450} - \frac{\zeta_5 y}{3780} + \frac{\zeta_7}{1440y} + \frac{23\zeta_9}{128y^3} - \frac{\zeta_5^2}{256y^4} + \frac{167\zeta_{11}}{512y^5}$$

From B-cycle to modular graph functions

Let $B_\Gamma(\tau) = \sum_{j\geq 0} b_\Gamma^{(j)}(\tau) q^j$, $\mathsf{b}_\Gamma(\tau) := b_\Gamma^{(0)}(\tau)$, $T := 2\pi i\tau$.

b[]

$$= \frac{808T^6}{638512875} + \frac{\zeta_2 T^4}{13365} + \frac{\zeta_3 T^3}{9450} + \frac{4\zeta_4 T^2}{1575} + \frac{\zeta_2\zeta_3 T}{378} - \frac{\zeta_5 T}{3780} + \frac{31\zeta_6}{1680} + \frac{\zeta_3\zeta_4}{16T} - \frac{\zeta_2\zeta_5}{144T} + \frac{\zeta_7}{1440T} - \frac{107\zeta_8}{2160T^2} - \frac{35\zeta_2\zeta_7}{96T^3} + \frac{23\zeta_9}{128T^3} + \frac{55\zeta_3\zeta_6}{96T^3} - \frac{15\zeta_3\zeta_7}{128T^4} - \frac{525\zeta_{10}}{512T^4} - \frac{45\zeta_3^2\zeta_4}{128T^4} + \frac{15\zeta_2\zeta_3\zeta_5}{32T^4} - \frac{\zeta_5^2}{256T^4} + \frac{5\zeta_2\zeta_{3,5}}{64T^4} - \frac{95\zeta_5\zeta_6}{512T^5} + \frac{333\zeta_4\zeta_7}{128T^5} - \frac{5\zeta_3\zeta_8}{48T^5} - \frac{317\zeta_2\zeta_9}{256T^5} + \frac{167\zeta_{11}}{512T^5} - \frac{3435\zeta_{12}}{11056T^6}$$

d[]

$$= \frac{808y^6}{638512875} + \frac{\zeta_3 y^3}{9450} - \frac{\zeta_5 y}{3780} + \frac{\zeta_7}{1440y} + \frac{23\zeta_9}{128y^3} - \frac{\zeta_5^2}{256y^4} + \frac{167\zeta_{11}}{512y^5}$$

"Rule 2": Take Laurent polys of B-cycle graph functions, send $\zeta(2k) \to 0$, $\zeta(2k+1) \to 2\zeta(2k+1)$ and $T = 2\pi i\tau \to \pi\tau_2 =: y$
⇢ Laurent polys of modular graph functions.

From B-cycle to modular graph functions

"Rule 3": Write B-cycle graph functions in terms of certain iterated Eisenstein integrals, whose q-expansion reads

$$\mathcal{E}_0(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \ldots, k_r, 0^{p_r-1}; \tau) := (-2)^r \left(\prod_{j=1}^{r} \frac{1}{(k_j-1)!} \right)$$

$$\times \sum_{m_i, n_i = 1}^{\infty} \frac{m_1^{k_1-1} m_2^{k_2-1} \ldots m_r^{k_r-1} q^{m_1 n_1 + m_2 n_2 + \ldots + m_r n_r}}{(m_1 n_1)^{p_1} (m_1 n_1 + m_2 n_2)^{p_2} \ldots (m_1 n_1 + m_2 n_2 + \ldots + m_r n_r)^{p_r}},$$

and send these iterated integrals to their real part.

From B-cycle to modular graph functions

"Rule 3": Write B-cycle graph functions in terms of certain iterated Eisenstein integrals, whose q-expansion reads

$$\mathcal{E}_0(k_1, 0^{p_1-1}, k_2, 0^{p_2-1}, \ldots, k_r, 0^{p_r-1}; \tau) := (-2)^r \left(\prod_{j=1}^{r} \frac{1}{(k_j-1)!} \right)$$

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and send these iterated integrals to their real part.

$3\,\mathbf{D}\Big[$  $\Big] + \mathbf{D}\Big[$  $\Big] - \frac{15}{14}\,\mathbf{D}\Big[$  $\Big]$

$$= \frac{2y^6}{6251175} + \frac{y\zeta(5)}{210} + \frac{\zeta(7)}{16y} - \frac{7\zeta(9)}{64y^3} + \frac{9\zeta(5)^2}{64y^4}$$

$$- \left(\frac{4y}{7} + \frac{135\zeta(5)}{4y^4} \right) \mathrm{Re}[\mathcal{E}_0(6,0,0,0,0)] + \frac{2025 \mathrm{Re}[\mathcal{E}_0(6,0,0,0,0)]^2}{y^4}$$

$$+ 21600 \mathrm{Re}[\mathcal{E}_0(6,6,0,0,0,0)] - \frac{20}{7} \mathrm{Re}[\mathcal{E}_0(6,0,0,0,0,0)] + \cdots$$

THE END

Thanks for your attention!