Analytic continuation of the equal mass sunrise and the kite

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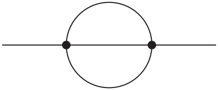
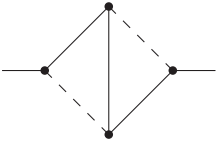
Outline:

- Results for the equal mass sunrise and the kite integral in the Euclidean region
- Short discussion of ELi-functions
- The underlying family of elliptic curves
- Monodromy of the periods
- Results and numerical evaluations
- Appendix: The Picard-Lefschetz theorem

Previously we have obtained results for

- the **equal mass sunrise** $S(2-2\epsilon, t)$ in 2 dimensions (Adams, CB, Weinzierl 2015)
- the **kite** integral $I(4-2\epsilon, t)$ in 4 dimensions (Adams, CB, Schweitzer, Weinzierl 2016)

Many other people have provided results. (Laporta, Remiddi, Tancredi, Bloch, Vanhove, Sabry, Bauberger, Berends, Böhm, Buza, Weiglein, Scharf, Broadhurst, Fleischer, Tarasov, Groote, Körner, Pivovarov, ...)

Using our framework of **ELi-functions** as elliptic generalizations of polylogarithms, we can recursively compute arbitrary orders in ϵ.

Open problem so far: Our results are given only for $t < 0$ where $t = p^2$.

As a generalization of classical polylogarithms $\mathrm{Li}_n(x) = \sum_{j=1}^{\infty} \frac{x^j}{j^n}$ we define **basic ELi-functions**

$$\mathrm{ELi}_{n;m}(x;y;q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk} = \sum_{k=1}^{\infty} \frac{y^k}{k^m} \mathrm{Li}_n(q^k x),$$

and **multi-variable generalization**:

$$\mathrm{ELi}_{n_1,\ldots,n_l;m_1,\ldots,m_l;2o_1,\ldots,2o_{l-1}}(x_1,\ldots,x_l;y_1,\ldots,y_l;q)$$

$$= \sum_{j_1=1}^{\infty}\cdots\sum_{j_l=1}^{\infty}\sum_{k_1=1}^{\infty}\cdots\sum_{k_l=1}^{\infty} \frac{x_1^{j_1}}{j_1^{n_1}}\cdots\frac{x_l^{j_l}}{j_l^{n_l}}\frac{y_1^{k_1}}{k_1^{m_1}}\cdots\frac{y_l^{k_l}}{k_l^{m_l}} \frac{q^{j_1k_1+\ldots+j_lk_l}}{\prod_{i=1}^{l-1}(j_ik_i+\ldots+j_lk_l)^{o_i}}$$

Dependences in our results:

q is a function of t and of masses; the x_i only depend on masses; $y_i \in \{-1, 1\}$

Our results involve **combinations** like

$$\mathrm{E}_{n;m}(x;\,y;\,q) =$$

$$\begin{cases} \frac{1}{i}\left(\frac{1}{2}\mathrm{Li}_n(x) - \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x;\,y;\,q) - \mathrm{ELi}_{n;m}(x^{-1};\,y^{-1};\,q)\right) & ,\, n+m \text{ even}, \\ \frac{1}{2}\mathrm{Li}_n(x) + \frac{1}{2}\mathrm{Li}_n(x^{-1}) + \mathrm{ELi}_{n;m}(x;\,y;\,q) + \mathrm{ELi}_{n;m}(x^{-1};\,y^{-1};\,q) & ,\, n+m \text{ odd}. \end{cases}$$

These are closer to elliptic polylogarithms of the mathematical literature, such as the functions of Brown, Levin 2011.

Multiplication property:

$$\mathrm{ELi}_{\tilde{n},\tilde{m}}(\tilde{x};\tilde{y};q)\cdot\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q)$$

$$=\mathrm{ELi}_{n_1,\dots,n_l,\tilde{n};m_1,\dots,m_l,\tilde{m};2o_1,\dots,2o_{l-1},0}(x_1,\dots,x_l,\tilde{x};y_1,\dots,y_l,\tilde{y};q)$$

Integration property:

$$\int^q\frac{dq'}{q'}\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}}(x_1,\dots,x_l;y_1,\dots,y_l;q')$$

$$=\mathrm{ELi}_{n_1,\dots,n_l;m_1,\dots,m_l;2o_1,\dots,2o_{l-1}+1}(x_1,\dots,x_l;y_1,\dots,y_l;q)$$

$\Rightarrow$We can **multiply** with $\mathrm{ELi}_{n;m}(x;y;q)$ and **integrate** over $\frac{dq}{q}$ **without leaving this class of functions**.

For the **sunrise**

$$S(2-2\epsilon,\ t) = S^{(0)} + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + ...$$

we **re-write the differential equations** in terms of q.

$\Rightarrow$ We obtain a **recursive relation** of the form

$$S^{(j)} = -\frac{\psi_1}{\pi} \int_0^q \frac{dq_1}{q_1} \int_0^{q_1} \frac{dq_2}{q_2} \left(a_j + b \cdot S^{(j-2)} \right).$$

- From previous work we know $S^{(0)}$ and $S^{(1)}$ in terms of $\mathrm{ELi}_{n_1,...;m_1,...;2o_1,...}$.
- We can express all a_j and b in terms of $\mathrm{ELi}_{n;m}$.

$\Rightarrow$ Using the multiplication- and integration-properties, we can compute $S^{(j)}$ to **arbitrary** j in terms of $\mathrm{ELi}_{n_1,...;m_1,...;2o_1,...}$.

In a **similar** way, we obtain a result for the **kite** integral to arbitrary order.

Example: For the equal mass sunrise $S(2-2\epsilon,\ t) = S^{(0)} + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + \ldots$ we have

$$S^{(j)}(2,t) = \frac{\psi_1}{\pi} E^{(j)}$$

with $\psi_1 = \frac{4}{(m^2-t)^{\frac{3}{4}}(9m^2-t)^{\frac{1}{4}}} K(k)$ and with

$$\begin{aligned}
E^{(0)} &= 3\mathrm{E}_{2;0}\left(r_3;-1;-q\right), \\
E^{(1)} &= 3\mathrm{E}_{3;1}\left(r_3;-1;-q\right) + 3\mathrm{E}_{0,1;-2,0;4}\left(r_3,r_3;-1,-1;-q\right) \\
&\quad -9\mathrm{E}_{0,1;-2,0;4}\left(r_3,r_3;-1,1;-q\right) + 18\mathrm{E}_{0,1;-2,0;4}\left(r_3,-1;-1,1;-q\right) \\
&\quad +\frac{3}{2i}\left(-2\mathrm{Li}_{2,1}\left(r_3,1\right) - 2\mathrm{Li}_3\left(r_3\right) + 2\mathrm{Li}_{2,1}\left(r_3^{-1},1\right)\right. \\
&\quad \left. +2\mathrm{Li}_3\left(r_3^{-1}\right) + 6\mathrm{Li}_1\left(-1\right)\left(\mathrm{Li}_2\left(r_3\right) - \mathrm{Li}_2\left(r_3^{-1}\right)\right)\right) + L_{1;0}E^{(0)}_{111}
\end{aligned}$$

where $r_3 = e^{2\pi i/3}$.

The underlying **family of elliptic curves** is given by $\mathcal{F} = 0$ for the sunrise's second Symanzik polynomial

$$\mathcal{F} = -x_1 x_2 x_3 t + m^2 \left(x_1 + x_2 + x_3\right)\left(x_1 x_2 + x_2 x_3 + x_1 x_3\right).$$

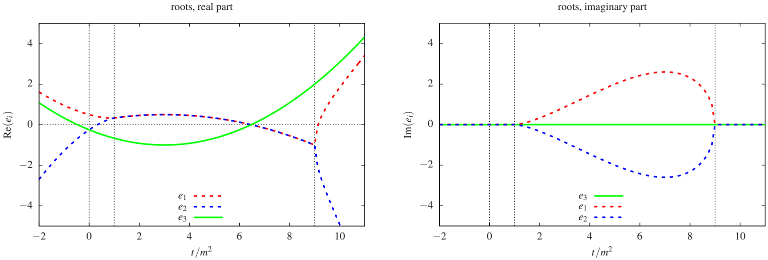
We change variables to Weierstrass normal form (in the chart $z = 1$):

$$y^2 = 4\left(x - e_1\right)\left(x - e_2\right)\left(x - e_3\right)$$

with the three roots

$$\begin{aligned} e_1 &= \frac{1}{24}\left(-t^2 + 6m^2 t + 3m^4 + 3\left(m^2 - t\right)^{\frac{3}{2}}\left(9m^2 - t\right)^{\frac{1}{2}}\right) \\ e_2 &= \frac{1}{24}\left(-t^2 + 6m^2 t + 3m^4 - 3\left(m^2 - t\right)^{\frac{3}{2}}\left(9m^2 - t\right)^{\frac{1}{2}}\right) \\ e_3 &= \frac{1}{24}\left(2t^2 - 12m^2 t - 6m^4\right). \end{aligned}$$

For every value $t \in \mathbb{R}$ this defines an elliptic curve, except for values where two of the roots coincide (i.e. the family degenerates).



Roots coincide at

$$
\begin{aligned}
t = 0 \quad &: \quad e_2 = e_3, \\
t = m^2 \quad &: \quad e_1 = e_2, \\
t = 9m^2 \quad &: \quad e_1 = e_2, \\
t = \infty \quad &: \quad e_1 = e_3.
\end{aligned}
$$

Remark: These are the singularities of the second order Picard-Fuchs differential operator of the sunrise.

For $t < 0$ we have $e_2 < e_3 < e_1$. Here we define the **period integrals**

$$\psi_1 = 2\int_{e_2}^{e_3} \frac{dx}{y},\ \psi_2 = 2\int_{e_1}^{e_3} \frac{dx}{y}$$

with x, y of the Weierstrass normal form.

One can derive explicitly

$$\psi_1 = \frac{4}{(m^2 - t)^{\frac{3}{4}}\,(9m^2 - t)^{\frac{1}{4}}} K(k),\ \psi_2 = \frac{4i}{(m^2 - t)^{\frac{3}{4}}\,(9m^2 - t)^{\frac{1}{4}}} K(k')$$

with the complete elliptic integral of the first kind

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)\,(1 - k^2 t^2)}}$$

with moduli

$$k = \frac{e_3 - e_2}{e_1 - e_2},\ k' = 1 - k^2 = \frac{e_1 - e_3}{e_1 - e_2}.$$

With these periods, we define

$$q = e^{i\frac{\psi_2}{\psi_1}} = e^{-\frac{K(k')}{K(k)}}.$$

What happens when we vary t?

$\Rightarrow$ The points e_1, e_2, e_3 will will move around and the moduli $k = \frac{e_3 - e_2}{e_1 - e_2}$, $k' = \frac{e_1 - e_3}{e_1 - e_2}$ will change.

$\Rightarrow q = e^{i\frac{\psi_2}{\psi_1}} = e^{-\frac{K(k')}{K(k)}}$ will change.

Important detail: Segments of the real t-axis correspond to **branch-cuts** of these functions.

We always need to control "**on which side** of the cut we are"

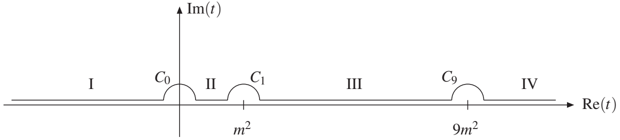
and how to evaluate **on** the cut.

$\Rightarrow$ We use **Feynman's prescription**: replace $t \to t + i\delta$ with a small, positive $\delta \in \mathbb{R}$ which is sent to zero.

Example:

$$\sqrt{-t} \to \sqrt{-t - i\delta} = \begin{cases} -i\sqrt{|t|} & \text{for } t > 0, \\ \sqrt{|t|} & \text{for } t \leq 0. \end{cases}$$

⇒Contour of the variation of t



Our goal: Derive explicit expressions for $q = e^{i\frac{\psi_2}{\psi_1}}$ in the regions II, III and IV

i.e. explicit expressions for ψ_i in all regions.

Observation: This **suffices for the analytic continuation** of the sunrise and kite results! (compare work of Remiddi, Tancredi,...)

Check of the observation:

The relation

$$t = -9m^2 \frac{\eta(\tau)^4 \eta\left(\frac{3\tau}{2}\right)^4 \eta(6\tau)^4}{\eta\left(\frac{\tau}{2}\right)^4 \eta(2\tau)^4 \eta(3\tau)^4}$$

with $\tau = \frac{\psi_2}{\psi_1}$ remains true for regions II, III and IV after these replacements.

When does q change in a non-trivial way?

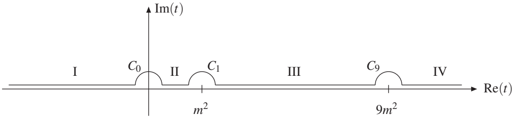
$$q = e^{i\frac{\psi_2}{\psi_1}} = e^{-\frac{K(k')}{K(k)}}$$

$\Rightarrow$ Only when the moduli k, k' **cross branch cuts** of the complete elliptic integral

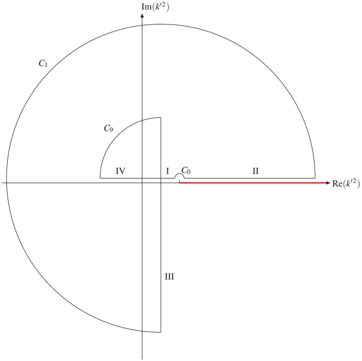
$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.$$

Simplification: Consider $\tilde{K}\left(k^2\right) = K(k)$ as a function of k^2.

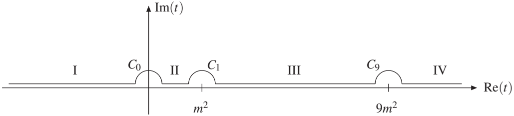
This function has **only one branch cut** in the k^2-plane at $[1, \infty[$.



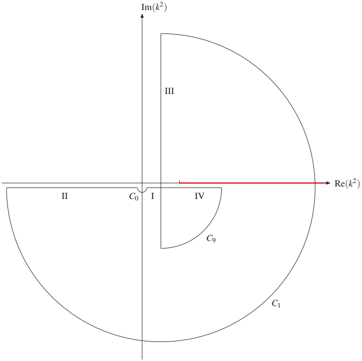
Along the path of t we have for k'^2 :



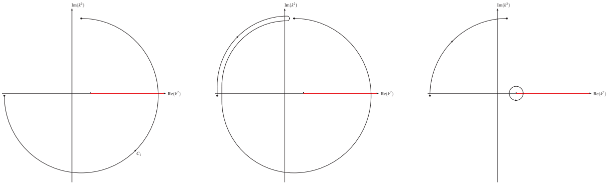
Branch-cut is **not** crossed. $\Rightarrow \psi_2\left(t+i\delta\right)=\frac{4i}{\left(m^2-t\right)^{\frac{3}{4}}\left(9m^2-t\right)^{\frac{1}{4}}}K\left(k'\left(t+i\delta\right)\right)$ for all regions.



Along the path of t we have for k^2 :



$\Rightarrow$ We only have to study the monodromy behaviour of ψ_1 along C_1.



We vary t along C_1 :

The path for k^2 can be decomposed into:

- A full, small **circle around 1** (the finite end-point of the branch-cut) in anti-clockwise direction.
- A quarter-circle in clockwise direction.

The full circle is the relevant part. How does ψ_1 change on this path?

Consider the Legendre form

$$y^2 = x(x-\lambda)(x-1) \text{ with } \lambda = k^2.$$

We have k^2 rotating (anti-clockwise) around 1.

Equivalently consider

$$y^2 = x(x-e_1(\phi))(x-e_2(\phi))$$

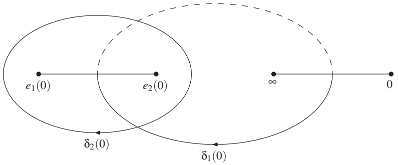
where two points $e_1(\phi) = 1 - re^{i\phi}$ and $e_2(\phi) = 1 + re^{i\phi}$ **rotate around each other** as we send ϕ from 0 to 2π.

Furthermore: Consider **periods as integrals over cycles** δ_1, δ_2 on the elliptic curve:

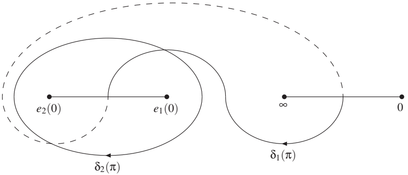
$$\psi_1 = \int_{\delta_1(\phi)} \frac{dx}{y}, \; \psi_2 = \int_{\delta_2(\phi)} \frac{dx}{y}$$

with $y = -\sqrt{x}\sqrt{x-e_1(\phi)}\sqrt{x-e_2(\phi)}$.

At $\phi = 0$, before the rotation of e_1 and e_2 around each other, we have:

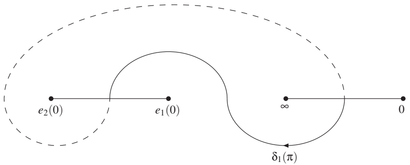


At $\phi = \pi$, after the first half of the rotation, we obtain:

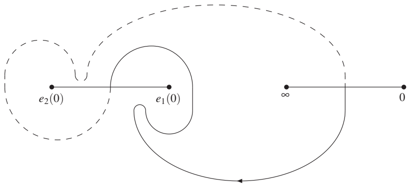


δ_2 does not change ($\Longleftrightarrow$ ψ_2 does not change). Can we express $\delta_1(\pi)$ by $\delta_1(0)$, $\delta_2(0)$?

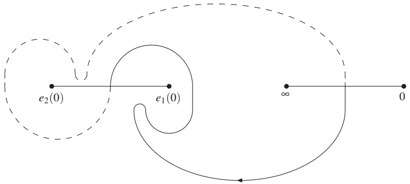
Consider the contour $\delta_1(\pi)$:



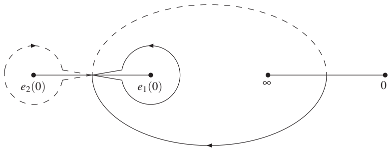
Without changing the integral, we may deform it to



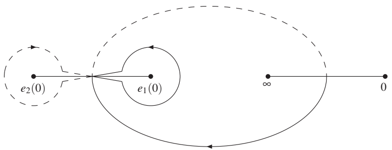
... and furthermore from



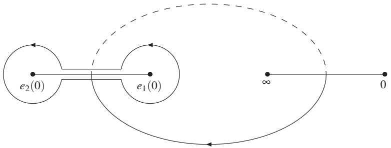
to



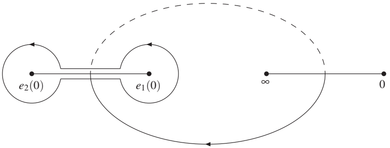
Notice: The two circles around e_1 and e_2 are in different Riemann sheets.



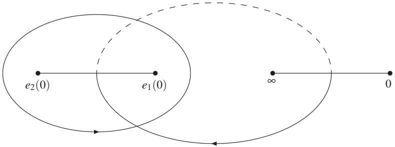
Now pull the circle around e_2 to the other sheet:



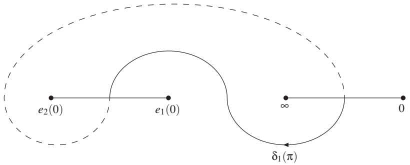
Notice: Both circles have the same orientation.

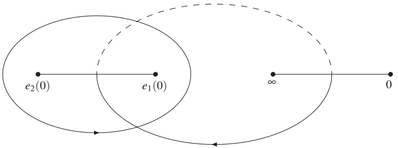


We can therefore merge the two circles to one contour:



The result of this operation is the contour $-\delta_2(0)$ (minus sign: changed orientation).





Fazit:

$$\begin{aligned} \delta_1(\pi) &= \delta_1(0) - \delta_2(0), \\ \delta_2(\pi) &= \delta_2(0). \end{aligned}$$

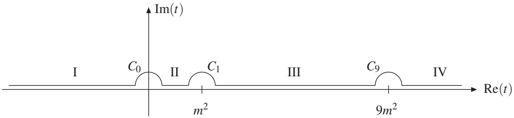
After a full rotation of e_1, e_2 around each other (i.e. of k^2 around 1) we have

$$\begin{aligned} \delta_1(2\pi) &= \delta_1(0) - 2\delta_2(0) \\ \delta_2(2\pi) &= \delta_2(0). \end{aligned}$$

$\Rightarrow$ As t varies along C_1 the periods change as

$$\begin{aligned} \psi_1 &\to \psi_1 - 2\psi_2, \\ \psi_2 &\to \psi_2. \end{aligned}$$

Result: To obtain a result for the eq. mass sunrise $S(2-2\epsilon, t)$ and kite integral $I(4-2\epsilon, t)$ for any t on the path



evaluate our results of the Euclidean region at

$$q(t+i\delta) = e^{i\frac{\psi_2(t+i\delta)}{\psi_1(t+i\delta)}}$$

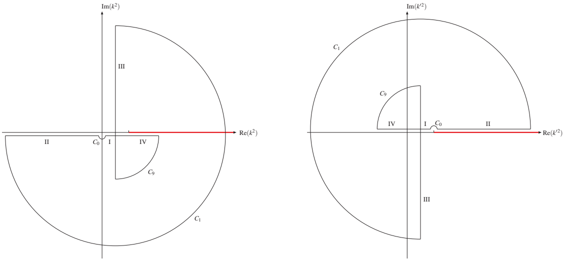
with

$$\begin{pmatrix} \psi_2(t+i\delta) \\ \psi_1(t+i\delta) \end{pmatrix} = \frac{4}{(m^2-t-i\delta)^{\frac{3}{4}}\,(9m^2-t-i\delta)^{\frac{1}{4}}} M_t \begin{pmatrix} iK\left(k'\left(t+i\delta\right)\right) \\ K\left(k\left(t+i\delta\right)\right) \end{pmatrix}$$

and

$$M_t = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } -\infty < t < m^2, \\ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} & \text{for } m^2 < t < \infty. \end{cases}$$

Preparational remark on the numerical evaluation:



In regions II and IV we have to evaluate elliptic integrals **on the cut**, approaching it from above or below.

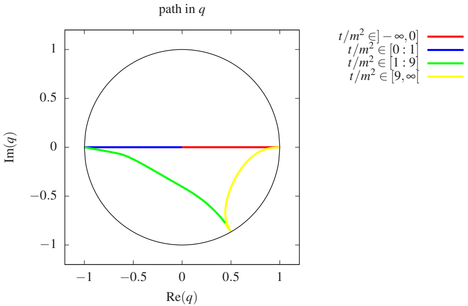
We can use the relation

$$K\left(k \pm i\delta\right) \stackrel{\delta \to 0}{=} \frac{1}{k}\left(K\left(\frac{1}{k}\right) \pm iK\left(\sqrt{1-\frac{1}{k^2}}\right)\right)$$

Remark: This can be used to obtain the analytic continuation as well.

Numerical evaluation:

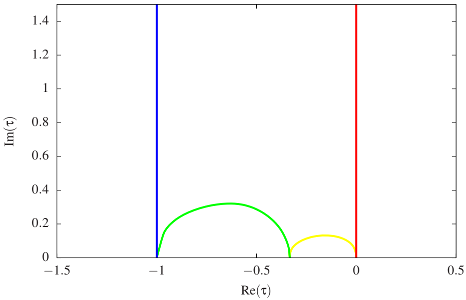
$$q(t+i\delta) = e^{i\frac{\psi_2(t+i\delta)}{\psi_1(t+i\delta)}}$$



Notice: $|q| < 1$ for finite $t \in \mathbb{R} \setminus \{m^2, 9m^2\} \Rightarrow$ ELi-functions converge

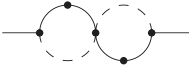
Numerical evaluation:

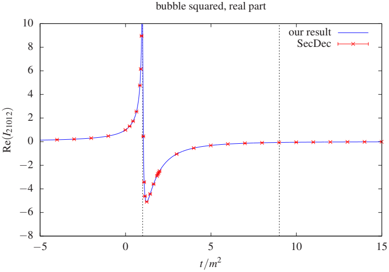
$$\tau(t + i\delta) = \frac{\psi_2(t + i\delta)}{\psi_1(t + i\delta)}$$

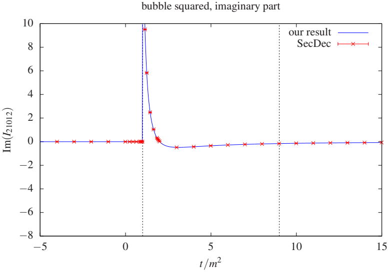


Notice: $\mathrm{Im}(\tau) > 0$ for finite $t \in \mathbb{R} \backslash \{m^2, 9m^2\}$

Numerical evaluation: The ϵ^0-term of a non-elliptic master integral (cross-check), expressed in ELi-functions

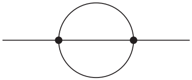


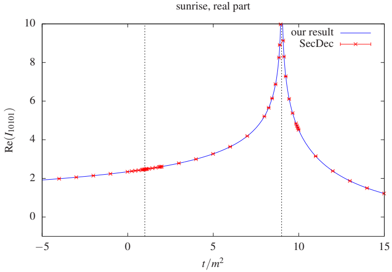


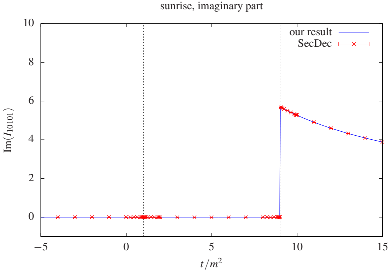


blue line: our result, red dots: SecDec (Borowka et al 2015)

Numerical evaluation: The ϵ^0-term of the equal mass sunrise integral

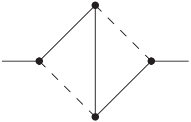


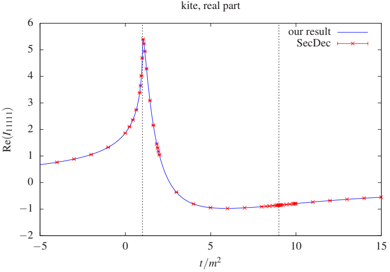


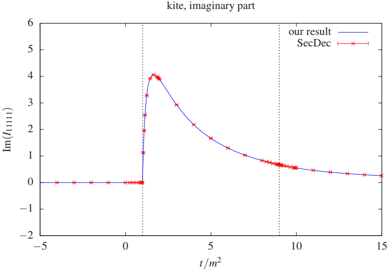


blue line: our result, red dots: SecDec (Borowka et al 2015)

Numerical evaluation: The ϵ^0-term of the kite integral

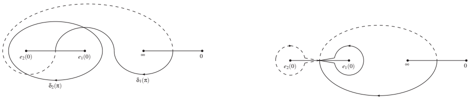






blue line: our result, red dots: SecDec (Borowka et al 2015)

Appendix: The perspective of Picard-Lefschetz theory



We return to our only non-trivial operation: the variation of t along C_1 (near $t = m^2$) with

$$\delta_1(\pi) \quad = \quad \delta_1(0) - \delta_2(0),\ \psi_1 \to \psi_1 - \psi_2$$

after a half-rotation of the **line between** e_1 **and** e_2.

- The situation at $t = m^2$ is known as **pinch singularity**: here $e_1 = e_2$ and the contour δ_1 is trapped
- Let s be the oriented line (1-simplex) from e_1 to e_2. It is called a **vanishing cycle** w.r.t. the pinch singularity.
- Its boundary is $\partial s = e_2 - e_1$.
- The **Leray co-boundary** $\delta(e)$ of a point e is an oriented circle around e.
$\Rightarrow \delta(\partial s) = \delta e_2 - \delta e_1$ are two circles with opposite orientations (as in our picture).

Assume a **pinch singularity** at $t = t_0$ and let s be a corresponding **vanishing cycle**.

Consider a **variation** of t in a circle around $t = t_0$ such that s transforms into itself.

The **Picard-Lefschetz theorem states**: Any cycle c transforms under this variation like

$$c \to c + n \cdot K(s,c) \cdot h.$$

- $n \in \{-1, 1\}$, depending on the dimension of the space and of c,
- $K(s,c) \in \{-1, 0, 1\}$: the intersection number or Kronecker index of s and c, depending on their relative orientation,
- h : a cycle defined by the Leray co-boundary of a vanishing cycle or its boundary. In our simple case: $h = \delta\left(\partial s\right)$.

Our case: $c = \delta_1$, $h = \delta_2$, $n \cdot K(s,c) = -1$, and therefore $\delta_1 \to \delta_1 - \delta_2$. This reproduces our $\delta_1(\pi) = \delta_1(0) - \delta_2(0)$.

The theorem applies in **much greater generality**. (See e.g. Hwa, Teplitz 1966)

Conclusions:

- A nice feature of results in terms of ELi-functions: in the analytical continuation w.r.t. t **only** q **has to be discussed**.
- For sunrise and kite: **only at the point** $t = m^2$ there is a non-trivial behaviour of one of the periods ψ_1.
- We have derived this behaviour **purely geometrically** on the elliptic curve.
- We can numerically evaluate our **results** to arbitrary order for every $t \in \mathbb{R} \backslash \left\{ m^2, 9m^2 \right\}$.
- The **Picard-Lefschetz theorem** explains the observed monodromy at $t = m^2$ and applies to much more general cases.