Elliptic Integrals and Two-Loop $t\bar{t}$ Production in QCD

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Plan of the Talk

- General Introduction
  - Top Quark at hadron colliders
- Status of the Theoretical Calculations
  - The General Framework
  - The NLO Corrections
  - Higher-order Corrections
  - Analytic calculation of matrix elements
- Conclusions

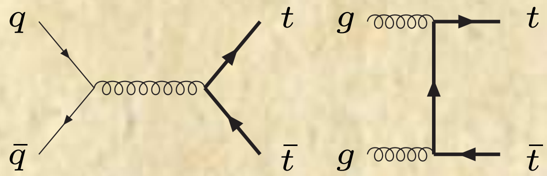
Top Quark

Top Quark

- With a mass of $m_t = 173.34 \pm 0.76$ GeV (March 2014), the TOP quark (the up-type quark of the third generation) is the heaviest elementary particle produced so far at colliders
- Because of its mass, top quark plays a unique role in understanding the EW symmetry breaking $\Rightarrow$ top physics is crucial at the LHC
- The top quark is produced (@ hadron colliders) via two mechanisms

Pair Production

  - $pp(\bar{p}) \to t\bar{t}$



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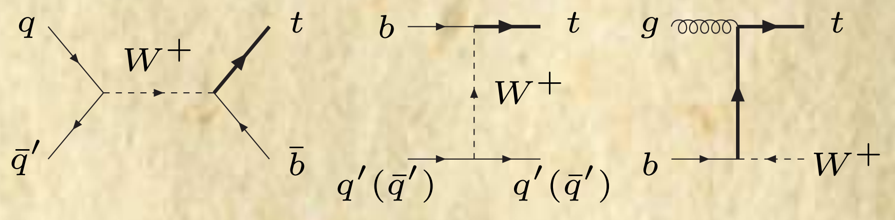
Single Top

  - $pp(\bar{p}) \to t\bar{b}, tq'(\bar{q}'), tW^-$



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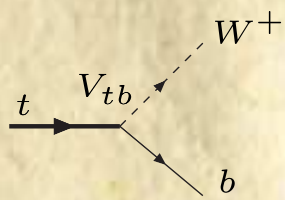
- The top quark plays a double role: signal or background for new physics
- Top quark does not hadronize, since it decays in about $5 \cdot 10^{-25}$s (one order of magnitude smaller than the hadronization time) $\Longrightarrow$ opportunity to study the quark as single particle
  - Spin properties
  - Interaction vertices
  - Top quark mass
- Decay products: almost exclusively $t \to W^+ b \quad (|V_{tb}| \gg |V_{td}|, |V_{ts}|)$



Top Quark: Event Selection

Top Pair events

$\boxed{\sigma_{\mathbf{t\bar{t}}}^{\mathbf{Tevatron}} \sim \text{7pb}}$

$\boxed{\sigma_{\mathbf{t\bar{t}}}^{\mathbf{LHC\,8}} \sim \text{250pb}}$

- $p\bar{p}(p) \to t\bar{t} \to W^+ b W^- \bar{b} \to l\nu l\nu b\bar{b}$
- $p\bar{p}(p) \to t\bar{t} \to W^+ b W^- \bar{b} \to l\nu q\bar{q}' b\bar{b}$
- $p\bar{p}(p) \to t\bar{t} \to W^+ b W^- \bar{b} \to q\bar{q}' q\bar{q}' b\bar{b}$

Top Quark: Event Selection

Top Pair events

Tevatron $\begin{cases} gg \text{ fusion} & \sim 15\% \\ q\bar{q} \text{ annihilation} & \sim 85\% \end{cases}$

LHC $\begin{cases} gg \text{ fusion} & \sim 90\% \\ q\bar{q} \text{ annihilation} & \sim 10\% \end{cases}$

$\boxed{\sigma^{\mathbf{Tevatron}}_{\mathbf{t\bar{t}}} \sim \mathbf{7pb}}$

$\boxed{\sigma^{\mathbf{LHC\,8}}_{\mathbf{t\bar{t}}} \sim \mathbf{250pb}}$

- $p\bar{p}(p) \to t\bar{t} \to W^+bW^-\bar{b} \to l\nu l\nu b\bar{b}$
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$\sigma_{\mathbf{t\bar{t}}}^{\mathbf{Tevatron}} \sim$ **7pb**

$\sigma_{\mathbf{t\bar{t}}}^{\mathbf{LHC\,8}} \sim$ **250pb**

- $p\bar{p}(p) \to t\bar{t} \to W^+bW^-\bar{b} \to l\nu l\nu b\bar{b}$ — Dilepton $\sim 10\%$
- $p\bar{p}(p) \to t\bar{t} \to W^+bW^-\bar{b} \to l\nu q\bar{q}'b\bar{b}$ — Lep+jets $\sim 44\%$
- $p\bar{p}(p) \to t\bar{t} \to W^+bW^-\bar{b} \to q\bar{q}'q\bar{q}'b\bar{b}$ — All jets $\sim 46\%$

Top Quark: Event Selection

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LHC $\begin{cases} gg \text{ fusion} & \sim 90\% \\ q\bar{q} \text{ annihilation} & \sim 10\% \end{cases}$

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2 high-p_T lept, $\geq$ 2 jets and ME

Top Quark: Event Selection

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LHC $\left\{ \begin{array}{ll} gg \text{ fusion} & \sim 90\% \\ q\bar{q} \text{ annihilation} & \sim 10\% \end{array} \right.$

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2 high-p_T lept, $\geq$ 2 jets and ME

1 isol high-p_T lept, $\geq$ 4 jets and ME

Top Quark: Event Selection

Top Pair events

Tevatron $\begin{cases} gg \text{ fusion} & \sim 15\% \\ q\bar{q} \text{ annihilation} & \sim 85\% \end{cases}$

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2 high-p_T lept, $\geq$ 2 jets and ME

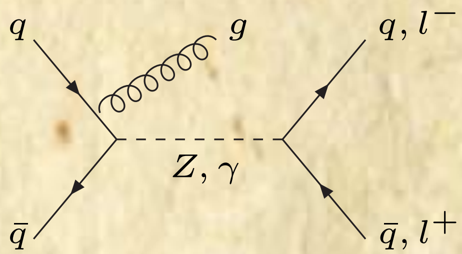
1 isol high-p_T lept, $\geq$ 4 jets and ME

NO lept, $\geq$ 6 jets and low ME

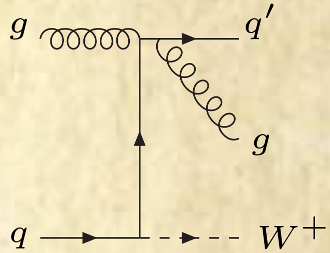
Top Quark: Background

Top Quark: Background

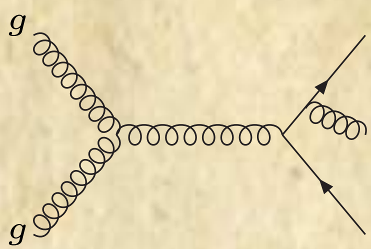
Background Processes



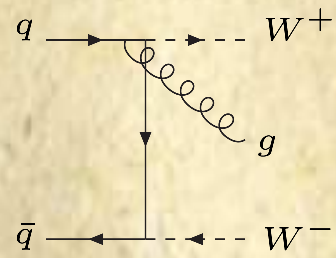
Drell-Yan



W+jets



QCD

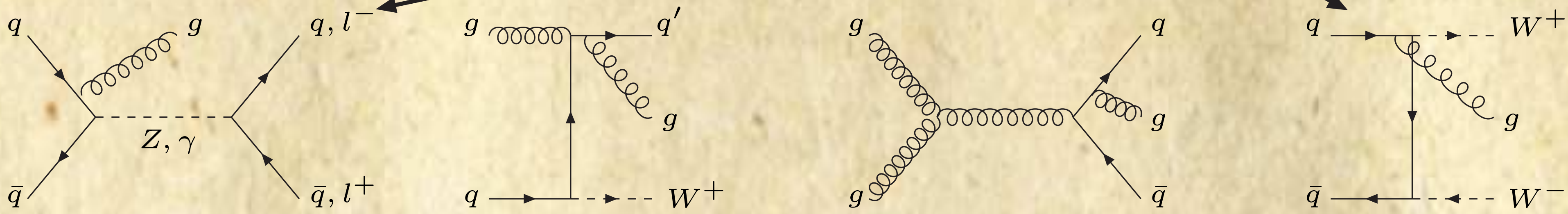


Di-boson

Top Quark: Background

Background Processes

main bg for Dilepton final state

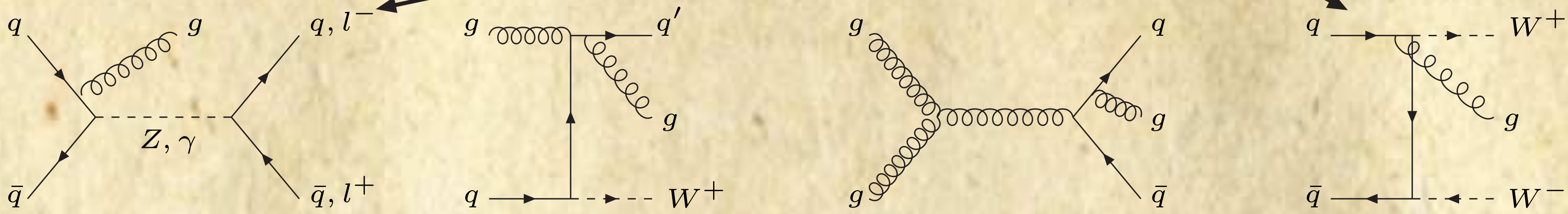


Drell-Yan W+jets QCD Di-boson

Top Quark: Background

Background Processes

main bg for Dilepton final state



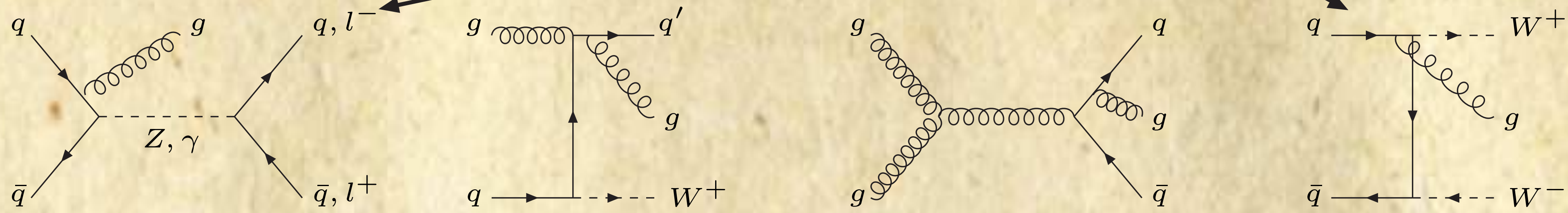
Drell-Yan W+jets QCD Di-boson

main bg for lepton+jets

Top Quark: Background

Background Processes

main bg for Dilepton final state



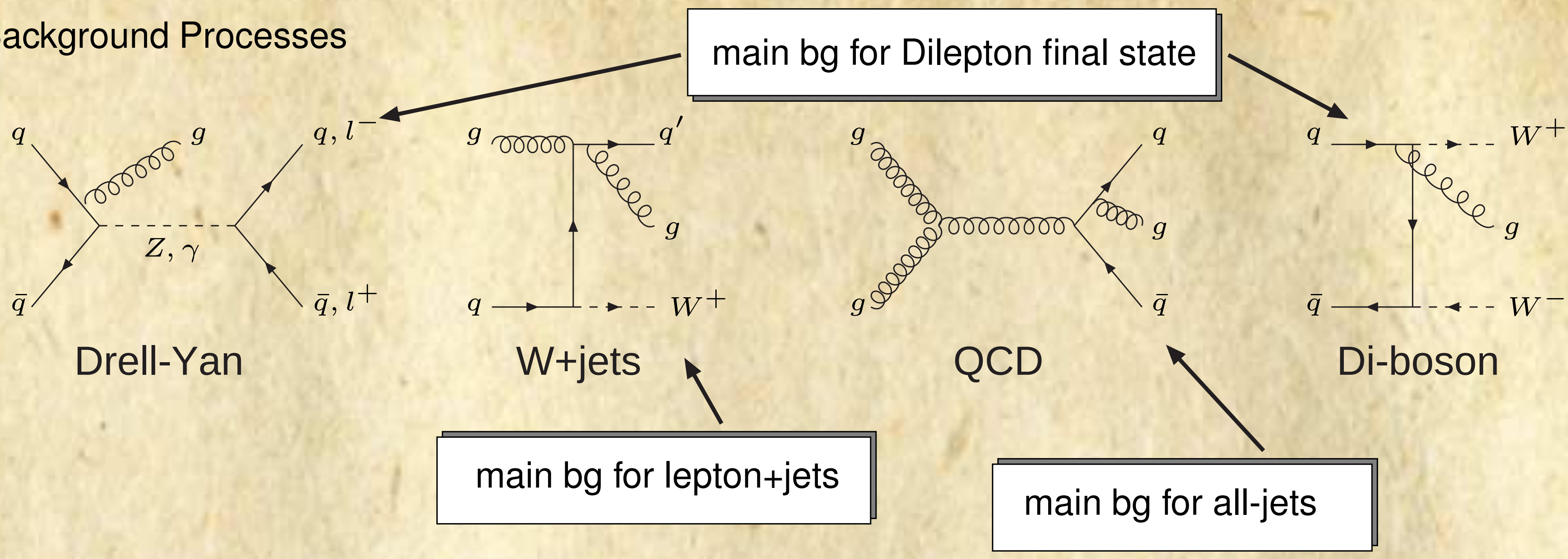
Drell-Yan | W+jets | QCD | Di-boson

main bg for lepton+jets

main bg for all-jets

Top Quark: Background

Background Processes



- Note: background processes known theoretically at NLO and NNLO. Simulated for CDF and D0 using Monte Carlo event generators (PYTHIA or HERWIG), normalized to the NLO cross section (for instance calculated with MCFM). For ATLAS and CMS using FEWZ, HATOR, and as CDF+D0.
- Background also evaluated using data-driven methods (for multi-jet bg).
- In order to increase the signal-to-background ratio is crucial the tagging of the b jets (not present in the background).

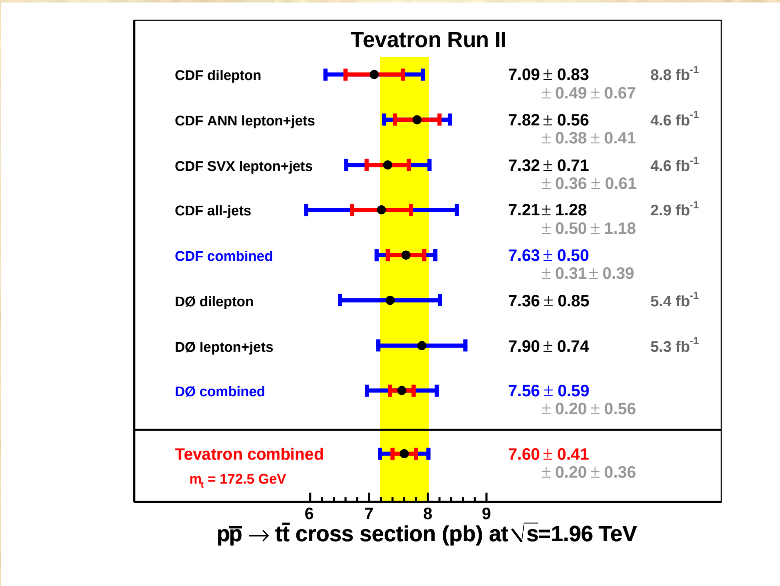
Top Quark: $t\bar{t}$ Cross Section

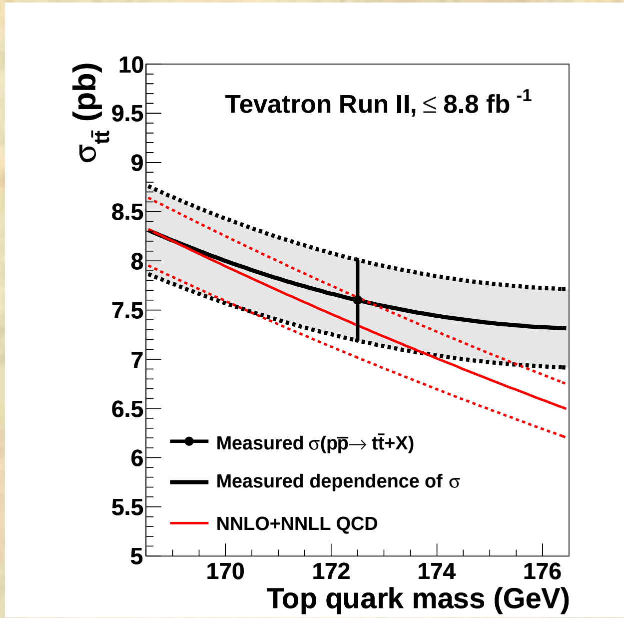
Top Quark: $t\bar{t}$ Cross Section

- Total $t\bar{t}$-pair Cross Section

$$\boxed{\sigma_{t\bar{t}} = \frac{N_{data} - N_{bkgr}}{\epsilon\, L}}$$

  - Test for the SM (in particular QCD)





Combination CDF-D0 September 2013 ($m_t = 172.5$ GeV)

$$\boxed{\sigma_{t\bar{t}} = 7.60 \pm 0.41\,\text{pb} \qquad (\Delta\sigma_{t\bar{t}}/\sigma_{t\bar{t}} \sim 5.4\%)}$$

using up to 8.8 fb^{-1} of data

Top Quark: $t\bar{t}$ Cross Section

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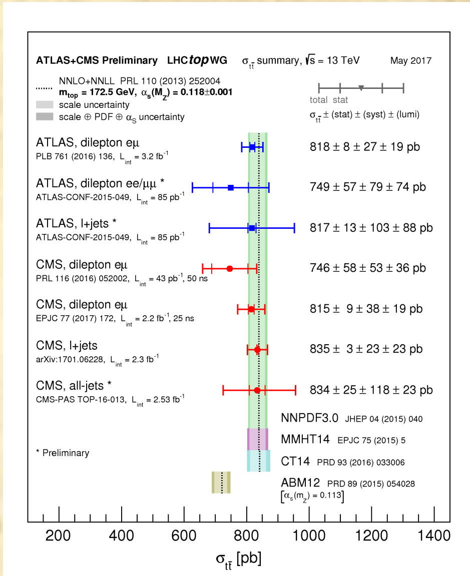


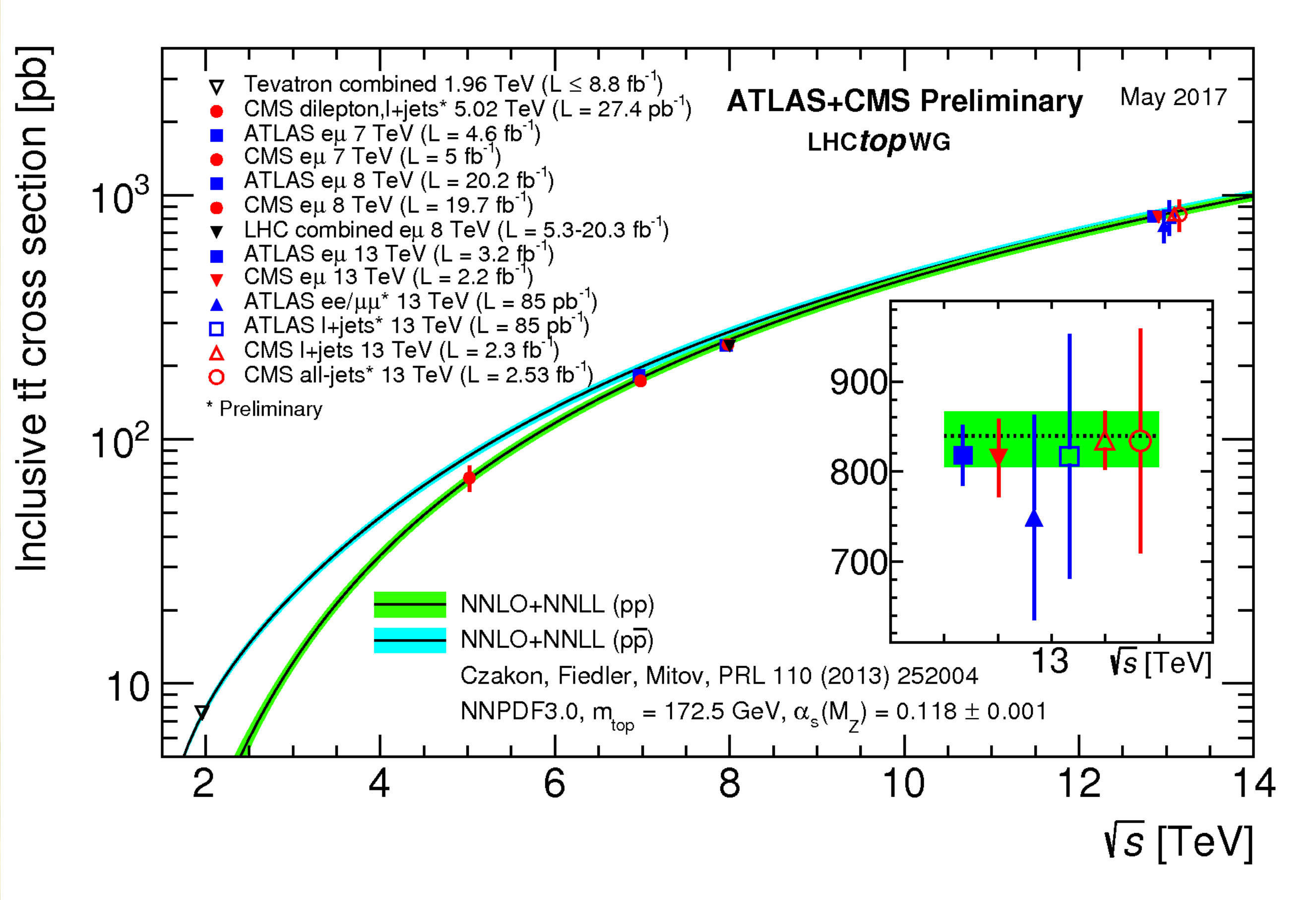
ATLAS+CMS ($m_t = 172.5$ GeV) @ $\sqrt{s} = 7$ TeV: $\boxed{\sigma_{t\bar{t}} = 173.3 \pm 2.3 \pm 7.6 \pm 6.3\,\text{pb}}$

ATLAS in lepton+jets ($m_t = 172.5$ GeV) @ $\sqrt{s} = 8$ TeV: $\boxed{\sigma_{t\bar{t}} = 260 \pm 1^{+22}_{-23} \pm 8\,\text{pb}}$

CMS in lepton+jets ($m_t = 172.5$ GeV) @ $\sqrt{s} = 13$ TeV: $\boxed{\sigma_{t\bar{t}} = 835 \pm 3 \pm 23 \pm 23\,\text{pb}}$

Top Quark: $t\bar{t}$ Cross Section

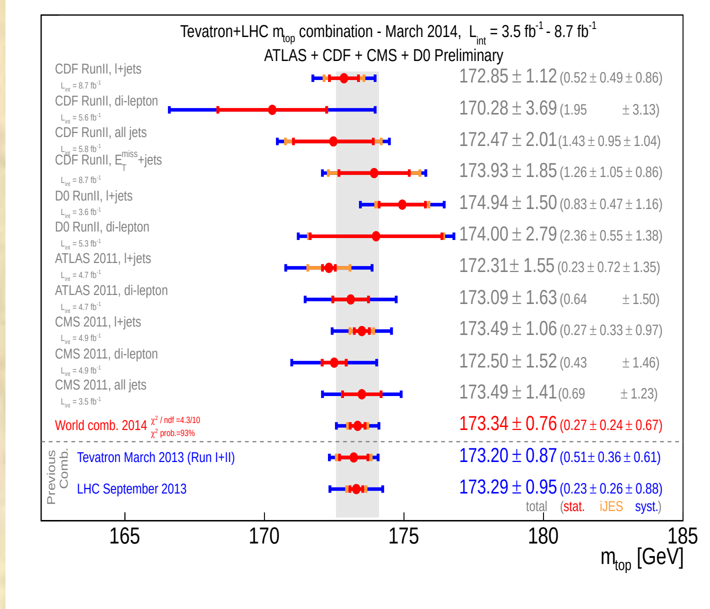
$t\bar{t}$ cross section measurements in agreement with theoretical predictions at NNLO+NNLL



Top Quark Mass

Top Quark Mass

- Top-quark Mass
  - Fundamental parameter of the SM. A precise measurement has impacts on EW precision fits. It was useful to constraint Higgs mass from radiative corrections (Δr) before the direct detection



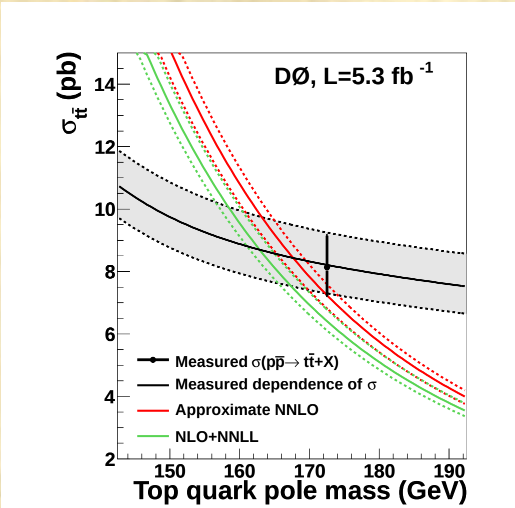
World Combination: ATLAS, CDF, CMS, D0 (18 March 2014)

$$\boxed{m_t = 173.34 \pm 0.27(stat) \pm 0.71(sys)\,\text{GeV} \quad (0.44\%)}$$

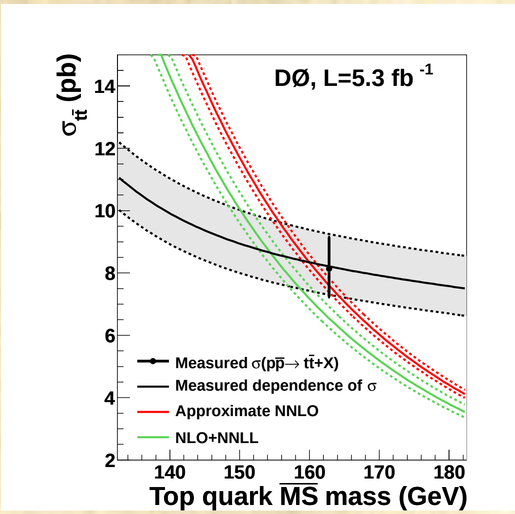
- In spite of the high precision is not totally clear which mass corresponds to the parameter measured by Tevatron and LHC (matching using LO+LL MC, reconstruction of the final state, hadronization modeling, bound-state effects near threshold ...). It is generally believed to be “something near” the “pole mass”.
  - Partial solution: extraction from observables known with good th accuracy
- However, top-quark pole mass is “physically” not well defined (although in pQCD it has a precise meaning) due to non-PT effects: $\mathcal{O}(\Lambda_{QCD})$ ambiguity.
  - Possible solution: change mass definition: short-distance mass def (for instance $\overline{\text{MS}}$)

Top Quark Mass

- Top-quark Mass from Theoretical $\sigma_{t\bar{t}}$ calculations, using NLO, NLO+NNLL, and approximate NNLO calculations (Tevatron)



Th. prediction	$m_t^{\rm pole}$ (GeV)	$\Delta m_t^{\rm pole}$ (GeV)
MC assumpt.	$m_t^{\rm MC} = m_t^{\rm pole}$	$m_t^{\rm MC} = m_t^{\overline{\rm MS}}$
NLO	$164.8^{+5.7}_{-5.4}$	-3.0
NLO+NLL	$166.5^{+5.5}_{-4.8}$	-2.7
NLO+NNLL	$163.0^{+5.1}_{-4.6}$	-3.3
Appr. NNLO$_1$	$167.5^{+5.2}_{-4.7}$	-2.7
Appr. NNLO$_2$	$166.7^{+5.2}_{-4.5}$	-2.8

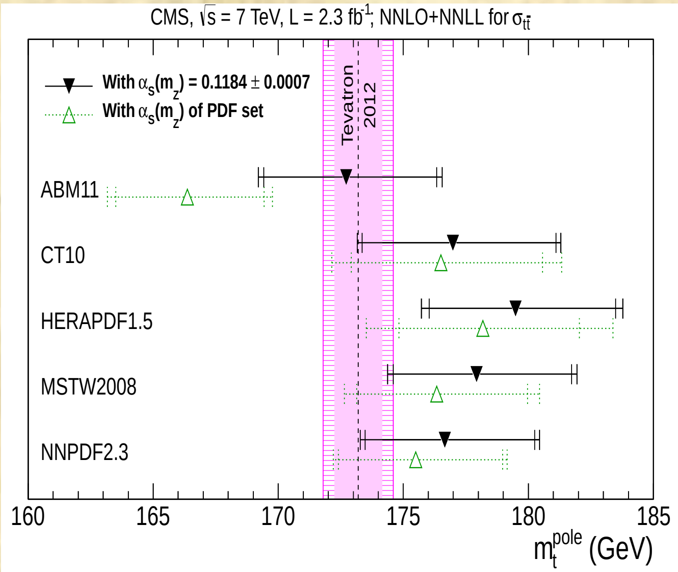


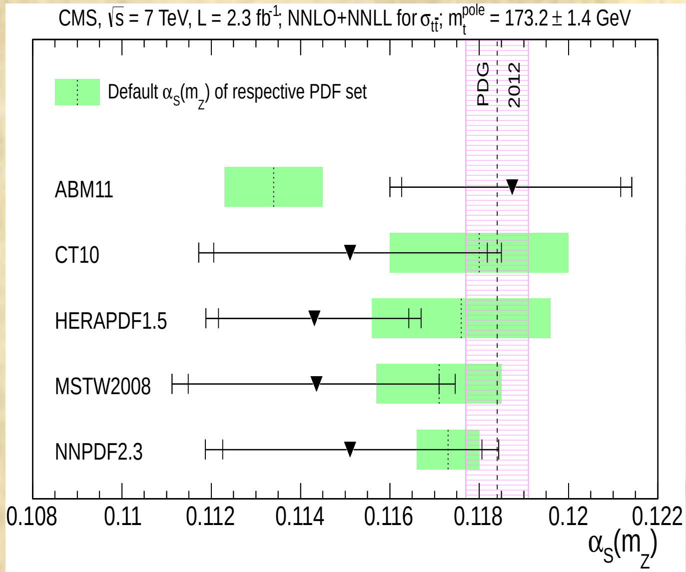
Th. prediction	$m_t^{\rm MS}$ (GeV)	$\Delta m_t^{\rm MS}$ (GeV)
MC assumpt.	$m_t^{\rm MC} = m_t^{\rm pole}$	$m_t^{\rm MC} = m_t^{\overline{\rm MS}}$
NLO+NNLL	$154.5^{+5.0}_{-4.3}$	-2.9
Appr. NNLO$_1$	$160.0^{+4.8}_{-4.3}$	-2.6

D0 Collaboration, Phys. Lett. B 703 (2011) 422

Top Quark Mass

- Same analysis done by CMS with Run I sample (7 TeV), constraining alternatively $\alpha_S(M_Z)$ to be the world average getting m_t^{pole} and vice versa. They use NNLO+NNLL accuracy $t\bar{t}$ cross section.





$$m_t^{pole} = 176.7^{+3.8}_{-3.4}\ \text{GeV} \qquad \alpha_S(m_Z) = 0.1151^{+0.0033}_{-0.0032}$$

S. Chatrchyan *et al.* [CMS Collaboration], Phys. Lett. B **728** (2014) 496

Top Quark: other properties

- Top-Anti Top Mass Difference
  - CDF $\boxed{\Delta m_t = -1.95 \pm 1.11(stat) \pm 0.59(syst)\text{ GeV}}$ with 8.7 fb^{-1}.
  - D0 $\boxed{\Delta m_t = 0.8 \pm 1.8\,(stat) \pm 0.5\,(syst)\text{ GeV}}$ with 3.6 fb^{-1}.
  - CMS $\boxed{\Delta m_t = -0.44 \pm 0.46(stat) \pm 0.27(syst)\text{ GeV}}$ with 4.7 fb^{-1} @ 7 TeV.
  - ATLAS $\boxed{\Delta m_t = 0.67 \pm 0.61(stat) \pm 0.41(syst)\text{ GeV}}$ with 4.7 fb^{-1} @ 7 TeV.
- Top-quark Width
  - CDF $\boxed{\Gamma_t = 2.21^{+1.84}_{-1.11}\text{ GeV}}$ with 8.7 fb^{-1}.
  - D0 $\boxed{\Gamma_t = 2.00^{+0.47}_{-0.43}\text{ GeV}}$ with 5.4 fb^{-1}.
- Top-quark Charge
  - Both at Tevatron and LHC 7, lepton+jets events compatible with the charge of the top of $+2/3$. Exotic top with charge $-4/3$ is excluded at 99% CL.
- W helicity fractions
  - Tevatron $\boxed{F_0 = 0.722 \pm 0.081 \quad F_+ = -0.033 \pm 0.046}$
  - LHC 7 (CMS)

    $$\boxed{F_0 = 0.682 \pm 0.030 \pm 0.033 \quad F_+ = 0.008 \pm 0.012 \pm 0.014 \quad F_- = 0.310 \pm 0.022 \pm 0.022}$$

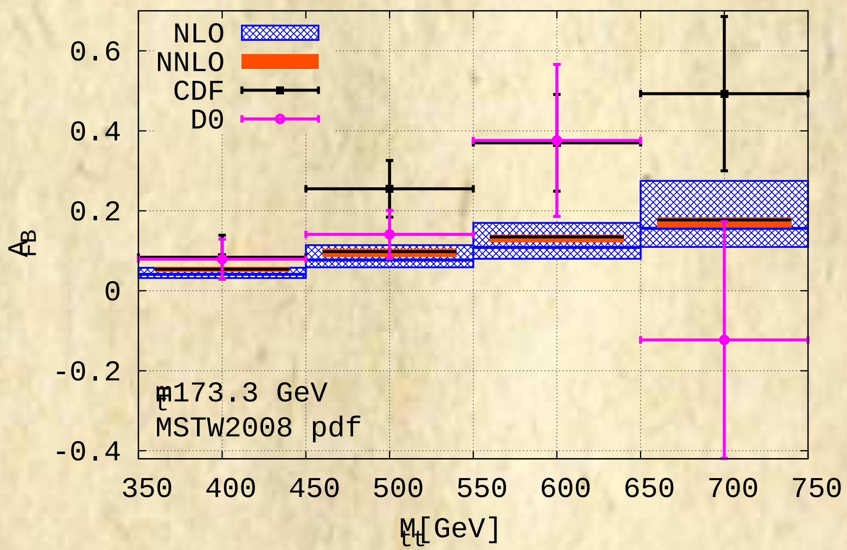
    Ok with SM prediction at NNLO (A. Czarnecki, J. G. Körner, J. H. Piclum, Phys. Rev. D 81 (2010) 111503)

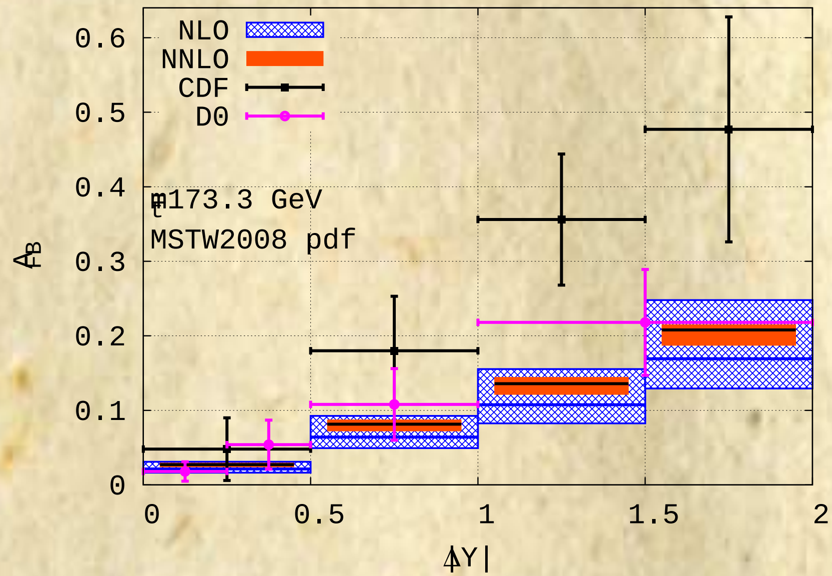
Top Quark: A_{FB} @ Tevatron

- FB Asymmetry

$$A_{FB}^{lab\,(\text{or }t\bar{t})} = \frac{N(y_t(\text{or }\Delta y) > 0) - N(y_t(\text{or }\Delta y) < 0)}{N(y_t(\text{or }\Delta y) > 0) + N(y_t(\text{or }\Delta y) < 0)}$$

After years of discrepancy, the situation changed quite considerably in 2014.





- D0 data are consistent with the SM prediction. CDF still has $< 2\sigma$ discrepancy

M. Czakon, P. Fiedler, A. Mitov, Phys. Rev. Lett. 115 (2015) 5, 052001

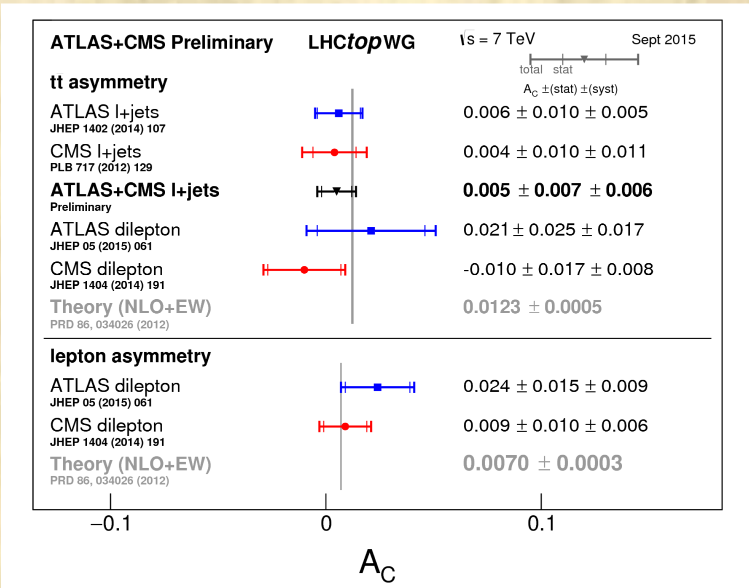
Top Quark: Charge asymmetries @ LHC

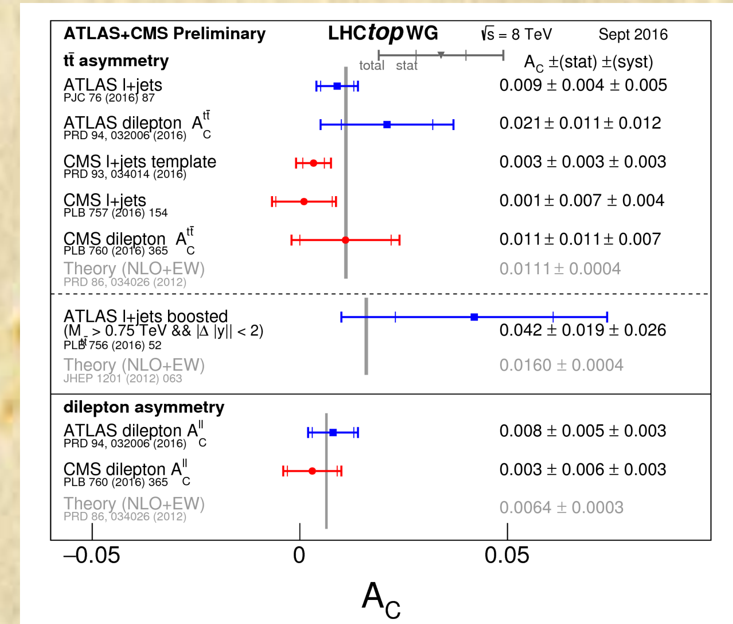
- Charge Asymmetry

$$\mathbf{A_C} = \frac{N(\Delta|y| > 0) - N(\Delta|y| < 0)}{N(\Delta|y| > 0) + N(\Delta|y| < 0)} \qquad \Delta|y| = |y_t| - |y_{\bar{t}}|$$

and in dilepton events:

$$\mathbf{A_C^{lep}} = \frac{N(\Delta|\eta_l| > 0) - N(\Delta|\eta_l| < 0)}{N(\Delta|\eta_l| > 0) + N(\Delta|\eta_l| < 0)} \qquad \Delta|\eta_l| = |\eta_{l+}| - |\eta_{l-}|$$



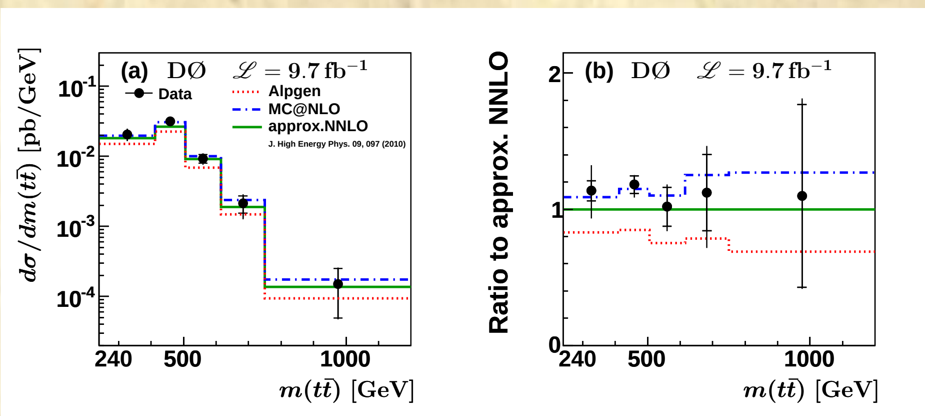


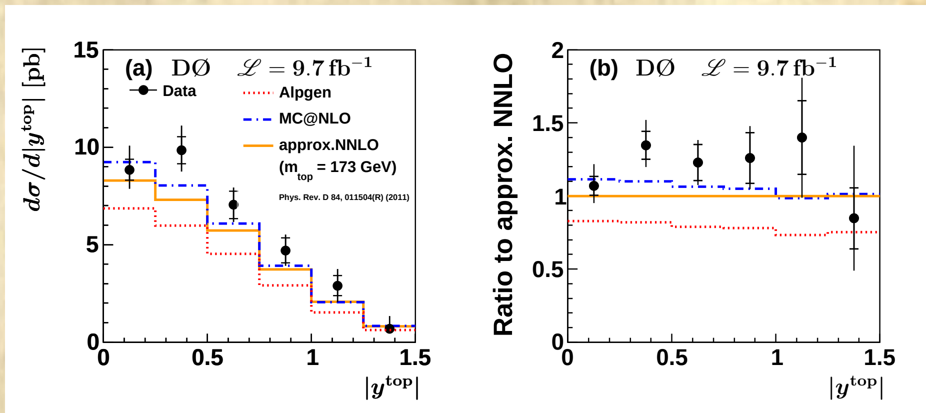
W. Bernreuther, Z. G. Si, Phys. Rev. D **86** (2012) 034026
J. H. Kuhn, G. Rodrigo, JHEP 1201 (2012) 063

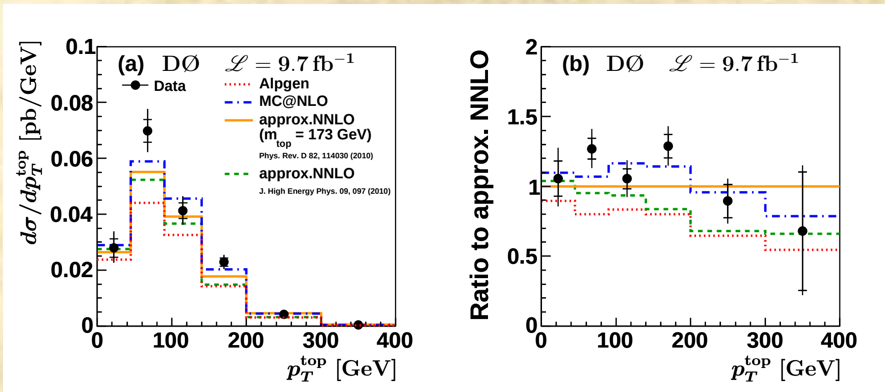
Top Quark: Distributions

Also differential distributions were studied at Tevatron and LHC

- Invariant mass, top rapidity and p_T distributions in $t\bar{t}$ events, with 9.7 fb^{-1}





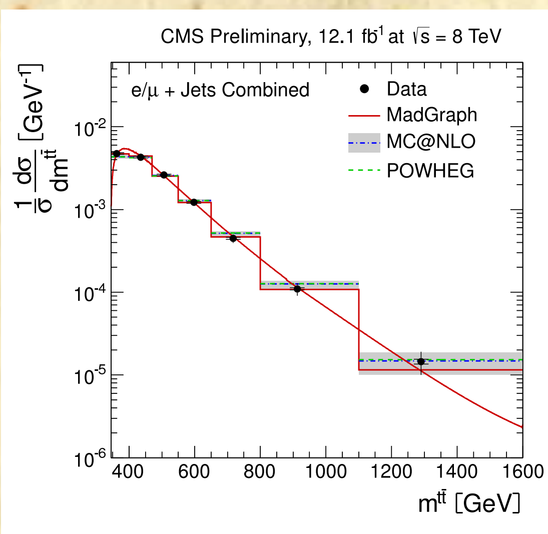


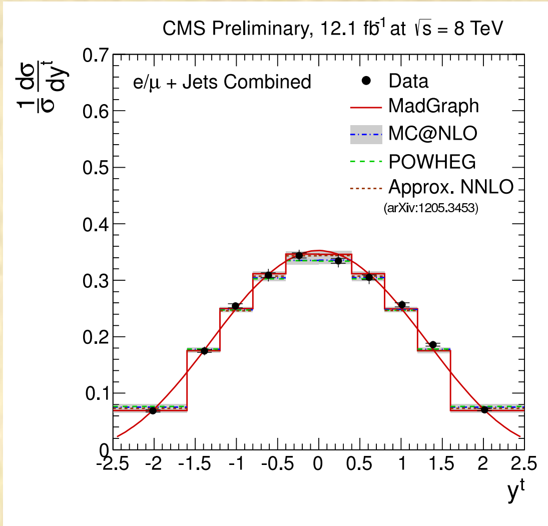
In very good agreement with the SM predictions

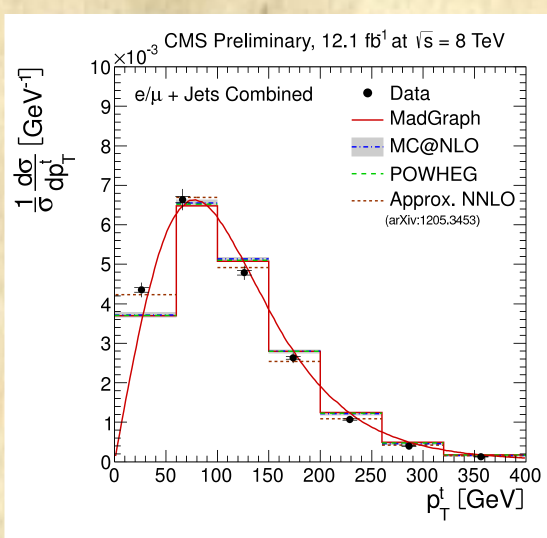
Top Quark: Distributions

Also differential distributions were studied at Tevatron and LHC

- Invariant mass, top rapidity and p_T distributions in $t\bar{t}$ events @ 8 TeV







Again, in very good agreement with the SM predictions

Top Quark: searches for NP

Both @ Tevatron and @ LHC rich program of BSM searches in top quark evants:

- New production mechanisms via new spin-1 or spin-2 resonances: $q\bar{q} \to Z' \to t\bar{t}$ in lepton+jets and all hadronic events. Bumps in the invariant-mass distribution (excluded vector resonances, Z', with masses below $\sim$ 900 GeV and W' with masses below $\sim$ 800 GeV @ 95% CL)
- Top charge measurements (excluded exotic top-quark with $Q_t = -4/3$ @ 99% CL)
- Anomalous couplings
  $$L = -\frac{g}{\sqrt{2}}\bar{b}\left\{\gamma^\mu(V_L P_L + V_R P_R) + \frac{i\sigma^{\mu\nu}(p_t - p_b)_\nu}{M_W}(g_L P_L + g_R P_R)\right\} t W_\mu^-$$
  - From helicity fractions
  - From asymmetries in the final state
- Forward-backward asymmetry (now consistent with the SM prediction)
- Non SM Top decays. Search for charged Higgs: $t \to H^+ b \to q\bar{q}' b(\tau\nu b)$
- Search for heavy $t' \to W^+ b$ in lepton+jets (recently excluded t' with $m_{t'} < 360$ GeV and b' with $m_{b'} < 385$ GeV @ 95% CL)

Theoretical Framework: pQCD

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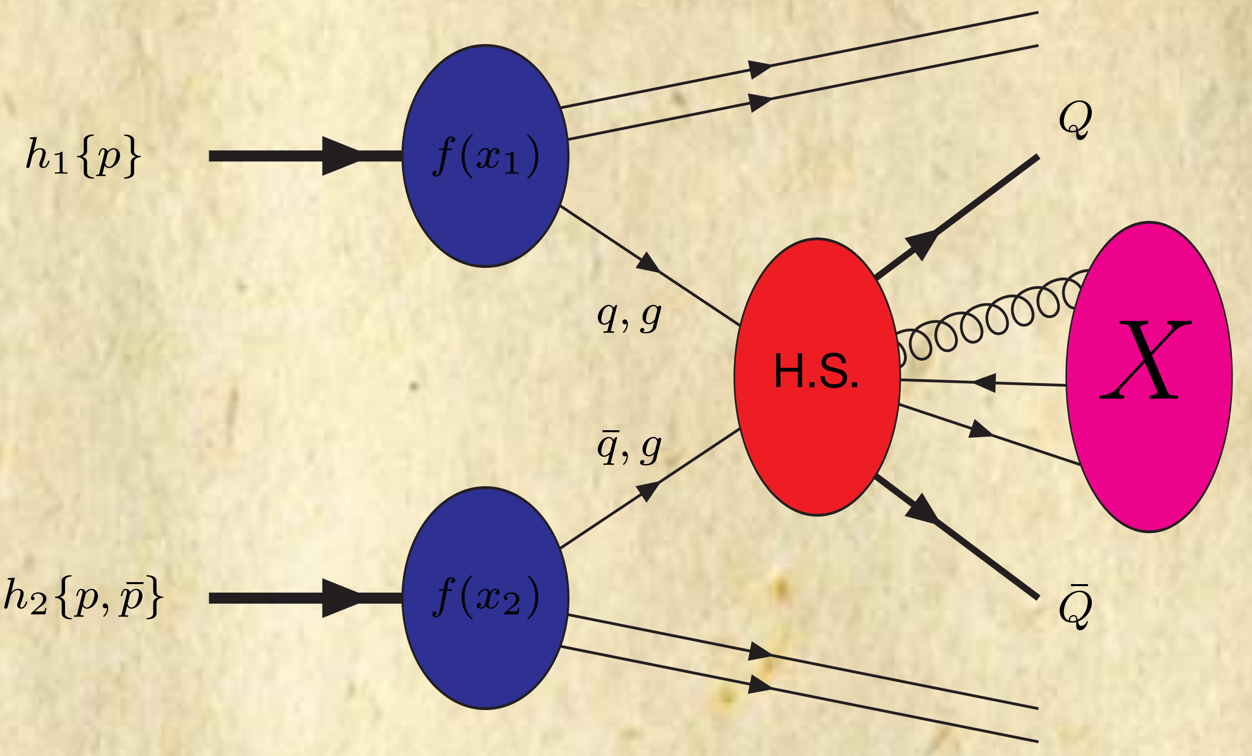
Let us consider the heavy-quark production in hadron collisions $h_1 + h_2 \to Q\bar{Q} + X$

According to the $\boxed{\text{FACTORIZATION THEOREM}}$ the process can be sketched as follows:



$$\sigma_{h_1,h_2} = \sum_{i,j} \int_0^1 dx_1 \int_0^1 dx_2 \, f_{h_1,i}(x_1, \mu_F) f_{h_2,j}(x_2, \mu_F) \; \hat{\sigma}_{ij}\left(\hat{s}, m_t, \alpha_s(\mu_R), \mu_F, \mu_R\right)$$

$$s = \left(p_{h_1} + p_{h_2}\right)^2 \; , \; \hat{s} = x_1 x_2 s$$

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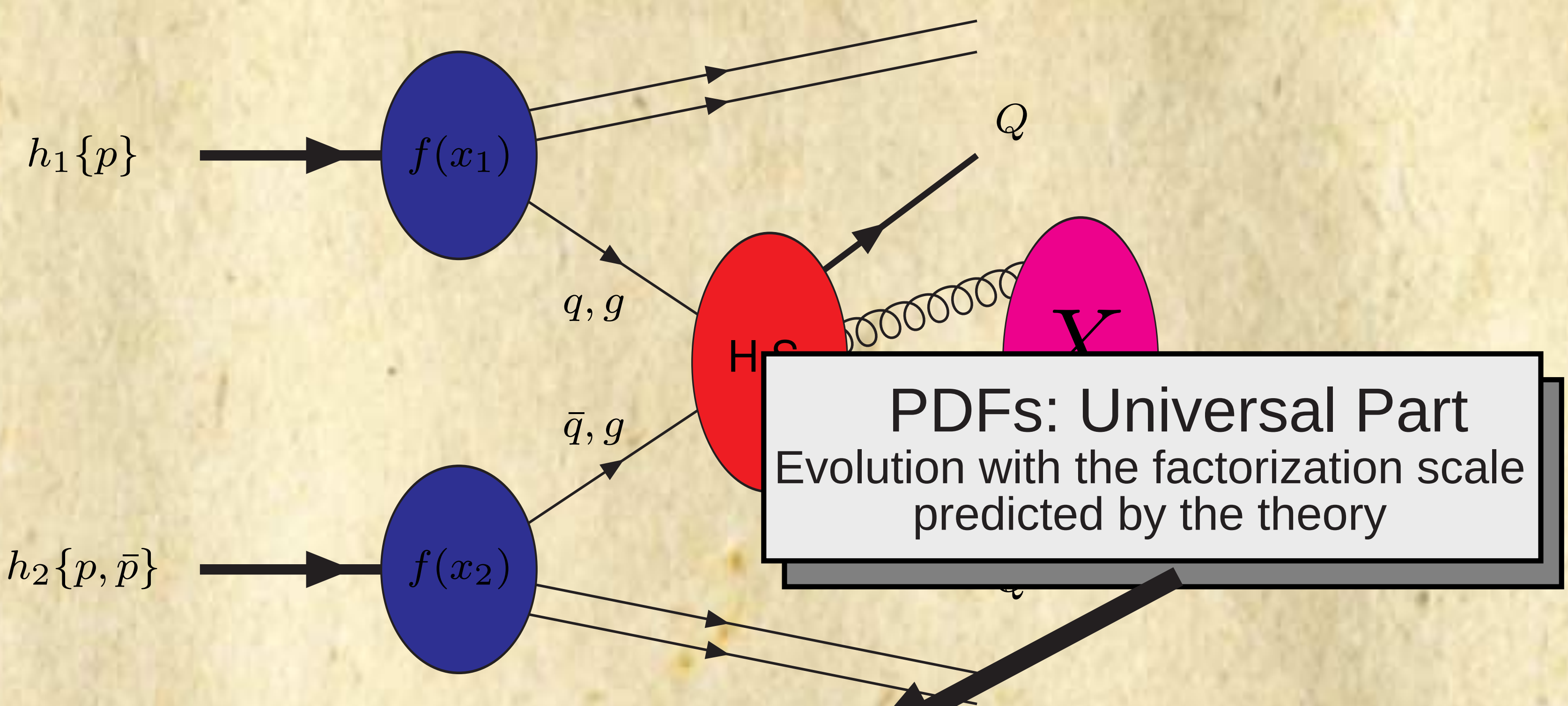

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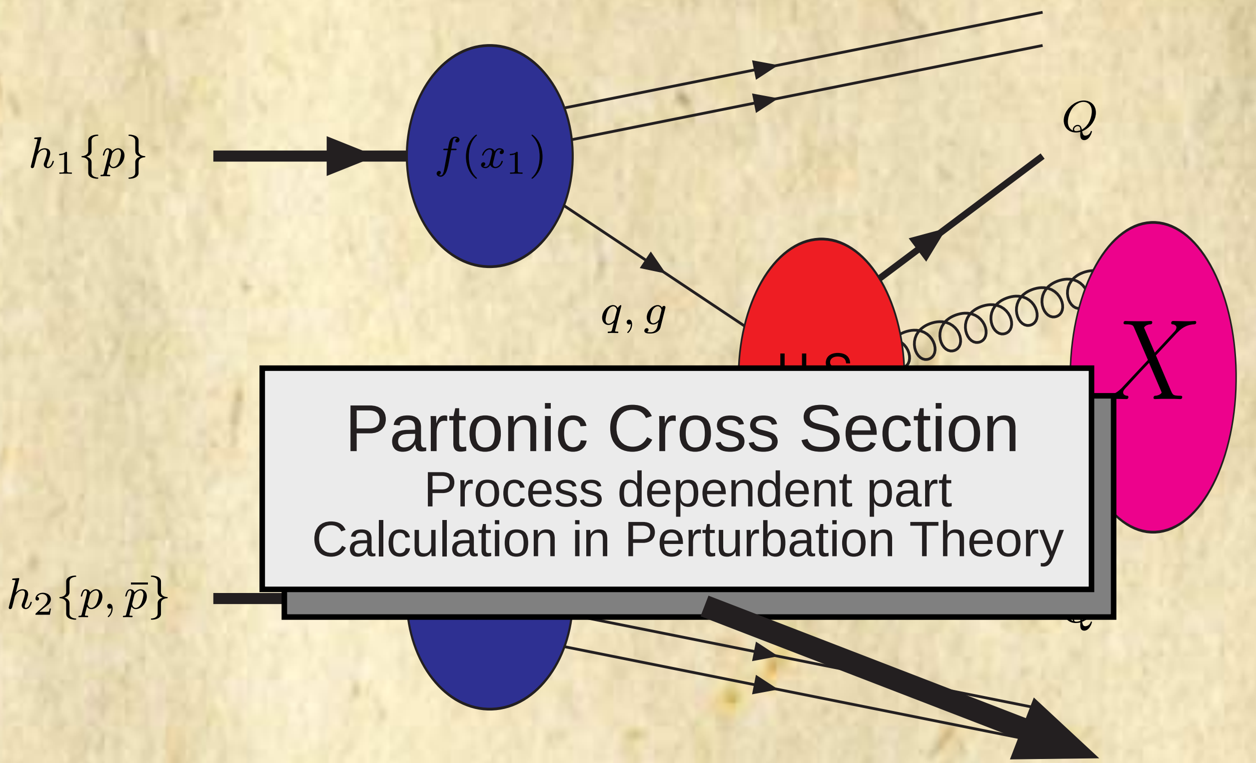

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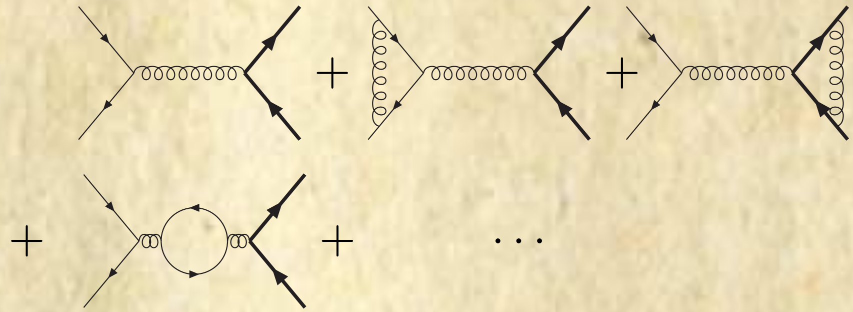
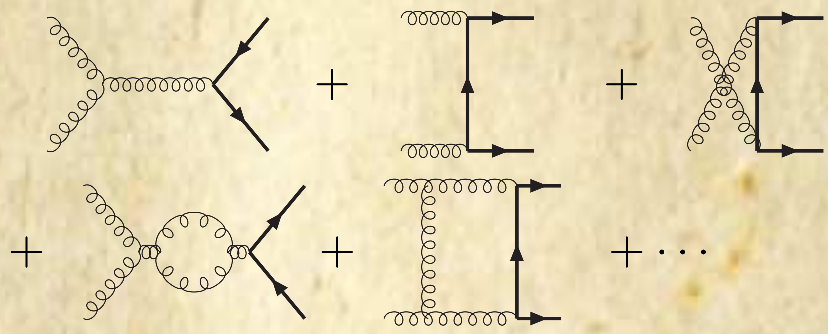
Partonic Cross Section: PT Expansion

$$\hat{\sigma}_{ij}^{Q\bar{Q}} \propto \left|\mathcal{M}_{ij}^{Q\bar{Q}}\right|^2 = \left|\mathcal{M}_{ij,0}^{Q\bar{Q}} + \alpha_S\,\mathcal{M}_{ij,1}^{Q\bar{Q}} + \alpha_S^2\,\mathcal{M}_{ij,2}^{Q\bar{Q}} + \cdots\right|^2$$

$\mathcal{M}_{q\bar{q}}^{Q\bar{Q}} =$



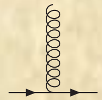
$\mathcal{M}_{g\bar{g}}^{Q\bar{Q}} =$

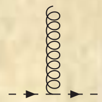


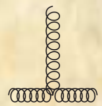
 $\rightarrow \dfrac{\delta_{ij}(-i\,\not{k}+m)}{k^2+m^2-i\epsilon}$

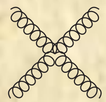
 $\rightarrow \dfrac{\delta_{ab}}{k^2-i\epsilon}$

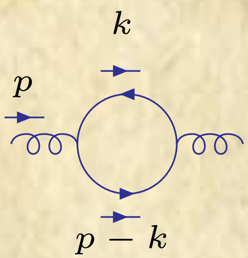
 $\rightarrow \dfrac{\delta_{\mu\nu}\,\delta_{ab}}{k^2-i\epsilon}$

 $\rightarrow i g_S\, t_{ij}^a\,\gamma^\mu$

 $\rightarrow -i g_S\, f^{cab}\, p^\mu$

 $\rightarrow i g_S\, f^{abc}[\delta_{\mu\nu}(p_\sigma - q_\sigma) + \delta_{\nu\sigma}(q_\mu - k_\mu) + \delta_{\mu\sigma}(k_\nu - p_\nu)]$

 $\rightarrow -g_S^2[f^{gac}f^{gbd}(2\delta_{\mu\nu}\delta_{\sigma\tau} - \delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma}) + \cdots$



$$\propto \frac{\alpha_S}{\pi}\int d^4k\,\frac{tr\{t^a t^b\}\,tr\{\gamma^\mu(-i\,\not{k}+m)\gamma^\nu[i(\not{p}-\not{k})+m]\}}{(k^2+m^2)[(p-k)^2+m^2]}$$

Cross Section: LO (stable top)

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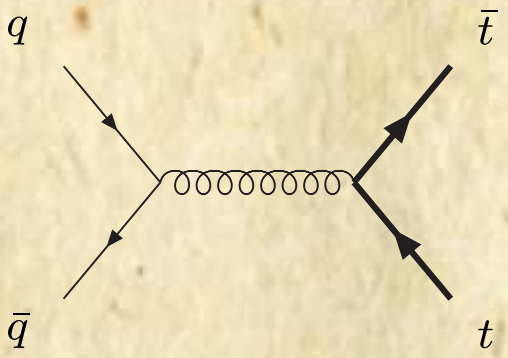
Top-Antitop production at leading order, partonic diagrams:

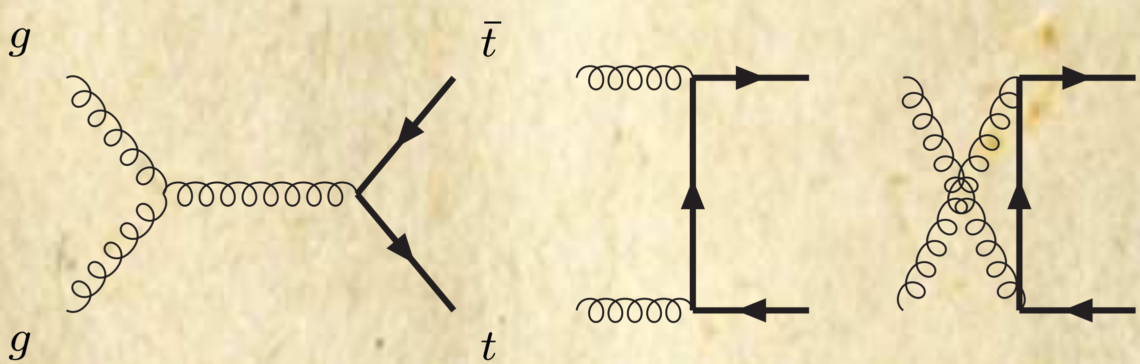
$$q(p_1) + \bar{q}(p_2) \longrightarrow t(p_3) + \bar{t}(p_4)$$



Dominant at Tevatron

$\sim 85\%$

$$g(p_1) + g(p_2) \longrightarrow t(p_3) + \bar{t}(p_4)$$



Dominant at LHC

$\sim 90\%$

Cross Section: NLO (stable top)

Cross Section: NLO (stable top)

Fixed Order

- The NLO QCD corrections are quite sizable: + 25% at Tevatron and +50% at LHC. Scales variation to $\pm 15\%$.

  Nason, Dawson, Ellis '88-'90; Beenakker, Kuijf, van Neerven, Smith '89-'91; Mangano, Nason, Ridolfi '92; Frixione et al. '95; Czakon and Mitov '08.

- Mixed NLO QCD-EW corrections are small: - 1% at Tevatron and -0.5% at LHC.

  Beenakker *et al.* '94 Bernreuther, Fuecker, and Si '05-'08
  Kühn, Scharf, and Uwer '05-'06; Moretti, Nolten, and Ross '06.

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- The QCD corrections to processes involving at least two large energy scales ($\hat{s}, m_t^2 \gg \Lambda_{QCD}^2$) are characterized by a logarithmic behavior in the vicinity of the boundary of the phase space

$$\sigma \sim \sum_{n,m} C_{n,m} \alpha_S^n \ln^m (1-\rho) \qquad m \leq 2n$$

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$$\sigma \sim \sum_{n,m} C_{n,m} \alpha_S^n \ln^m (1-\rho) \qquad m \leq 2n$$

- Even if $\alpha_S \ll 1$ (perturbative region) we can have at all orders $\boxed{\alpha_S^n \ln^m (1-\rho) \sim \mathcal{O}(1)}$

  Resummation $\Longrightarrow$ improved perturbation theory

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All-order Soft-Gluon Resummation

- Leading-Logs (LL)

  Laenen et al. '92-'95; Berger and Contopanagos '95-'96; Catani et al. '96.

- Next-to-Leading-Logs (NLL)

  Kidonakis and Sterman '97; R. B., Catani, Mangano, and Nason '98.

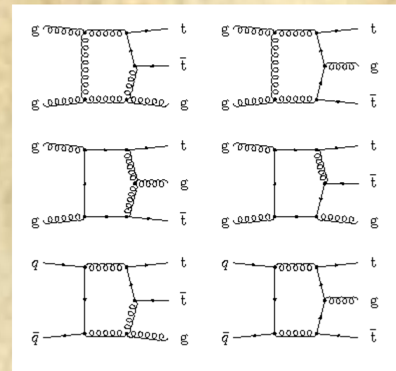
- Next-to-Next-to-Leading-Logs (NNLL)

  Moch and Uwer '08; Beneke et al. '09-'10; Czakon et al. '09; Kidonakis '09; Ahrens et al. '10; Cacciari-Czakon-Mangano-Mitov-Nason '12

Distributions (stable top)

Distributions (stable top)

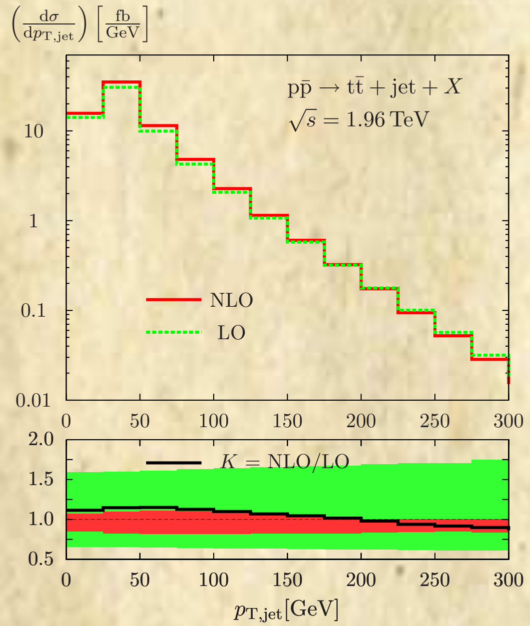
- $\boxed{pp(\bar{p}) \to t\bar{t} + 1\, jet}$
  - Important for a deeper understanding of the $t\bar{t}$ prod (possible structure of the top-quark)
  - Technically complex involving multi-leg NLO diagrams

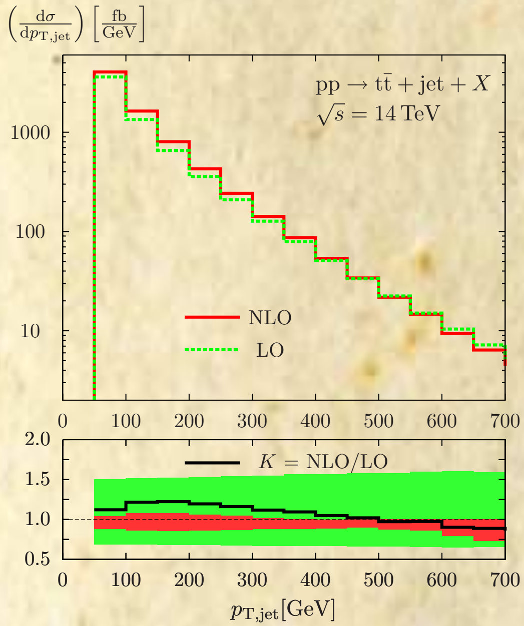


$$\boxed{\sigma_{t\bar{t}+j}(LHC) = 376.2^{+17}_{-48}\ \text{pb}}$$

$$\boxed{\sigma_{t\bar{t}+j}(Tev) = 1.79^{+0.16}_{-0.31}\ \text{pb}}$$

$$\boxed{\sigma^{CDF}_{t\bar{t}+j} = 1.6 \pm 0.2\,(\text{stat}) \pm 0.5\,(\text{syst})\ \text{pb}}$$



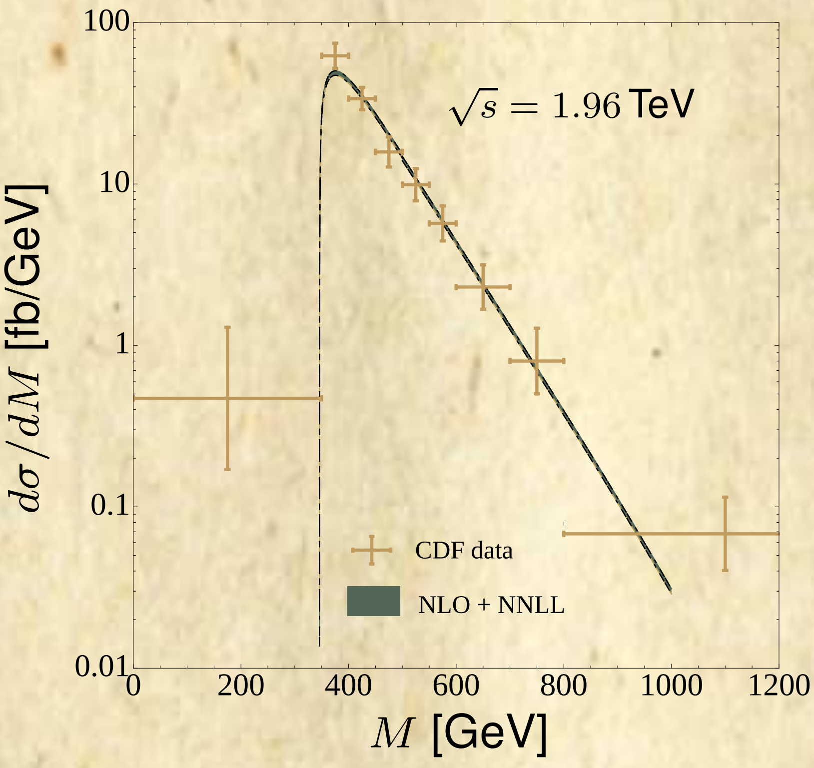


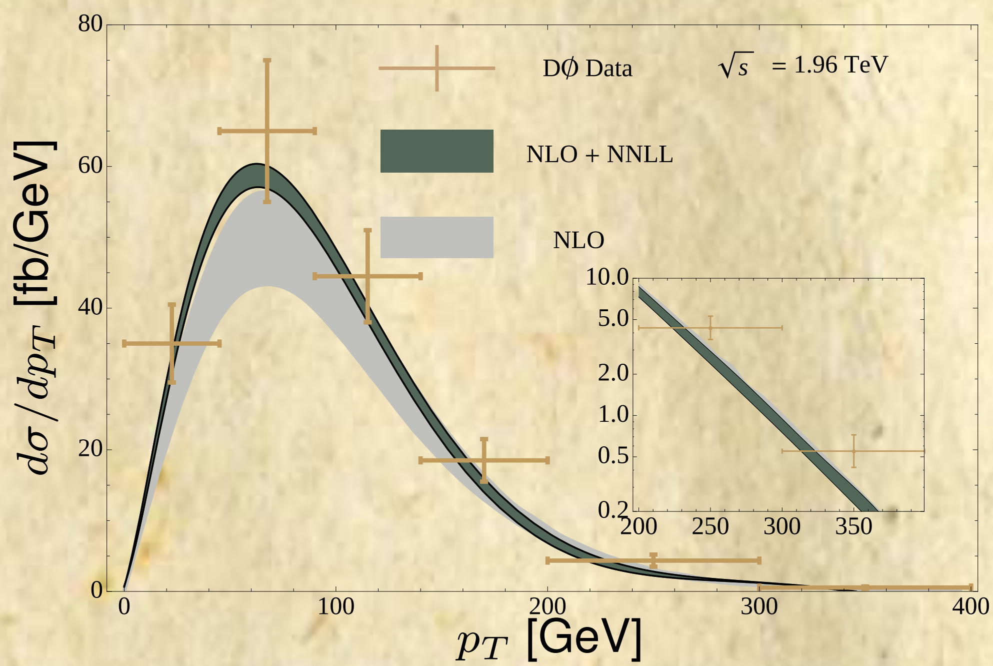
S. Dittmaier, P. Uwer and S. Weinzierl,
Phys. Rev. Lett. 98 (2007) 262002
Eur. Phys. J. C **59** (2009) 625

confirmed by G. Bevilacqua, M. Czakon, C.G. Papadopoulos, M. Worek, Phys.Rev.Lett. 104 (2010) 162002
K. Melnikov and M. Schulze, Nucl.Phys. B840 (2010) 129-159

Distributions (stable top)

- Invariant mass and p_T distributions in $t\bar{t}$ events: NLO + resummed (SCET) NNLL comparison with CDF and D0 data





V. Ahrens, A. Ferroglia, M. Neubert, B. D. Pecjak, and L. L. Yang, JHEP 1009 (2010) 097
V. Ahrens, A. Ferroglia, M. Neubert, B. D. Pecjak, and L. L. Yang, JHEP 1109 (2011) 070

Tools @ NLO

The corrections at NLO for the $t\bar{t}$ (and single-top productions) are implemented in a series of public codes

- MCFM

  J. M. Campbell, R. K. Ellis, Phys. Rev. **D60** (1999) 113006

- MC@NLO

  S. Frixione, B. R. Webber, JHEP **0206** (2002) 029

- POWHEG

  S. Frixione, P. Nason, C. Oleari, JHEP **0711** (2007) 070

NLO with decay Products: Fact. Corrections

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- The calculations shown so far consider a stable top (anti-top) quark. Advantage: reduction in the complexity of a NLO calculation
- In "reality" the out states are leptons and hadrons $\Longrightarrow$ experiments put cuts on leptons and hadrons. Desirable a description of the process in terms of actual out states

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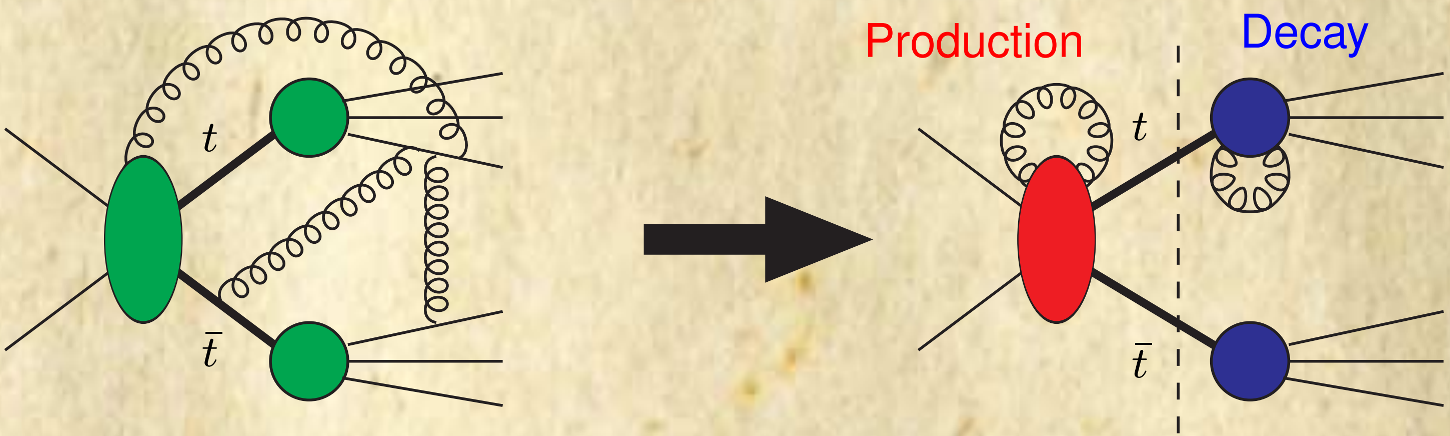
- In "reality" the out states are leptons and hadrons $\Longrightarrow$ experiments put cuts on leptons and hadrons. Desirable a description of the process in terms of actual out states
- $\boxed{\text{Factorizable corrections}}$ do not mix production and decay stages!

$$\lim_{\Gamma_t/m_t \to 0} \frac{1}{(p_t^2 - m_t^2)^2 + m_t^2 \Gamma_t^2} = \frac{\pi}{m_t \Gamma_t} \delta(p_t^2 - m_t^2)$$



$$+ \mathcal{O}(\Gamma_t/m_t)$$

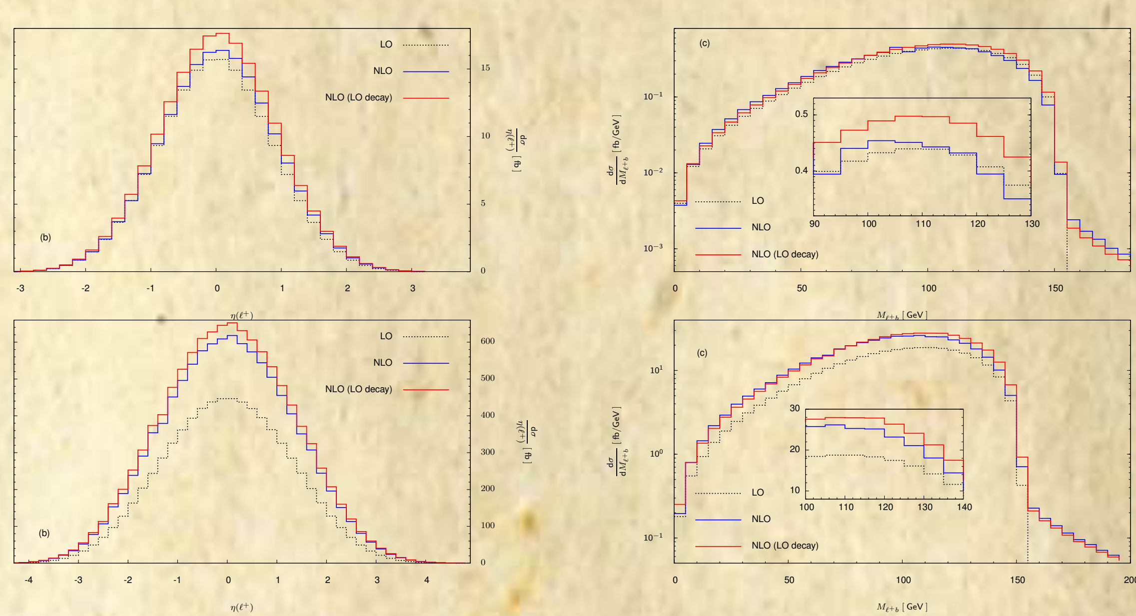
- The non-factorizable corrections do not decouple, but in sufficiently inclusive observables they become small: $\sim \mathcal{O}(\Gamma_t/m_t)$ Fadin, Khoze, Martin '94; Aeppli, van Oldenborgh, Wyler '94; Melnikov, Yakovlev '94; Beenakker, Berends, Chapovsky '99
- One can keep track of the spin of the top and anti-top and compute spin correlations

NLO with decay Products: Fact. Corrections

- NLO corrections to various kinematic distributions for Tevatron and LHC (Bernreuther et al. include also EW corrections)

Tevatron

LHC



W. Bernreuther et al., Nucl.Phys. B690 (2004) 81

W. Bernreuther and Z. Si, Nucl.Phys. B837 (2010) 90-121

K.Melnikov and M. Schulze, JHEP **0908** (2009) 049

- NB: the study could be extended at NNLO, using differential top quark decay at NNLO

M. Brucherseifer, F. Caola and K. Melnikov, JHEP **1304** (2013) 059

J. Gao, C. S. Li and H. X. Zhu, Phys. Rev. Lett. **110** (2013) 042001

NLO with decay Products

- In 2011 two groups computed the full set of NLO corrections to $\boxed{pp \to WWbb}$
  - Calculation technically challenging (~ 1500 Feynman diagrams, up to 6 external legs)
  - The direct calculation confirms that for inclusive quantities the non-factorizable corrections are of $\mathcal{O}(\Gamma_t/m_t)$
  - Possibility to study many distributions imposing realistic experimental cuts

A. Denner, S. Dittmaier, S. Kallweit, and S. Pozzorini, Phys. Rev. Lett. 106 (2011) 052001
G. Bevilacqua, M. Czakon, A. van Hameren, C. G. Papadopoulos, M. Worek, JHEP 1102 (2011) 083

- Finally, very recently these corrections were matched with PS in the POWHEG-BOX frame

T. Jezo, J. M. Lindert, P. Nason, C. Oleari and S. Pozzorini, Eur. Phys. J. C **76** (2016) 12 691

$t\bar{t}$ Cross Section @ NNLO in QCD

In 2013 the total cross section was calculated in perturbative QCD at the NNLO!

- Outstanding calculation, at the edge of current techniques! Virtual part: numerical solution of the differential equations for the MIs; Real Part variation of sector decomposition. Numerical cancelation of remaining IR divergences

  P. Bärnreuther, M. Czakon and A. Mitov, Phys. Rev. Lett. **109** (2012) 132001
  M. Czakon and A. Mitov, JHEP **1212** (2012) 054, JHEP **1301** (2013) 080
  M. Czakon, P. Fiedler and A. Mitov, Phys. Rev. Lett. **110** (2013) 252004

- Numerical implementation very demanding, but fitted for different values of m_t in the program `Top++`

  M. Czakon and A. Mitov, Comput. Phys. Commun. **185** (2014) 2930

- Resummation of soft gluons included up to NNLL

  M. Cacciari, M. Czakon, M. Mangano, A. Mitov and P. Nason, Phys. Lett. B **710** (2012) 612

- Distributions were produced

  M. Czakon, D. Heymes and A. Mitov, Phys. Rev. Lett. **116** (2016) 8, 082003 ; JHEP **1605** (2016) 034

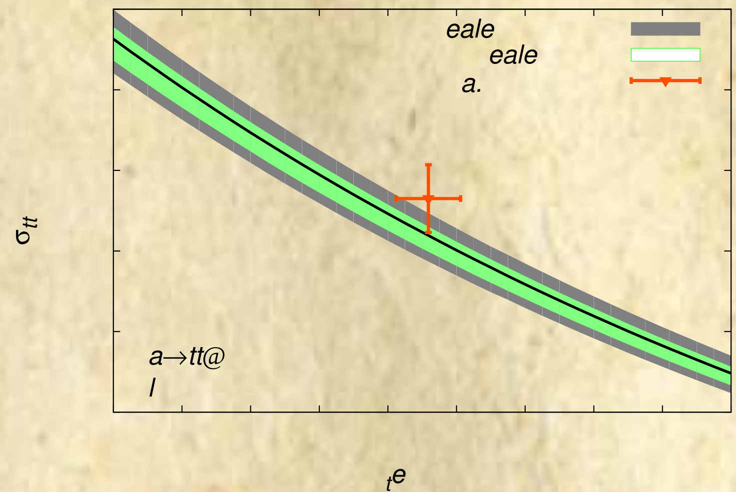
- Very recently NNLO QCD corrections were implemented by NLO EW corrections

  M. Czakon, D. Heymes, A. Mitov, D. Pagani, I. Tsinikos and M. Zaro, arXiv:1705.04105 [hep-ph].

$t\bar{t}$ Cross Section @ NNLO in QCD

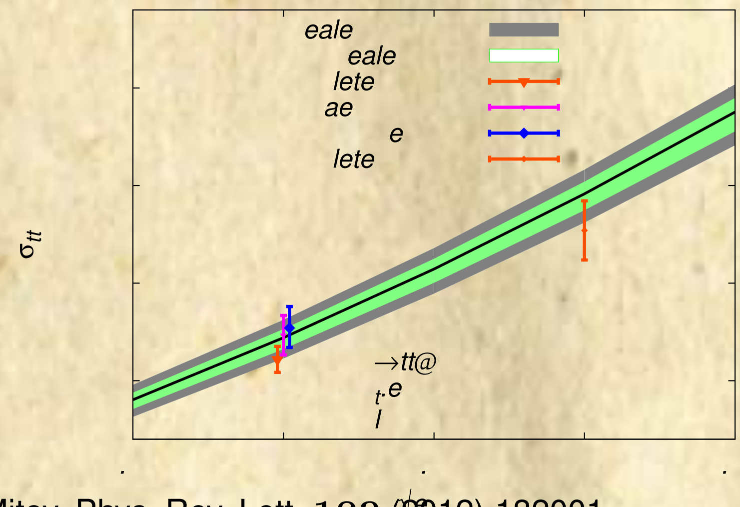
Pure NNLO

Collider	$\sigma_{\rm tot}$ [pb]	scales [pb]	pdf [pb]
Tevatron	7.009	$+0.259(3.7\%)$ $-0.374(5.3\%)$	$+0.169(2.4\%)$ $-0.121(1.7\%)$
LHC 7 TeV	167.0	$+6.7(4.0\%)$ $-10.7(6.4\%)$	$+4.6(2.8\%)$ $-4.7(2.8\%)$
LHC 8 TeV	239.1	$+9.2(3.9\%)$ $-14.8(6.2\%)$	$+6.1(2.5\%)$ $-6.2(2.6\%)$
LHC 14 TeV	933.0	$+31.8(3.4\%)$ $-51.0(5.5\%)$	$+16.1(1.7\%)$ $-17.6(1.9\%)$



NNLO+NNLL

Collider	$\sigma_{\rm tot}$ [pb]	scales [pb]	pdf [pb]
Tevatron	7.164	$+0.110(1.5\%)$ $-0.200(2.8\%)$	$+0.169(2.4\%)$ $-0.122(1.7\%)$
LHC 7 TeV	172.0	$+4.4(2.6\%)$ $-5.8(3.4\%)$	$+4.7(2.7\%)$ $-4.8(2.8\%)$
LHC 8 TeV	245.8	$+6.2(2.5\%)$ $-8.4(3.4\%)$	$+6.2(2.5\%)$ $-6.4(2.6\%)$
LHC 14 TeV	953.6	$+22.7(2.4\%)$ $-33.9(3.6\%)$	$+16.2(1.7\%)$ $-17.8(1.9\%)$



P. Bärnreuther, M. Czakon and A. Mitov, Phys. Rev. Lett. **109** (2012) 132001
M. Czakon and A. Mitov, JHEP **1212** (2012) 054, JHEP **1301** (2013) 080
M. Czakon, P. Fiedler and A. Mitov, Phys. Rev. Lett. **110** (2013) 252004

Analytic Calculation: $t\bar{t}$ @ NNLO

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The NNLO calculation of the top-quark pair hadro-production requires several ingredients:

- Virtual Corrections
  - two-loop matrix elements for $q\bar{q} \to t\bar{t}$ and $gg \to t\bar{t}$

    Czakon '08, R. B., Ferroglia, Gehrmann, Maitre, von Manteuffel, Studerus '08-'13, Ferroglia, Neubert, Pecjak, Yang '09
  - interference of one-loop diagrams

    Körner et al. '05-'08; Anastasiou and Aybat '08
- Real Corrections
  - one-loop matrix elements for the hadronic production of $t\bar{t} + 1$ parton
  - tree-level matrix elements for the hadronic production of $t\bar{t} + 2$ partons

    Dittmaier, Uwer and Weinzierl '07-'08, Bevilacqua, Czakon, Papadopoulos, Worek '10, Melnikov, Schulze '10
- Subtraction Terms
  - In a complete NNLO computation of $\sigma_{t\bar{t}}$ we need subtraction terms with up to 2 unresolved partons.

    Different methods on the market at the NNLO

    Gehrmann-De Ridder, Ritzmann '09, Daleo et al. '09, Boughezal et al. '10, Glover, Pires '10, Del Duca, Somogyi, Trocsanyi '13, Catani Grazzini '07, B. Catani Grazzini Sargsyan Torre '15

    Double and single real in $\sigma_{t\bar{t}}$

    Czakon '10, Anastasiou, Herzog, Lazopoulos '10

Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned}\mathcal{A}_2^{(2\times 0)} \quad &= \quad N_c C_F\Big[N_c^2 A + B + \frac{C}{N_c^2} + N_l\left(N_c D_l + \frac{E_l}{N_c}\right) \\ &\quad + N_h\left(N_c D_h + \frac{E_h}{N_c}\right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h\Big]\end{aligned}$$

$\boxed{\text{218 two-loop diagrams}}$ contribute to the $\boxed{10}$ different color coefficients

- The whole $\mathcal{A}_2^{(2\times 0)}$ is known numerically

  Czakon '08.

- The coefficients D_i, E_i, F_i, and A are known analytically (agreement with num res)

  R. B., Ferroglia, Gehrmann, Maitre, and Studerus '08-'09

- The coefficients B and C can be calculated analytically (with the same techniques)

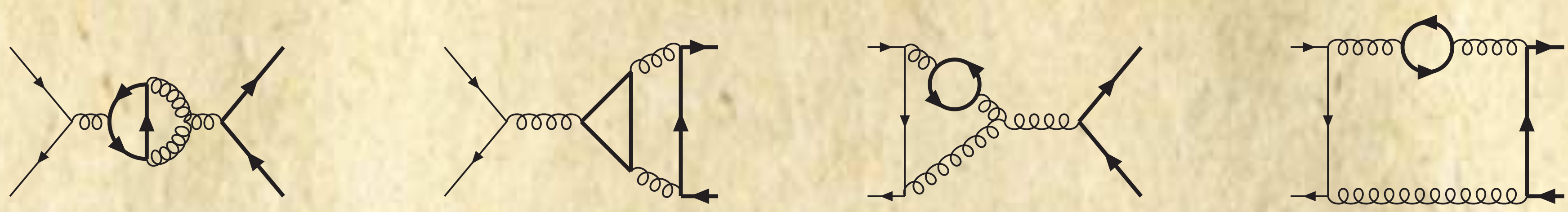
  A. von Manteuffel et al., in progress

- The poles of $\mathcal{A}_2^{(2\times 0)}$ (and therefore of B and C) are known analytically

  Ferroglia, Neubert, Pecjak, and Li Yang '09

Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

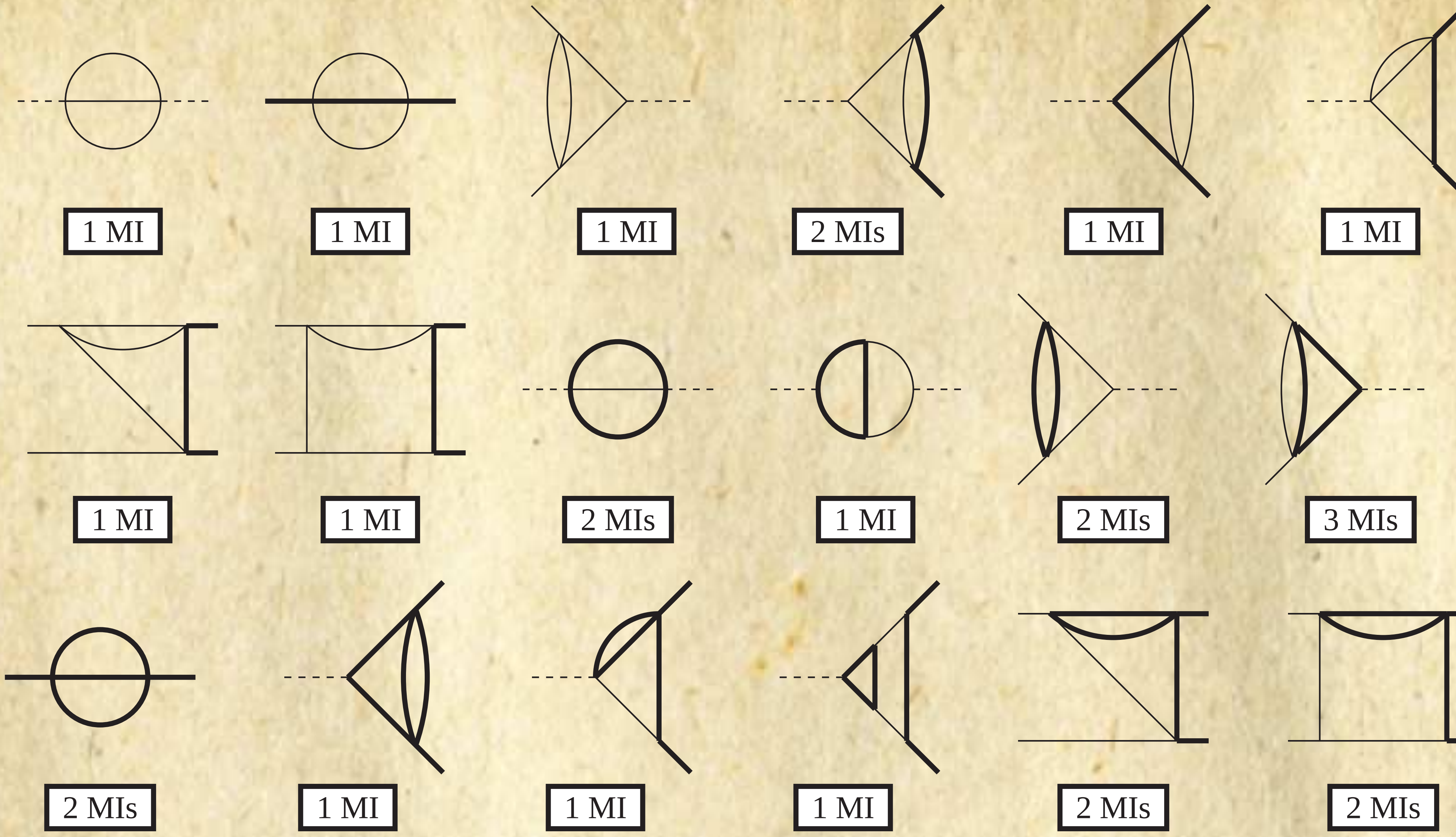
- $\boxed{D_i,\ E_i,\ F_i}$ come from the corrections involving a closed (light or heavy) fermionic loop:



- $\boxed{A}$ the leading-color coefficient, comes from the planar diagrams:

- The calculation is carried out analytically using:
  - Laporta Algorithm for the reduction of the dimensionally-regularized scalar integrals (in terms of which we express the $|\mathcal{M}|^2$) to the Master Integrals (MIs)
  - Differential Equations Method for the analytic solution of the MIs

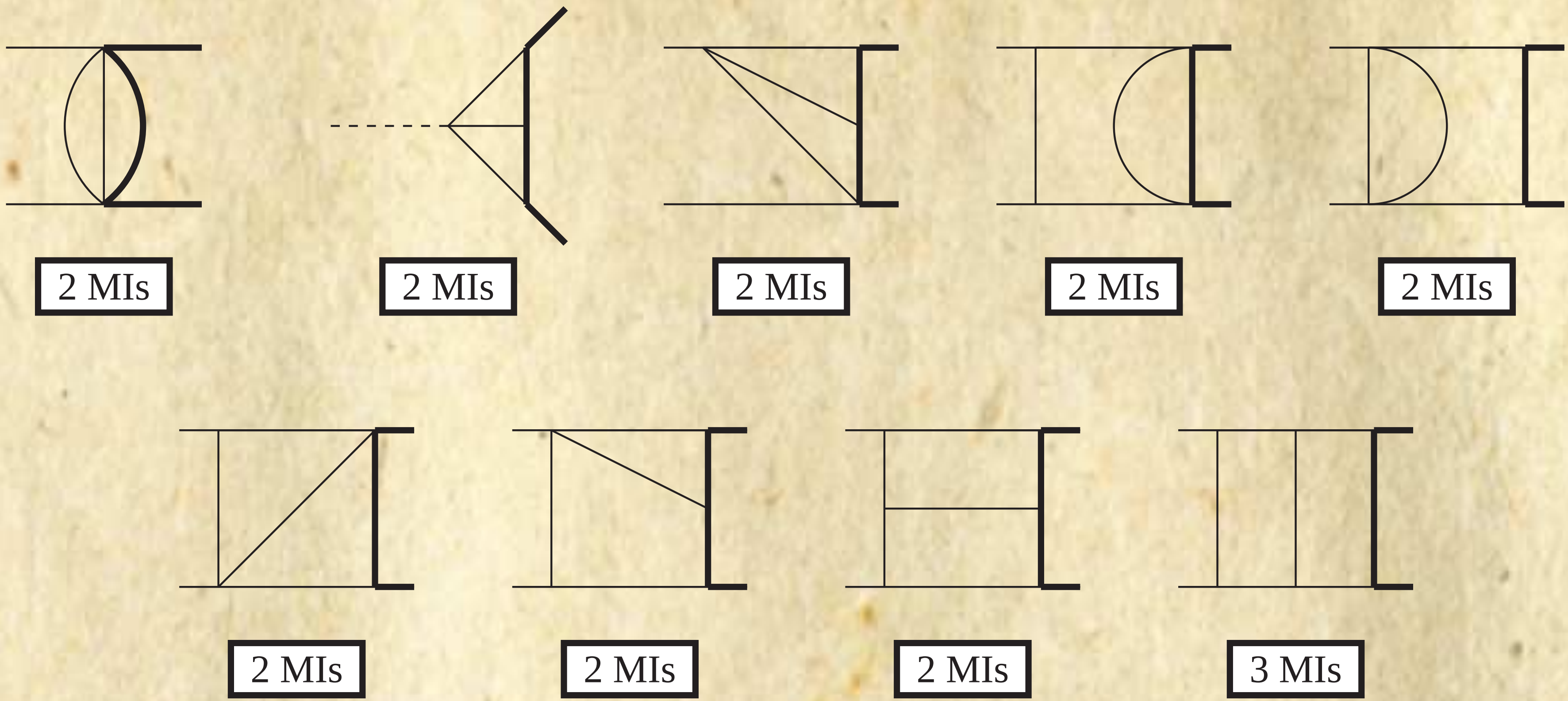
Master Integrals for N_l and N_h



18 irreducible two-loop topologies (26 MIs)

R. B., A. Ferroglia, T. Gehrmann, D. Maitre, and C. Studerus, JHEP **0807** (2008) 129.

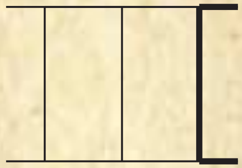
Master Integrals for the Leading Color Coeff



For the leading color coefficient there are 9 additional irreducible topologies (19 MIs)

R. B., A. Ferroglia, T. Gehrmann, and C. Studerus, JHEP **0908** (2009) 067.

Example: Box for the Leading Color Coeff



$$= \frac{1}{m^6} \sum_{i=-4}^{-1} A_i \epsilon^i + \mathcal{O}(\epsilon^0)$$

$$A_{-4} = \frac{x^2}{24(1-x)^4(1+y)} ,$$

$$A_{-3} = \frac{x^2}{96(1-x)^4(1+y)} \Big[-10G(-1;y) + 3G(0;x) - 6G \ldots$$

$$A_{-2} = \frac{x^2}{48(1-x)^4(1+y)} \Big[-5\zeta(2) - 6G(-1;y)G(0;x) + \ldots$$

$$A_{-1} = \frac{x^2}{48(1-x)^4(1+y)} \Big[-13\zeta(3) + 38\zeta(2)G(-1;y) + 9\zeta(2)G(0;x) + 6\zeta(2)G(1;x) - 24\zeta(2)G(-1/y;x)$$

$$+24G(0;x)G(-1,-1;y) - 24G(1;x)G(-1,-1;y) - 12G(-1/y;x)\,G(-1,-1;y)$$

$$-12G(-y;x)G(-1,-1;y) - 6G(0;x)G(0,-1;y) + 6G(-1/y;x)\,G(0,-1;y) + 6G(-y;x)G(0,-1;y)$$

$$+12G(-1;y)G(1,0;x) - 24G(-1;y)G(1,1;x) - 6G(-1;y)G(-1/y,0;x) + 12G(-1;y)G(-1/y,1;x$$

$$-6G(-1;y)G(-y,0;x) + 12G(-1;y)G(-y,1;x) + 16G(-1,-1,-1;y) - \ldots G(-1,0,-1;y)$$

$$-12G(0,-1,-1;y) + 6G(0,0,-1;y) + 6G(1,0,0;x) - 12G(1,0,1;x) - 12G(1,1,0;x) + 24G(1,1,1$$

$$-6G(-1/y,0,0;x) + 12G(-1/y,0,1;x) + 6G(-1/y,1,0;x) - 12G(-1/y,1,1;x) + 6G(-y,1,0;x)$$

$$-12G(-y,1,1;x) \Big]$$

1- and 2-dim GHPLs

$$\rho = \frac{4m^2}{\hat{s}} \to 1$$

GHPLs

- One- and two-dimensional Generalized Harmonic Polylogarithms (GHPLs) are defined as repeated integrations over set of basic functions. In the case at hand

$$f_w(x) = \frac{1}{x-w}\,, \quad \text{with} \quad w \in \left\{0, 1, -1, -y, -\frac{1}{y}, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right\}$$

$$f_w(y) = \frac{1}{y-w}\,, \quad \text{with} \quad w \in \left\{0, 1, -1, -x, -\frac{1}{x}, 1 - \frac{1}{x} - x\right\}$$

- The weight-one GHPLs are defined as

$$G(0;x) = \ln x\,, \qquad G(w;x) = \int_0^x dt f_w(t)$$

- Higher weight GHPLs are defined by iterated integrations

$$G(\underbrace{0,0,\cdots,0}_{n};x) = \frac{1}{n!}\ln^n x\,, \qquad G(w,\cdots;x) = \int_0^x dt f_w(t) G(\cdots;t)$$

- Shuffle algebra. Integration by parts identities

Goncharov '98, Remiddi and Vermaseren '99, Gehrmann and Remiddi '01-'02, Vollinga and Weinzierl '04

Two-Loop Corrections to $gg \rightarrow t\bar{t}$

Two-Loop Corrections to $gg \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2\mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned}\mathcal{A}_2^{(2\times 0)} \quad = \quad & (N_c^2-1)\Big(N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D + N_c^2 N_l E_l + N_c^2 N_h E_h \\ & + N_l F_l + N_h F_h + \frac{N_l}{N_c^2} G_l + \frac{N_h}{N_c^2} G_h + N_c N_l^2 H_l + N_c N_h^2 H_h \\ & + N_c N_l N_h H_{lh} + \frac{N_l^2}{N_c} I_l + \frac{N_h^2}{N_c} I_h + \frac{N_l N_h}{N_c} I_{lh}\Big)\end{aligned}$$

789 two-loop diagrams contribute to 16 different color coefficients

- Numeric result for $\mathcal{A}_2^{(2\times 0)}$ known

P. Bärnreuther, M. Czakon and P. Fiedler, '14

- The poles of $\mathcal{A}_2^{(2\times 0)}$ are known analytically

Ferroglia, Neubert, Pecjak, and Li Yang '09

- The leading color A, and light-quark E_l–I_l coefficients are known analytically

R. B., Ferroglia, Gehrmann, von Manteuffel and Studerus '11, '13

Two-Loop Corrections to $gg \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2\mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\mathcal{A}_2^{(2\times 0)} = (N_c^2-1)\Big(N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D + N_c^2 N_l E_l + N_c^2 N_h E_h$$

$$+ N_l F_l + N_h F_h + \frac{N_l}{N_c^2} G_l + \frac{N_h}{N_c^2} G_h + N_c N_l^2 H_l + N_c N_h^2 H_h$$

$$+ N_c N_l N_h H_{lh} + \frac{N_l^2}{N_c} \ldots$$

For the leading-color coefficient

NO additional MI

$\boxed{\text{789 two-loop diagrams}}$ contribute to $\boxed{16}$ different c…

- Numeric result for $\mathcal{A}_2^{(2\times 0)}$ recently published

  P. Bärnreuther, M. Czakon and P. Fiedler, '14

- The poles of $\mathcal{A}_2^{(2\times 0)}$ are known analytically

  Ferroglia, Neubert, Pecjak, and Li Yang '09

- The leading color A, and light-quark E_l–I_l coefficients are known analytically

  R. B., Ferroglia, Gehrmann, von Manteuffel and Studerus '11, '13

Two-Loop Corrections to $gg \to t\bar{t}$

$$|\mathcal{M}|^2(s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2\mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned}\mathcal{A}_2^{(2\times 0)} \quad = \quad (N_c^2-1)\Bigg(& N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D + N_c^2 N_l E_l + N_c^2 N_h E_h \\ & + N_l F_l + N_h F_h + \frac{N_l}{N_c^2} G_l + \frac{N_h}{N_c^2} G_h + N_c N_l^2 H_l + N_c N_h^2 H_h \\ & + N_c N_l N_h H_{lh} + \frac{N_l^2}{N_c} I_l + \frac{N_h^2}{N_c} I_h + \frac{N_l N_h}{N_c} I_{lh}\Bigg)\end{aligned}$$

- For the light-fermion contrib

9 additional MIs

different color coefficients

ublished

P. Bärnreuther, M. Czakon and P. Fiedler, ’14

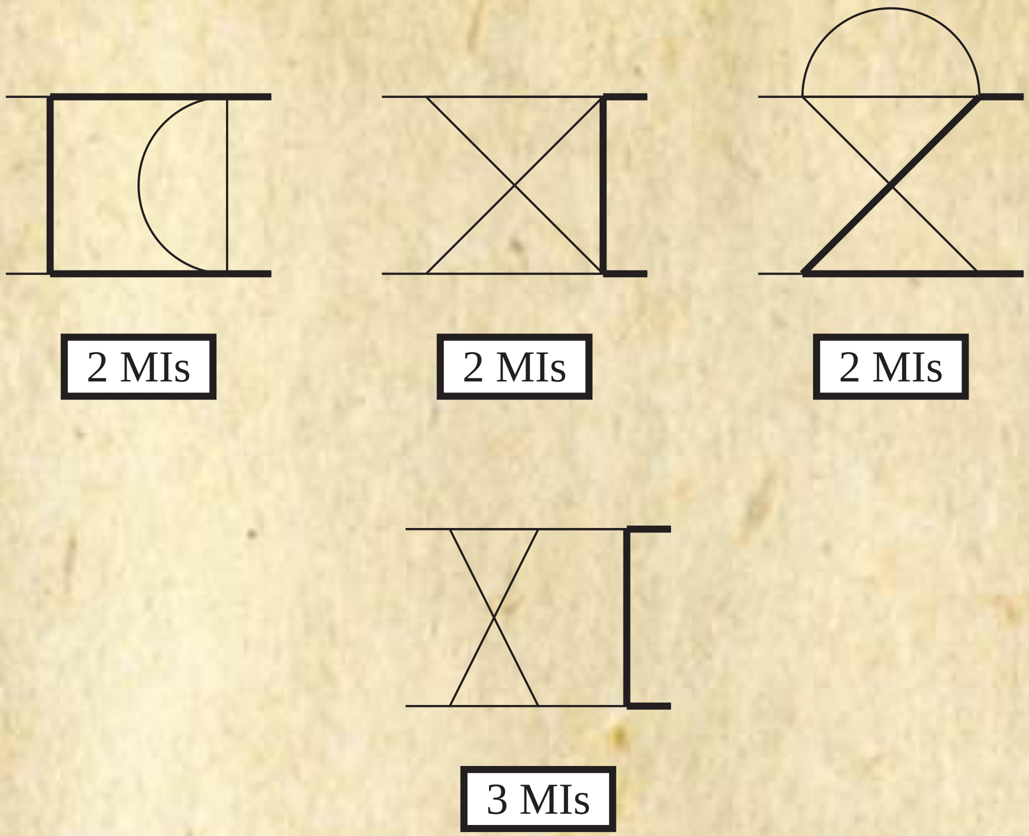
lytically

Ferroglia, Neubert, Pecjak, and Li Yang ’09

- The leading color A, and light-quark E_l–I_l coefficients are known analytically

R. B., Ferroglia, Gehrmann, von Manteuffel and Studerus ’11, ’13

Additional Master Integrals for the N_l Coeff



For the N_l coefficients in the gg channel there are 4 additional irreducible topologies (9 MIs)

A. von Manteuffel and C. Studerus, JHEP **1310** (2013) 037

Light Quark Coefficients in gg

Some considerations concerning the functional basis in which to express our analytic results are in order:

- The result can be written in terms of 289 GHPLs up to weight 4. They can be reduced to 221 using the algebra (3 MB of analytic formula)
- Alphabet in the naive case:

$$G(...;y) \in \left\{-1, 0, -\frac{1}{x}, -x, -\frac{(1+x^2)}{x}, -\frac{(1-x+x^2)}{x}\right\}$$

$$G(...;x) \in \left\{-1, 0, 1, [1+o^2], [1-o+o^2]\right\}$$

- NOTE: in this basis, 200 s for the numerical evaluation of a single phase space point! Hopeless! No way to use it in a Monte Carlo. What to do?

From complicated functions of simple arguments x, y $\longrightarrow$ To simpler functions of complicated arguments

R. B., A. Ferroglia, T. Gehrmann, A. von Manteuffel, and C. Studerus, JHEP **1312** (2013) 038

Optimized Functional Basis

- It turns actually out that a good choice is to express the result in terms ONLY of logarithms, polylogarithms Li_n with $n = 2, 3, 4$, and a single type of multiple polylogarithms, the $\mathrm{Li}_{2,2}$:

$$\mathrm{Li}_n(x) = -G(\underbrace{0, \cdots, 0, 1}_{n}; x)\,, \qquad \mathrm{Li}_{2,2}(x_1, x_2) = G\left(0, \frac{1}{x_1}, 0, \frac{1}{x_1 x_2}; 1\right)$$

  of arguments

$$\pm x,\ \pm x^2,\ -\frac{1}{y},\ -y,\ -\frac{y}{x},\ -x(x+y),\ \frac{x+y}{y},\ -\frac{x+z(x,y)}{x+y}, \cdots$$

  these arguments are such that the multiple polylogarithms are real valued in the Minkowski region

- We find again 225 multipole polylogarithms, out of which 57 $\mathrm{Li}_{2,2}$. Moreover the size of the analytic expression is always about 3 MB. However, the numerical evaluation now takes a fraction of a second!!
- Part of this transformation was done using symbols and co-products (Duhr, Gangl, Rhodes '12)

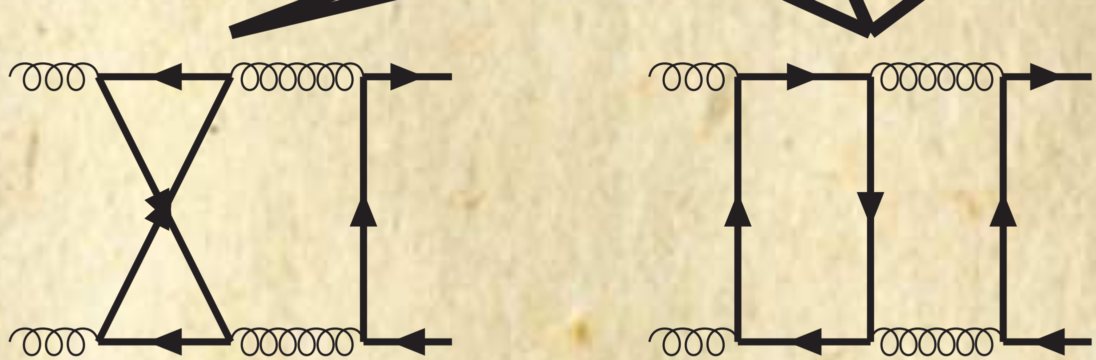
R. B., A. Ferroglia, T. Gehrmann, A. von Manteuffel, and C. Studerus, JHEP **1312** (2013) 038

Heavy-Quark Loop Coefficients

Heavy-Quark Loop Coefficients

The color structure of the heavy-quark loop coefficients is the following

$$A_2^{2\times 0} \quad = \quad (N_c^2-1)\left(\quad N_c^2 N_h\, E_h \qquad + N_h\, F_h \qquad + \frac{N_h}{N_c^2}\, G_h \quad \right)$$

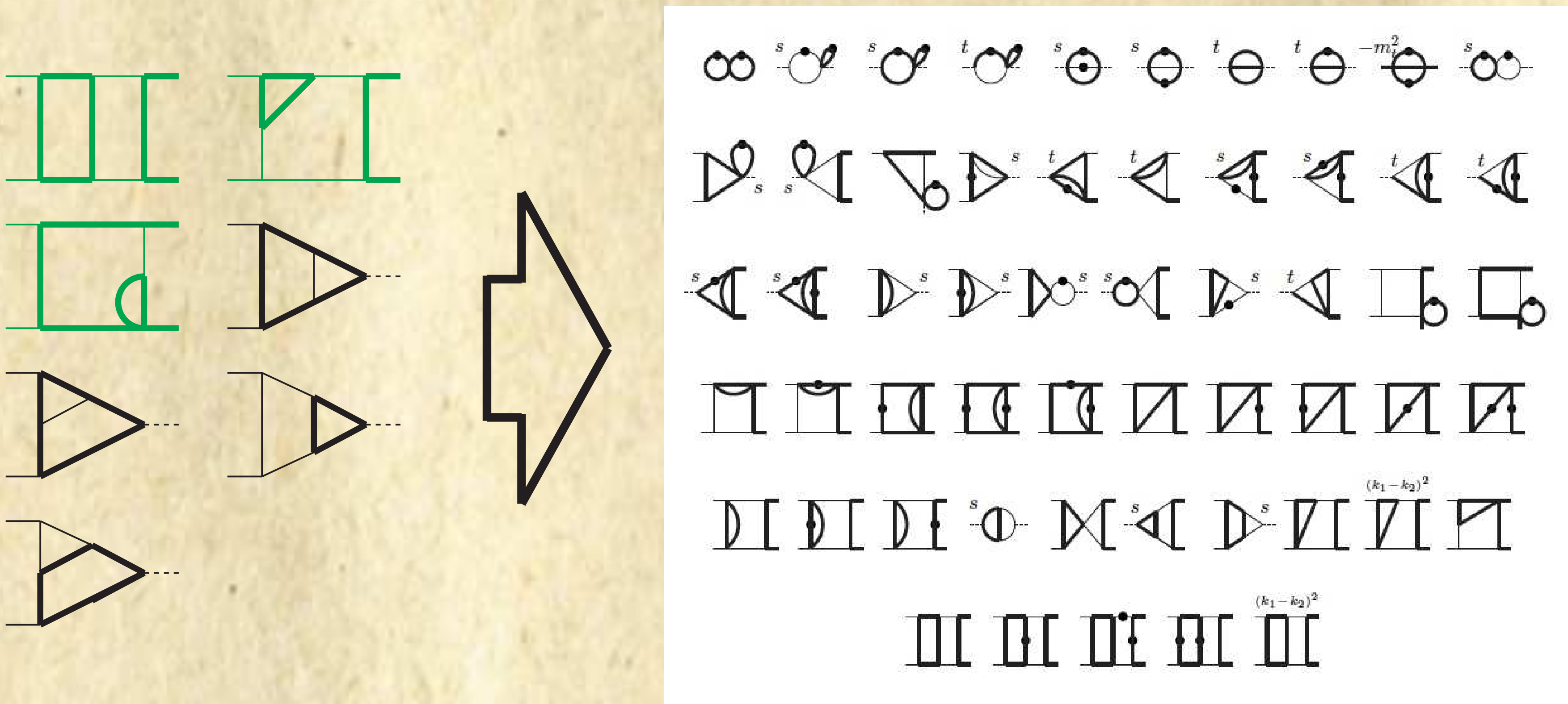


- The planar diagrams contribute to all the three color factors, while the crossed diagrams only to two of them
- Therefore, calculation of planar diagrams gives one gauge independent color factors out of three

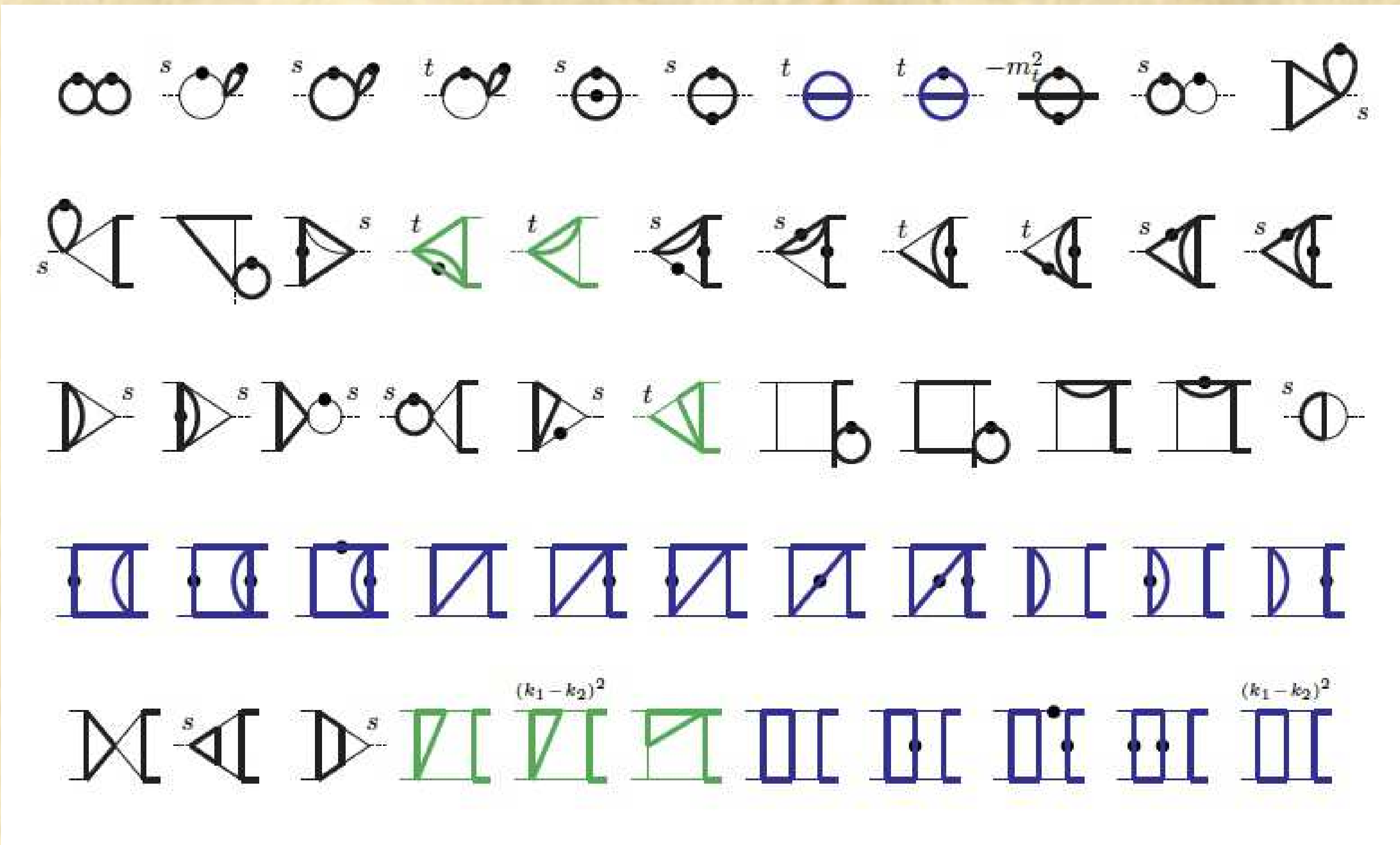
In collaboration with P. Caucal and M. Capozi

Planar Corrections

- The planar Feynman diagrams can be described in terms of dim-reg scalar integrals belonging to 7 topologies: 2 at 7 denominators and 5 at 6 denominators
- The 7-denom topologies are reduced to a set of 55 Master Integrals using IBP's
- The MIs are calculated with the Diff Eqs Method



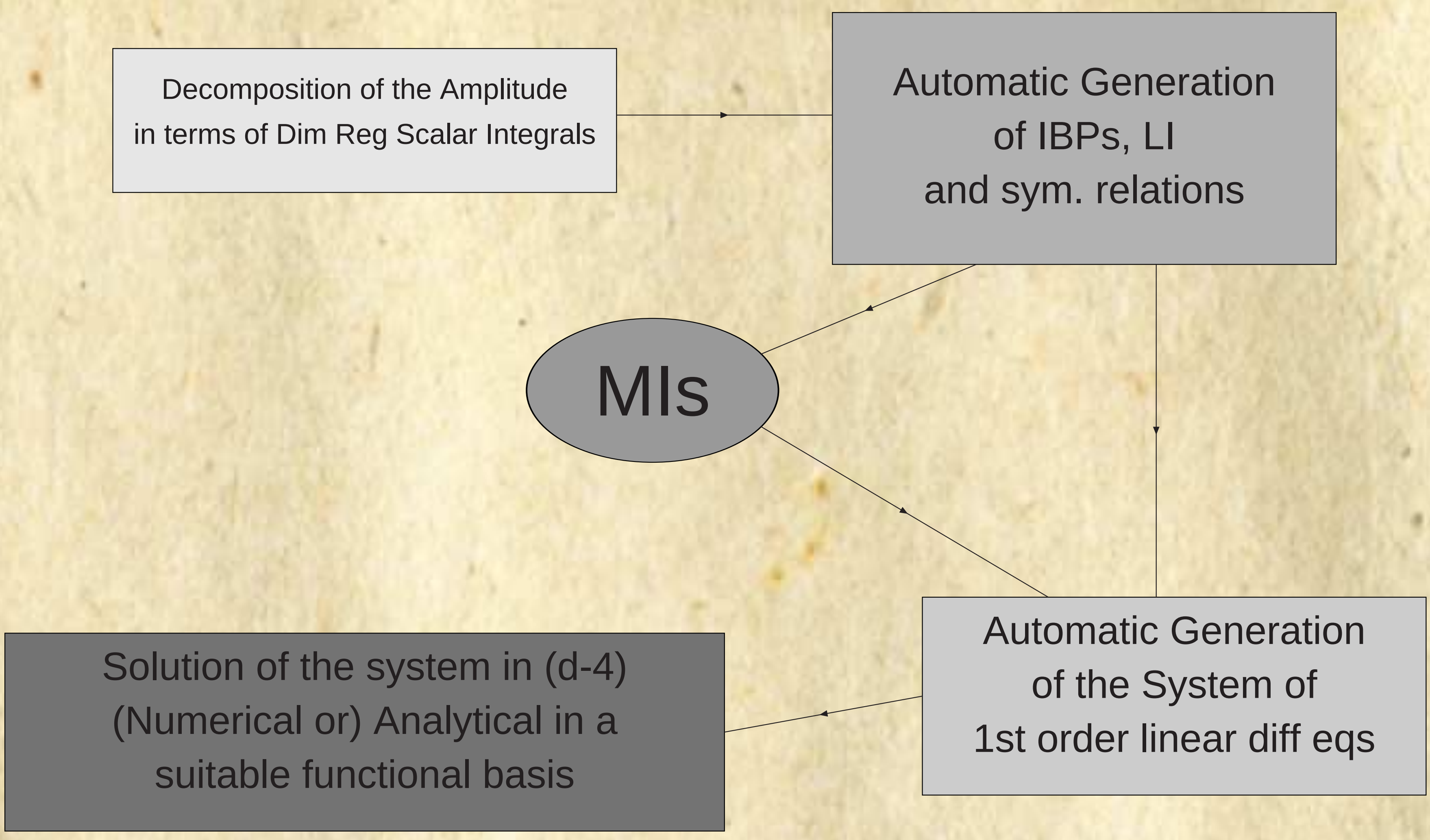
Planar Master Integrals



- Blue diagrams have homogeneous solutions expressed in terms of Elliptic Integrals
- Green diagrams contain non-homogeneous elliptic terms

Differential Equations Method

- One of the more successful techniques for the computation of multi-loop Feynman diagrams in the last years is the Differential Equations Method



Integration-by-Parts Identities

One of the building blocks of the method is constituted by the $\boxed{\text{REDUCTION PROCEDURE}}$

- Using IBP identities and LIs the scalar integrals in terms of which our observable is expressed are "REDUCED" to a set of l.i. ones: the MASTER INTEGRALS
- Different algorithms used for this goal

S. Laporta '96, R. N. Lee '08, A. von Manteuffel and R. M. Schabinger '15, A. Georgoudis and Y. Zhang '15

PUBLIC PROGRAMS

- AIR – Maple package (C. Anastasiou, A. Lazopoulos, JHEP **0407** (2004) 046)
- FIRE – Mathematica package (A. V. Smirnov, JHEP **0810** (2008) 107)
- REDUZE – REDUZE2 C++/GiNaC packages (C. Studerus, Comput. Phys. Commun. **181** (2010) 1293; A. von Manteuffel and C. Studerus, arXiv:1201.4330 [hep-ph].)
- LiteRed – Mathematica package (R. N. Lee arXiv:1212.2685 [hep-ph])
- Kira – C++/GiNaC (P. Maierhöfer, J. Usovitsch, P. Uwer, arXiv:1705.05610)

F.V. Tkachov, *Phys. Lett.* **B100** (1981) 65.
K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys.* **B192** (1981) 159.

Differential Equations for the MIs

The Master Integrals are function of the Mandelstam invariants ($x = s/m^2, t/m^2, ...$)

$$F_i = \int d^D k_1 d^D k_2 \, \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} = F_i(x)$$

They obey systems of first-order linear differential equations in the invariants

$$\frac{dF_i}{dx} = \sum_j h_j(x, D) \; F_j + \Omega_i(x, D)$$

where $i, j = 1, ..., N_{MIs}$ and $\Omega_i(x, D)$ involves subtopologies.

- The choice of the masters is arbitrary, but crucial for the solution of the system!
- We look for solutions in $(D-4) \sim 0$ (Laurent expansion)
- The system can be solved analytically (but also numerically ...)
- Analytical solutions need a suitable functional basis, that depends on the problem

V. Kotikov, $Phys.\ Lett.$ $\mathbf{B254}$ (1991) 158; $\mathbf{B259}$ (1991) 314; $\mathbf{B267}$ (1991) 123.
E. Remiddi, $Nuovo\ Cim.$ $\mathbf{110A}$ (1997) 1435.
E. Remiddi and T. Gehrmann, $Nucl.\ Phys.$$\mathbf{B580}$ (2000) 485.

Decoupling and Non-Decoupling Systems

- In almost all the cases treated so far at NNLO and beyond (mainly massless corrections) the idea is to reduce the systems order-by-order in ϵ at a triangular matrix form for the homogeneous part

$$\partial_x h(x) = \begin{pmatrix} a_{1,1} & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} h(x) + \text{non homogeneous terms}$$

- However, not all the systems follow this behaviour. In some (more and more numerous) cases we are in the situation in which the simplification of the system cannot be better than this

$$\partial_x h(x) = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & 0 & a_{4,4} \end{pmatrix} h(x) + \text{non homogeneous terms}$$

  - In this case, although two of the masters can be solved using only first order differential equations, the other two are coupled and their sub-system is equivalent to a
Second Order Differential Equation
  - Solution: two sol for the homogeneous and the particular with the variation of constants

Functional Basis for the Solutions

If the system of differential equations can be cast in canonical form (triangularized in ϵ), then

- when all possible square roots are removed (with changes of variables), the appropriate functional basis for the analytic solutions is the one of Multiple Polylogarithms (MPLs)

$$G(a_1, a_2, ..., a_n, x) = \int_0^x \frac{1}{t - a_1} G(a_2, ..., a_n, t) dt$$

Goncharov '98, Remiddi-Vermaseren '99,
Ablinger-Bluemlein-Schneider '13, Duhr-Gangl-Rhodes '12

  - MPLs (or GPLs) can be evaluated numerically with dedicated C++ fast and precise numerical routines

Vollinga-Weinzierl '05

- In the case the alphabet cannot be fully linearized, we can find a solution in terms of repeated integrals that involve square roots. In particular, we can find a solution at weight 2 in terms of logarithms and Li_2 functions. The weight 3 will be an integration over known functions, while the weight 4 would involve a two-fold integration. However, integrating by parts we can make in such a way that we are left with a single one-fold integration to be done numerically.

Henn-Caron Huot '14

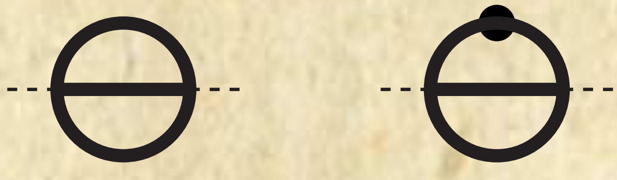
Two-Point Functions

The first case of Master Integrals that cannot be expressed in terms of generalized polylogarithms is the two-loop equal masses Sunrise

Two-Point Functions

The first case of Master Integrals that cannot be expressed in terms of generalized polylogarithms is the two-loop equal masses Sunrise

- Reducing the corresponding topology we find two MIs that obey a coupled system of first order linear differential equations in the dimensionless variable $z = p^2/m^2$



- The second-order linear diff eq for the scalar diagram in d dimensions is:

$$\frac{d^2}{dz^2}F + \frac{(3(4-d)z^2 + 10(6-d)z + 9d}{2z(z+1)(z+9)}\frac{d}{dz}F + \frac{(d-3)[(d-4)z - d - 4]}{2z(z+1)(z+9)}F = \Omega(z,d)$$

- Expanding in $(d-4)$ we find

$$F = -\frac{3}{8(d-4)^2} + \frac{(z+18)}{32(d-4)} + F_0 + \ldots$$

- The solution of F_0, F_1, etc ... is more easily found from the 2-dimensional solution using Tarasov's dimensional relations

Two-Point Functions

- The solutions of the homogeneous equation in $d = 2$ are given in terms of complete elliptic integral of the first kind

$$\psi_1(z) = \frac{K(m^2(z))}{[(z+1)^3(z+9)]^{\frac{1}{4}}} \qquad \psi_2(z) = \frac{K(1-m^2(z))}{[(z+1)^3(z+9)]^{\frac{1}{4}}}$$

where

$$K(m^2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-m^2x^2)}} \qquad m^2 = \frac{z^2+6z-3+\sqrt{(z+1)^3(z+9)}}{2\sqrt{(z+1)^3(z+9)}}$$

- Therefore, the particular solution is expressed via Euler's variation of constants in terms of integrals over the elliptic kernel represented by the homogeneous solutions

$$F(z) = c_1\psi_1(z) + c_2\psi_2(z) - \psi_1(z)\int^z \frac{dx}{W}\psi_2(x)\,\Omega(x) + \psi_2(z)\int^z \frac{dx}{W}\psi_1(x)\,\Omega(x)$$

S. Laporta and E. Remiddi, $Nucl.Phys.$ $\mathbf{B704}$ (2005) 349
L. Adams, C. Bogner, S. Weinzierl, $J.\ Math.\ Phys.$ $\mathbf{54}$ (2013) 052303
E. Remiddi and L. Tancredi, $Nucl.Phys.$ $\mathbf{B907}$ (2016) 400

Two-Point Functions

- Recently proposal of expressing the solution in terms of Elliptic Polylogarithms

$$\mathrm{ELi}_{n;m}(x,y,q) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{x^j}{j^n}\frac{y^k}{k^m} q^{jk}$$

where $q = Exp(i\pi\psi_2/\psi_1)$ is the nome of the elliptic curve and it is always $|q| < 1$
In terms of ELi the sunrise in $d = 2$ dimensions is

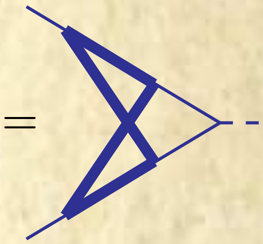
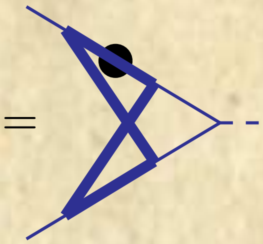
$$S^{(0)}_{1,1,1}(t) = \frac{3\psi_1}{i\pi}\left[\frac{1}{2}\mathrm{Li}_2(e^{2\pi i/3}) - \frac{1}{2}\mathrm{Li}_2(e^{-2\pi i/3}) + \mathrm{ELi}_{2,0}(e^{2\pi i/3},-1,-q) - \mathrm{ELi}_{2,0}(e^{-2\pi i/3},-1,-q)\right]$$

- Numeric evaluation of the Elliptic Polylogarithms in all the real t axis
- Dispersion relations (Remiddi and Tancredi) and E-Polylogarithms.
- Another two-loop two-point function was studied: Kite Integral (homogeneous non elliptic, sunrise in the non homogeneous part of the diff eq)
- Even more: three-loop “banana” graph! Homogeneous solutions as products of elliptic integrals

L. Adams, C. Bogner, S. Weinzierl, *J. Math. Phys.* **57** (2016) 032304
C. Bogner, A. Schweitzer, S. Weinzierl, Nucl. Phys. B **922** (2017) 528
A. Primo and L. Tancredi, Nucl. Phys. B **921** (2017) 316 ’17
J. Ablinger et al. arXiv:1706.01299 [hep-th]
E. Remiddi and L. Tancredi, arXiv:1709.03622

Three-Point Functions

Also three-point functions can exhibit an "elliptic behaviour". Very recently an elliptic three-point function was studied in detail

$F_1 =$  $F_2 =$ 

$$\frac{d^2}{dx^2} f(x) + \left(\frac{1}{x} + \frac{1}{x-16} \right) \frac{d}{dx} f(x) - \frac{1}{64} \left(\frac{1}{x} - \frac{1}{x-16} \right) f(x) = 0\,, \qquad f(x) = x^{\frac{3}{2}} F_1$$

- The homogeneous solutions for the two masters are expressed in terms of the complete elliptic integrals of the first and second kind

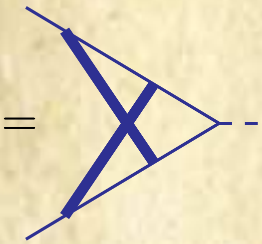
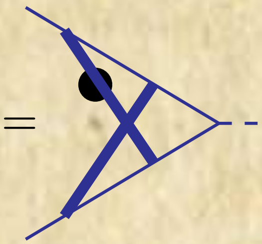
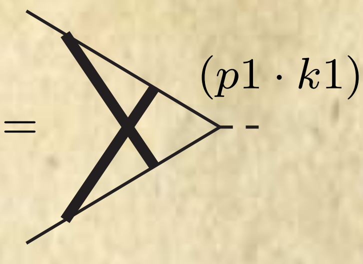
$$K(f(x)) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-fx^2)}} \qquad E(f(x)) = \int_0^1 \frac{\sqrt{1-fx^2}}{\sqrt{1-x^2}} dx$$

- The complete solution is found integrating in the different kinematic regions the non homogeneous part (previously expressed in terms of GPLs) over the elliptic homogeneous solutions. Excellent numerical performance

A. von Manteuffel and L. Tancredi, '17

Three-Point Functions

Another example: the two-massive exchange has three MIs

$F_1 =$  $F_2 =$  $F_3 =$ 

With this choice, the third one decouples from the other two. Therefore, we can write a second order differential equation for F_1 (for instance)

$$\frac{d^2 F_1}{dx^2} + \left[\frac{3}{x} + \frac{1}{x+1} + \frac{1}{x-8}\right]\frac{dF_1}{dx} + \left[\frac{1}{x^2} + \frac{9}{8x} - \frac{4}{3(x+1)} + \frac{5}{24(x-8)}\right] F_1 = \Omega(x)$$

Since the $d = 2$ homogeneous equations for the Sunrise $S(z)$ was

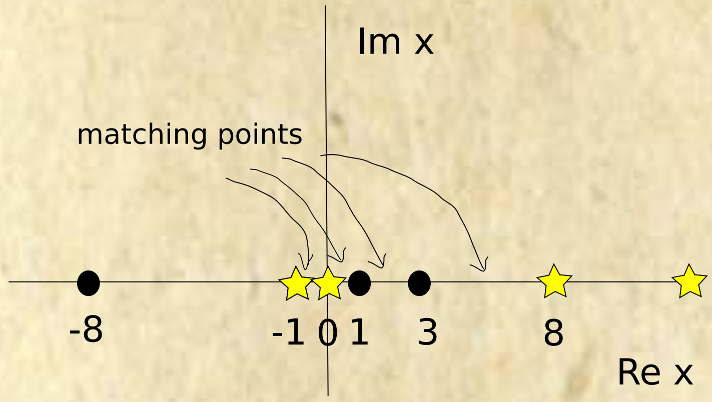
$$\frac{\partial^2}{\partial z^2} S(z) + \left[\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right]\frac{\partial}{\partial z} S(z) + \left[\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)}\right] S(z) = 0$$

it means that there is a simple relation between $S(z)$ and $F_1(z)$: $S(z) = -(z+1)\, F_1(-z-1)$

U. Aglietti, R. B., L. Grassi, E. Remiddi, *Nucl.Phys.* **B789** (2008) 45.

Semi-Numerical Evaluation

In the case of One dimensionless variable, one can adopt a Semi-Numerical evaluation of the masters, based on the differential equation

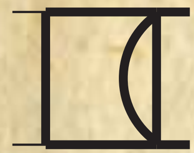


- We expand the diff eq and the solution in series of x around the singular points: $x = 0, 8, \infty, -1$. Every series depends on 2 arbitrary constants $\Rightarrow$ we impose the matching conditions expressing all of them in terms of 2 of them.
- Imposing the initial conditions we fix the constants and we find the solution in series representation. We construct a Fortran routine that gives $F_1(x)$ for every value of x with the desired precision.

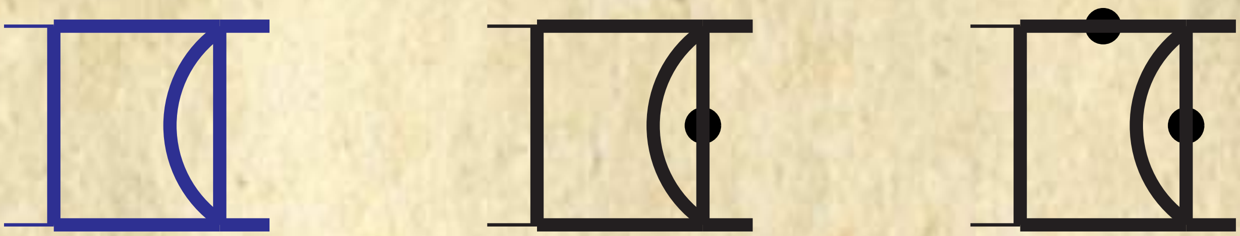
S. Pozzorini and E. Remiddi, *Comput.Phys.Commun.* **175** (2006) 381
U. Aglietti, R. B., L. Grassi, E. Remiddi, *Nucl.Phys.* **B789** (2008) 45
R. N. Lee, A. V. Smirnov, V. A. Smirnov, arXiv:1709.07525

- Unfortunately difficult to generalize to 3 scales (two variables) ...

$t\bar{t}$ 5-Den Elliptic Box

The first unknown four-point function is the 5-denominator Elliptic Box 

- The reduction procedure gives three MIs
  With the following choice we succeed to disentangle one of them:



- The system of first order differential equations becomes, at each order in epsilon, constituted by a single first order equation and two coupled equations (equivalent to a second order diff eq)
- We contruct the second order differential equation for one of the two masters (we choose the second) in s and t. We find the two independent solutions of the homogeneous equation
- We compute the Wronskian and we determine the particular solution via Euler's variation of constants

Second order hom differential equations

The equations in s and t of the master integral are ($m_t = 1$):

$$\frac{d^2}{ds^2}F + p(s,t)\frac{d}{ds}F + q(s,t)F = 0$$
$$\frac{d^2}{dt^2}F + r(s,t)\frac{d}{dt}F + u(s,t)F = 0$$

$$
\begin{aligned}
p(s,t) &= -\frac{1}{(s-4)} - \frac{2}{s} - \frac{1}{(s-4\frac{t-1}{t-9})} - \frac{1}{(s+\frac{(t-1)^2}{t})} + \frac{1}{(s+4\frac{t+1}{t+3})} \\
q(s,t) &= -\frac{1}{4s^2} - \frac{(t-9)^5}{(256(t-3)^3(4-9s-4t+st))} - \frac{(3+t)^5}{(64(-4+3s+4t+st)(-3-2t+t^2)^2)} \\
&+ \frac{(5-10t+2t^2)}{(4s(t-1)^2)} + \frac{(-25-77t-27t^2+t^3)}{(128(-4+s)(1+t)^2)} \\
&- \frac{((t-9)^2(-1971+1944t-534t^2+48t^3+t^4))}{(256(4+s(t-9)-4t)(t-3)^3(t-1))} + \frac{(9t^2+6t^3+2t^4-6t^5+t^6)}{((t-3)^2(t-1)^2(1+t)^2(1-2t+st+t^2))} \\
&- \frac{((3+t)^2(135+192t-10t^2-72t^3+11t^4))}{(64(t-3)^2(t-1)(1+t)^2(-4+4t+s(3+t)))}
\end{aligned}
$$

and similar coefficients for the equation in t ...

- Many singular points ... difficult direct solution! The parametrization trick does not help.

Cuts and Solutions of the Homogeneous Eq

- Another possible approach to the solution of the Homogeneous Diff Eq is the direct calculation of the maximal cut:
  - Simultaneously replace propagators with their δ-functions

$$\frac{1}{(p^2+m^2)} \to \delta(p^2+m^2)$$

  - If the propagator is squared, we cut it in the IBP sense (reduction to integrals with single prop and scalar prods)
  - The observation is based on the fact that if the masters under consideration obey a system

$$\partial_x M_i(\epsilon, x) = A_{ij}(\epsilon, x) M_j(\epsilon, x) + \Omega_i(\epsilon, x)$$

    then

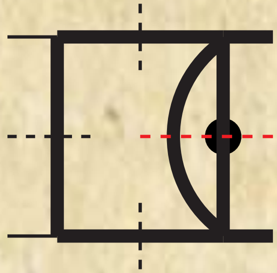
$$\partial_x Cut(M_i(\epsilon, x)) = A_{ij}(\epsilon, x) Cut(M_j(\epsilon, x))$$

    because $Cut(\Omega_i(\epsilon, x) = 0 \quad \Longrightarrow \quad$ the MaxCut is solution of the Hom Eq
  - Integrate directly finite MaxCut can help to solve the system of Diff Eqs

R. N. Lee and V. A. Smirnov, *JHEP* **12** (2012) 104.
A. Primo and L. Tancredi, *Nucl. Phys.* **B916** (2017) 94.
H. Frellesvig and C. G. Papadopoulos, *JHEP* **04** (2017) 083.
M. Harley, F. Moriello, R. M. Schabinger, ’17

Maximal Cut

We move to "PLAN B" which consists on the calculation of the $d = 4$ maximal cut (Primo and Tancredi), which is solution of the differential equation.



$$Cut(s,t) = \frac{K\left(\frac{16(t-1)(s+t-1)\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}}{4(t-1)^2\left(2\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-1\right)+s\left(t^2+8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}t-6t-8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-3\right)}\right)}{2s\sqrt{\frac{4(t-1)^2\left(2\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-1\right)+s\left(t^2+8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}t-6t-8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-3\right)}{s}}}$$

The two solutions of the homogeneous equation are then

$$\psi_1 = \frac{1}{R(s,t)}K(\omega) \qquad \psi_2 = \frac{1}{R(s,t)}K(1-\omega)$$

Solution

- Since the subtopologies entering the non-homogeneous part of the Diff Eq are expressed in terms of the variables x and y such that

$$s = -m^2 \frac{(1-x)^2}{x} \qquad t = -m^2 y$$

we move to x and y

- Knowing the two solutions of the homogeneous equation, the particular solution can be found with the Euler variation of constants method

$$\begin{aligned} F &= c_1 \psi_1(x,y) + c_2 \psi_2(x,y) \\ &\quad -\psi_1(x,y) \int^x \frac{d\xi}{W} \psi_2(\xi,y)\, \Omega(\xi,y) + \psi_2(x,y) \int^x \frac{d\xi}{W} \psi_1(\xi,y)\, \Omega(\xi,y) \end{aligned}$$

- The Wronskian W of the solutions is

$$W(x,y) = \frac{\pi}{32} \frac{x^2[y-3-2x(3y-1)+x^2(y-3)]}{(x-1)^3(x+1)(x+y+x^2y+xy^2)[y+9+2x(y-7)+x^2(y+3)]}$$

- Imposing the regularity at $s=0$ we find $c_1 = c_2 = 0$

Solution

- The non-homogeneous terms $\Omega(x, y)$ contain polylogarithmic functions and elliptic integrals
- At $\epsilon = 0$ we have:

$$\Omega(x, y) = P(x, y)/Q(x, y)\,;\ \log x\,;\ K(f(y))$$

  so, $K(f(y))$ that comes from the sunrise does not enter the integration in $d\xi$!
- The iterated integrations that we have at this order in ϵ are of the kind

$$F_2 \sim \int_1^x d\xi \left\{ \frac{P(\xi, y)}{Q(\xi, y)}\,;\ \log \xi \right\} \frac{1}{R(\xi, y)} K(\omega(\xi, y))$$

- At $\mathcal{O}(\epsilon)$ (which is required in the amplitude) we also have $\text{Li}_2(f(\xi, y))$ and $\log^2$ at the place of the $\log$
- Note: we have a single integration in x (and y behaves as a parameter).
- Numerical evaluation extremely fast (for the moment with Mathematica). We are in agreement with FIESTA4 (5 digits).
- This representation is also suitable for analytic continuation in the Minkowski physical region.

The decoupled Masters

In principle, once the solution of the coupled masters is found, the problem is completely solved

- We solve the second order linear diff eq for one of the coupled MIs (homogeneous solutions and particular solution as repeated integrations over the elliptic kernel)
- The solution of the other coupled MI comes just performing derivatives
- The ϵ-decoupled MIs of the same set can be calculated solving a first order linear diff eq

However, this implies an additional integration over the solution of the coupled MIs

$\Longrightarrow$ even more complicated functional structure!

- Since the set of Masters can be chosen freely, we can find different basis in which we decouple one master and solve a second order diff eq for one of the coupled.
- We found two basis constituted by (F_1, F_2, F_3) and (F_1, F_2, F_4), with F_2, F_3 and F_4 constituting a basis of integrals finite in 4 dimensions. Having solved F_2, we can get the solutions of F_3 and F_4 just by derivatives

We calculated numerically also the finite parts of F_3 and F_4 in the Euclidean region and found agreement with FIESTA4 (5 digits)

Conclusions

- Analytic computations received a big boost in the last years. In particular the reduction to the MIs and the method of differential equations for their calculation seams to be very powerful (many calculations more and more complicated)
- The paradigm at the moment seams to be the following
  - The masters that can be expressed in terms of multiple polylogarithms satisfy a system of diff eqs in canonical form
  - Increasing the complexity of the calculations, we start to find cases in which the system does not decouple in ϵ. In these cases, higher-order differential equations (for the moment second-order) have to be solved. The basis of functions involved points in the direction of generalized hypergeometric functions (and particular subcases)
- We discussed the calculation of the planar corrections to $gg \to t\bar{t}$ that involve a closed heavy-quark loop, in perturbative QCD. We afforded the calculation of 55 MIs: 31 are expressed in term of multiple polylogarithms (or more in general repeated integrations over a limited alphabet); 24 of them involves elliptic integrals.
- For the masters involving elliptic integrals, we calculated the homogeneous solutions for the corresponding second order differential equations using the maximal cut in $d = 4$ dimensions.
- The study of the structure of the new functions just started ....