

Closed Form Solutions

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What is a closed form solution?

Example: Solve this equation for $y = y(x)$.

$$y' = \frac{4 - x^3}{(1 - x)^2} e^x$$

Definition

A **closed form** solution is an expression for an **exact** solution given with a **finite amount of data**.

This is not a closed form solution:

$$y = 4x + 6x^2 + \frac{22}{3}x^3 + \frac{95}{12}x^4 + \dots$$

because making it exact requires infinitely many terms.

The Risch algorithm finds a closed form solution:

$$y = \frac{2 + x^2}{1 - x} e^x$$

Risch algorithm (1969)

Previous slide: A **closed form** solution is an **expression** for an exact solution with only a finite amount of data.

Risch algorithm finds (if it exists) a **closed form** solution y for:

$$y' = f$$

To make that well-defined, specify **which expressions** are allowed:

Define E_{in} and E_{out} such that:

- Any $f \in E_{\text{in}}$ is allowed as input.
- Output: a solution iff \exists solution $y \in E_{\text{out}}$.

Risch: $E_{\text{in}} = E_{\text{out}} = \{\text{elementary functions}\}$
 $= \{\text{expressions with } \mathbb{C}(x) \text{ exp log } + - \cdot \div$
composition and algebraic extensions}\}.

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Liouvillian solutions

Kovacic' algorithm (1986)

- 1 Solves homogeneous differential equations of order 2

$$a_2 y'' + a_1 y' + a_0 y = 0$$

(Risch: inhomogeneous equations of order 1)

- 2 It finds solutions in a larger class:

$$E_{\text{out}} = \{\text{Liouvillian functions}\} \supsetneq \{\text{elementary functions}\}$$

- 3 but it is more restrictive in the input:

$$a_0, a_1, a_2 \in \{\text{rational functions}\} \subsetneq \{\text{elementary functions}\}$$

Remark

\exists common functions that are not Liouvillian.

Allow those as closed form \rightsquigarrow need other solvers.

A non-Liouvilian example

Let

$$y := \oint_{\gamma} \exp\left(\frac{t^2 + x + 1}{x - t}\right) \frac{t}{x^2 + x + 1} dt$$

Zeilberger's telescoping algorithm \rightsquigarrow an equation for y :

$$(x^4 - x)y'' + (4x^4 + 2x^3 - 3x^2 - 7x + 1)y' + (6x^3 - 9x^2 - 12x + 3)y = 0$$

Closed form solutions were thought to be rare.

But (for order 2) **telescoping equations** often (always?) have closed form solutions:

$$\exp(-2x) \cdot \left(I_0\left(2\sqrt{x^2 + x + 1}\right) - \frac{x + 1}{\sqrt{x^2 + x + 1}} I_1\left(2\sqrt{x^2 + x + 1}\right) \right)$$

and

$$\exp(-2x) \cdot \left(K_0\left(2\sqrt{x^2 + x + 1}\right) + \frac{x + 1}{\sqrt{x^2 + x + 1}} K_1\left(2\sqrt{x^2 + x + 1}\right) \right)$$

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Diagonals of rational functions

Take a rational function in several variables, for example:

$$F = \frac{1}{1 - x - 2y - 3z - 4yz - 5xyz - 6xyz^2} \in \mathbb{Q}(x, y, z)$$

and write it as a multivariate power series

$$F = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{ijk} x^i y^j z^k \in \mathbb{Q}[[x, y, z]]$$

The **diagonal** is

$$\text{diag}(F) = \sum_{i=0}^{\infty} a_{iii} x^i \in \mathbb{Q}[[x]]$$

Fact: Diagonals of rational functions are **D-finite** (holonomic); they satisfy a homogeneous linear differential equation with polynomial coefficients.

Conjecture: If such an equation has order 2 then it has closed form solutions.

Diagonals of rational functions

For the example, the diagonal $D = \text{diag}(F)$ satisfies:

$$a_2 D'' + a_1 D' + a_0 D = 0$$

where

$$a_2 = x(2x + 1)(40x^3 + 4x^2 - 72x - 5)(500x^4 - 149x^3 + 939x^2 - 1061x + 5)$$

$$a_1 = 120000x^8 + 32160x^7 - 288416x^6 - 74344x^5 - 341206x^4 + 135372x^3 + 93397x^2 + 10510x - 25$$

$$a_0 = 40000x^7 + 10240x^6 - 210464x^5 + 4944x^4 - 58610x^3 + 15752x^2 - 3715x + 1225$$

To solve order 2 equations with $a_0, a_1, a_2 \in \mathbb{Q}[x]$ download:

www.math.fsu.edu/~eimamogl/hypergeometricsols

Result:

$${}_2F_1 \left(\begin{matrix} \frac{1}{12}, \frac{5}{12} \\ 1 \end{matrix} \middle| f \right) \cdot (625x^4 + 140x^3 + 1158x^2 - 196x + 1)^{-1/4}$$

where

$$f = 124416 \frac{x^3(2x + 1)^2(500x^4 - 149x^3 + 939x^2 - 1061x + 5)}{(625x^4 + 140x^3 + 1158x^2 - 196x + 1)^3}$$

Globally bounded equations

Random equations rarely have closed form solutions.

So-called **globally bounded** equations are common in:

- combinatorics (generating functions of sequences in oeis.org)
- physics (Ising model, Feynman integrals, etc.)
- Period integrals, creative telescoping, diagonals.

Conjecture

Globally bounded equations (of order 2) have closed form solutions.

In other words: [Closed form solutions are common](#)
(for non-random equations of order 2)

If Maple + Mathematica don't find them \rightsquigarrow [download our code!](#)

Local to global strategy

Risch: Given elementary function f , solve:

$$y' = f$$

$\{\text{poles of } y\} \subseteq \{\text{poles of } f\} = \text{known}$

Kovacic: Given polynomials $a_0, a_1, a_2 \in \mathbb{C}[x]$, solve:

$$a_2 y'' + a_1 y' + a_0 y = 0$$

$\{\text{poles of } y\} \subseteq \{\text{roots of } a_2\} = \text{known}$

Local to global strategy

$\{\text{poles of } y\} + \{\text{terms in polar parts}\} (+ \text{other data}) \rightsquigarrow y$

Local data: Classifying singularities

Example:

$$y = \exp(r) \quad \text{where} \quad r = \frac{1}{x^3} + \frac{5}{x^2} + \frac{3}{x-1} + 5 + 7x$$

y has **essential singularities** at the poles of r .

Definition

$y_1, y_2 \neq 0$ have an **equivalent singularity** at $x = p$ when y_1/y_2 is meromorphic at $x = p$.

Equivalence class of y at $x = 0$, $x = 1$, $x = \infty$ (local data)

\rightsquigarrow Polar part of r at $x = 0$, $x = 1$, $x = \infty$ (local data)

\rightsquigarrow $\frac{1}{x^3} + \frac{5}{x^2}$ $\frac{3}{x-1}$ $7x$ (local data)

$\rightsquigarrow r$ (up to a constant term) (global data)

$\rightsquigarrow y$ (up to a constant factor) (global data)

Reconstructing solutions from local data

Recall: y_1, y_2 have **equivalent singularity** at $x = p$
if y_1/y_2 is meromorphic at $x = p$.

Hence:

y_1, y_2 equivalent at every $p \in \mathbb{C} \cup \{\infty\}$

\iff

y_1/y_2 meromorphic at every $p \in \mathbb{C} \cup \{\infty\}$

\iff

$y_1/y_2 \in \mathbb{C}(x)$

Hence:

{Eq. class of y at all p } \iff y up to a rational factor

For a differential equation L **can compute:**

{**generalized exponents** of L at p } \approx {Eq. classes of solutions}

Choose the right one at each $p \rightsquigarrow$ a solution (up to \approx)

Example: generalized exponents

Example: let L have singularities $\{0, 3, 4\}$, order 2, and solutions:

$$y_1 = (x^4 - 2x + 2) \cdot \exp\left(\int \frac{e_{0,1}}{x} + \frac{e_{3,1}}{x-3} + \frac{e_{4,1}}{x-4}\right)$$

$$y_2 = (x^3 + 3x - 7) \cdot \exp\left(\int \frac{e_{0,2}}{x} + \frac{e_{3,2}}{x-3} + \frac{e_{4,2}}{x-4}\right)$$

where $e_{p,i} \in \mathbb{C}\left[\frac{1}{x-p}\right]$ encodes the **polar part** $\frac{e_{p,i}}{x-p}$ at $x = p$.

These $e_{p,i}$ are the **generalized exponents** of L at $x = p$ and can be computed from L :

$$E_0 = \{e_{0,1}, e_{0,2}\}, \quad E_3 = \{e_{3,1}, e_{3,2}\}, \quad E_4 = \{e_{4,1}, e_{4,2}\}$$

To find y_1 we need to **choose the correct element** of each E_p .

The example has $2^3 = 8$ combinations.

One combination $\rightsquigarrow y_1$, another $\rightsquigarrow y_2$, other six \rightsquigarrow nothing.

Can reduce #combinations (e.g. Fuchs' relation)

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Generalized exponents \rightsquigarrow hyper-exponential solutions:

Let $a_0, \dots, a_n \in \mathbb{C}[x]$ and $L(y) := a_n y^{(n)} + \dots + a_1 y' + a_0 y = 0$.

Hyper-exponential solution: $y = \exp(\int r)$ for some $r \in \mathbb{C}(x)$.

{**generalized exponent** of such y **at all singularities** p of L }

\rightsquigarrow

y up to a **polynomial** factor (generalized exponent \approx eq. class)

Algorithm hyper-exponential solutions:

- 1 Compute generalized exponents $\{e_{p,1}, \dots, e_{p,n}\}$ at each singularity $p \in \mathbb{C} \cup \{\infty\}$ of L .
- 2 For each combination $e_p \in \{e_{p,1}, \dots, e_{p,n}\}$ (for all p) compute **polynomial solutions** of a related equation.

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Same strategy for difference equation

Combine **generalized exponents** \rightsquigarrow hyper-exponential solutions.

To do the same for **difference equations** we need the difference analogue of generalized exponents:

Difference case: $p = \infty$ is similar to the differential case.

But a finite singularity is not an element $p \in \mathbb{C}$.

Instead it is an element of \mathbb{C}/\mathbb{Z} because

$$y(x) \text{ singular at } p \iff y(x+1) \text{ singular at } p$$

is only true for $p = \infty$.

(1997): Generalized exponents

(1999): Difference case analogue:

generalized exponents at $p = \infty$ and

valuation growths at $p \in \mathbb{C}/\mathbb{Z}$

\rightsquigarrow Algorithm for **hypergeometric solutions**.

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\rightsquigarrow Algorithm for **hypergeometric solutions**.

Closed form solutions of linear differential equations:

Goal: define, then find, **closed form solutions** of:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \quad \text{with} \quad a_0, \dots, a_n \in \mathbb{C}(x). \quad (1)$$

The **order** is n (we assume $a_n \neq 0$).

Consider **closed form** expressions in terms of **functions** that are:

- 1 well known, and
- 2 **D-finite**: satisfies an equation of form (1).

D-finite of order 1 = hyper-exponential function.

Well known D-finite functions of order 2:

- Airy functions, Bessel functions, Kummer, Whittaker, ...
- Gauss hypergeometric function ${}_2F_1(a, b; c | x)$
Klein's theorem: Liouvillian solutions are ${}_2F_1$ expressible.

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Idea for constructing a Bessel-solver:

- Bessel functions have an essential singularity at $x = \infty$.
- Just like the function $\exp(x)$.
- So the strategy for hyper-exponential solutions may work for Bessel-type solutions as well.
- It also works for Airy, Kummer, Whittaker, and hypergeometric ${}_pF_q$ functions if $p + 1 \neq q$.

Later: Other strategies for Gauss hypergeometric ${}_2F_1$ function
(to solve **globally bounded** equations of order 2).

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Which Bessel expressions are D-finite?

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Bessel type closed form expressions

Let $B_\nu(x)$ be one of the Bessel functions, with parameter ν .

Bessel type closed form expressions should allow:

- algebraic functions
- exp and log
- composition
- field operations
- differentiation and integration
- and of course $B_\nu(x)$.

Example: $B_0(\exp(x))$ is a **Bessel type closed form expression** but is **not relevant** for (1) since it is **not D-finite**.

Question: which Bessel type expressions are D-finite?

D-finite functions:

A function $y = y(x)$ is **D-finite of order n** if it satisfies a differential equation of order n with rational function coefficients.

Operations that don't increase the order:

- 1 $y(x) \mapsto y(f)$ for some $f \in \mathbb{C}(x)$ called **pullback function**.
- 2 $y \mapsto r_0 y + r_1 y' + \cdots + r_{n-1} y^{(n-1)}$ for some $r_i \in \mathbb{C}(x)$.
- 3 $y \mapsto \exp(\int r) \cdot y$ for some $r \in \mathbb{C}(x)$.

Operations that can increase the order:

- 4 Same as (1),(2),(3) but with algebraic functions f, r_i, r .
- 5 $y_1, y_2 \mapsto y_1 + y_2$ order $n_1, n_2 \rightsquigarrow$ order $\leq n_1 + n_2$
- 6 $y_1, y_2 \mapsto y_1 \cdot y_2$ order $n_1, n_2 \rightsquigarrow$ order $\leq n_1 \cdot n_2$
Special case: $y \mapsto y^2$ order $n \rightsquigarrow$ order $\leq \frac{n(n+1)}{2}$

Have algorithms to recover any combination of: (2), (3), (5), and part of (6).

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Bessel type solutions of second order equations

Let $B_\nu(x)$ be one of the Bessel functions.

$B_\nu(\sqrt{x})$ is D-finite of order 2. Transformations (1), (2), (3) \rightsquigarrow

$$\exp\left(\int r\right) \cdot \left(r_0 \cdot B_\nu(\sqrt{f}) + r_1 \cdot B_\nu(\sqrt{f})'\right) \quad (2)$$

is D-finite of order 2 for any $r, r_0, r_1, f \in \mathbb{C}(x)$.

Theorem (Quan Yuan 2012)

Let k be a subfield of \mathbb{C} and let L be a linear homogeneous differential equation over $k(x)$ of **order 2**.

If \exists solution of form (2) with **algebraic functions** r, r_0, r_1, f then \exists solution with **rational functions** $r, r_0, r_1, f \in k(x)$.

Bessel-type solutions of higher order equations:

\rightsquigarrow Add transformations (4),(5),(6).

Finding Bessel type solutions

$$a_2 y'' + a_1 y' + a_0 y = 0 \quad \text{where} \quad a_0, a_1, a_2 \in \mathbb{C}[x].$$

Goal: Find Bessel-type solutions.

Idea: Recover the pullback function f in transformation (1) from data that is **invariant** under transformations (2),(3).

Hyper-exponential solutions:

Generalized exponents \rightsquigarrow {polar parts of f } \rightsquigarrow f

Bessel-type solutions:

Generalized exponents \rightsquigarrow {[half] of terms of polar parts of f }
 \rightsquigarrow need **more data** to find f .

More data: regular singularities \rightsquigarrow roots of order $\notin \text{denom}(\nu) \cdot \mathbb{Z}$

Combine data \rightsquigarrow f **except in one case:** $\text{denom}(\nu) = 2$
that "happens" to be solvable with Kovacic

Finding Bessel type solutions

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Local to global strategy for difference equations

Use local data that is **invariant** under the difference analogue of transformations (2),(3):

- Giles Levy (Ph.D 2009)
- Yongjae Cha (Ph.D 2010)

Example: oeis.org/A000179 (Ménage numbers)

Recurrence operator:

$$(\tau + 1) \circ (n\tau^2 - (n^2 + 2n)\tau - n - 2)$$

where τ is the **shift-operator**.

$$\text{solver} \rightsquigarrow c_1 \cdot n \cdot I_n(-2) + c_2 \cdot n \cdot K_n(2) + c_3 \cdot \epsilon(n)$$

where $I_n(x)$ and $K_n(x)$ are **Bessel functions** and $\epsilon(n)$ is a complicated expression that converges to 0 as $n \rightarrow \infty$.

Result:

$$A000179(n) = \text{round} \left(\frac{2n}{e^2} \cdot K_n(2) \right) \quad (\text{for } n > 0)$$

The **Gauss hypergeometric function** is:

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n$$

where $(a)_n = a \cdot (a+1) \cdots (a+n-1)$.

If $L(y) = 0$ is a **globally bounded** equation of order 2 then it conjecturally has algebraic or ${}_2F_1$ -type solutions:

$$y = \exp\left(\int r\right) \cdot \left(r_0 \cdot {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| f \right) + r_1 \cdot {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| f \right)' \right)$$

Problem: The local to global strategy:

invariant local data \rightsquigarrow pullback function $f \rightsquigarrow y$

works for many functions, but ${}_2F_1$ can be problematic because f can be large even if the amount of local data is small.

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where $(a)_n = a \cdot (a+1) \cdots (a+n-1)$.

If $L(y) = 0$ is a **globally bounded** equation of order 2 then it conjecturally has algebraic or ${}_2F_1$ -type solutions:

$$y = \exp\left(\int r\right) \cdot \left(r_0 \cdot {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| f \right) + r_1 \cdot {}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix} \middle| f \right)' \right)$$

Problem: The local to global strategy:

invariant local data \rightsquigarrow pullback function $f \rightsquigarrow y$

works for many functions, but ${}_2F_1$ can be problematic because f can be large even if the amount of local data is small.

Small equation:

$$4x(x^2 - 34x + 1)y'' + (8x^2 - 204x + 4)y' + (x - 10)y = 0$$

The smallest solution:

$$\frac{\sqrt{3 - 3x - \sqrt{x^2 - 34x + 1}}}{x + 1} \cdot {}_2F_1 \left(\begin{matrix} \frac{1}{3}, \frac{2}{3} \\ 1 \end{matrix} \middle| f \right)$$

has

$$f = \frac{(x^3 + 30x^2 - 24x + 1) - (x^2 - 7x + 1)\sqrt{x^2 - 34x + 1}}{2(x + 1)^3}$$

How to construct f from a **small amount** of **invariant local data**:

- Exponent-differences: 0 , 0 , $\frac{1}{2}$ (mod \mathbb{Z})
- at the singularities: $x = 0$, $x = \infty$, $x^2 - 34x + 1 = 0$

${}_2F_1$ -type solutions and related topics:

- Tingting Fang (Ph.D 2012)
 - Compute D -module automorphisms \rightsquigarrow descent.
(also useful for non ${}_2F_1$ cases and for order > 2)
- Vijay Kunwar (Ph.D 2014)
 - Small f : Construct from invariant local data.
 - Large f : Tabulate and use combinatorial objects (such as dessins d'enfant) to prove completeness.
- Erdal Imamoglu (Ph.D 2017)
 - If transformation (2) is not needed: quotient method.
 - Otherwise: Differential analogue of POLRED
 \rightsquigarrow simpler equations. Then use quotient method.
- Wen Xu (Ph.D in progress)
 - Multivariate generalizations of ${}_2F_1$ such as Appell F_1 .

POLRED: Cohen and Diaz Y Diaz (1991)

Algebraic computations often lead to an equation:

$$f(x) = 0$$

for some irreducible $f \in \mathbb{Q}[x]$. Such f defines a number field:

$$K = \mathbb{Q}[x]/(f)$$

In many computations there is no reason to assume that f is the simplest polynomial that defines K .

Algorithm POLRED

Input: Irreducible $f \in \mathbb{Q}[x]$.

Output: Monic $g \in \mathbb{Z}[x]$ for the same field:

$$K \cong \mathbb{Q}[x]/(g).$$

with **near-optimal size** for $\max(\text{abs}(\text{coefficients of } g))$.

Algorithm POLRED

Input: Polynomial f giving a number field $K = \mathbb{Q}[x]/(f)$.

Step 1: Compute a basis of \mathcal{O}_K

Step 2: Use LLL to find $\beta \in \mathcal{O}_K$ that are **small at infinity**

Step 3: Compute minimal equation for such β 's

Result: Good polynomial g with $K \cong \mathbb{Q}[x]/(g)$.

$$\begin{aligned} \mathcal{O}_K &= \{\mathbf{integral} \text{ elements of } K\} \\ &= \{\text{elements with no poles at finite points}\} \\ &= \mathbb{Z}\text{-module} \end{aligned}$$

point = valuation = prime ideal

pole = negative valuation

small at infinity = small values in \mathbb{C}

For more details see Cohen's book.

Let $D = \mathbb{C}(x)[\partial] =$ ring of differential operators.

Differential POLRED

Input: operator $L \in D$ giving a D -module $M := D/DL$

Step 1: Compute a basis of \mathcal{O}_M

Step 2: Find $\beta \in \mathcal{O}_M$ that are **small at infinity**

Step 3: Compute minimal operator for such β

Result: Good operator \tilde{L} with $M \cong D/D\tilde{L}$

$$\begin{aligned}\mathcal{O}_M &= \{\mathbf{integral} \text{ elements of } M\} \\ &= \{\text{elements with no poles at finite points}\} \\ &= \mathbb{C}[x]\text{-module}\end{aligned}$$

small at infinity = minimize pole order at infinity

For more details see Erdal Imamoglu's PhD thesis.

Differential POLRED, example

The following equation came from lattice path combinatorics
 \implies **globally bounded**, conjecturally implies \exists ${}_2F_1$ -type solutions

$$\begin{aligned} & x(8x^2 - 1)(8x^2 + 1)(896x^5 - 512x^4 + 832x^3 - 127x^2 - 6x - 12) \cdot y'' \\ & - (8x^2 + 1)(71680x^7 - 36864x^6 + 46080x^5 - 3528x^4 - 5280x^3 \\ & + 155x^2 + 24x + 36) \cdot y' + (1720320x^8 - 786432x^7 + 1078272x^6 \\ & - 183360x^5 + 48384x^4 - 12464x^3 - 4560x^2 - 928x - 96) \cdot y = 0 \end{aligned}$$

www.math.fsu.edu/~eimamogl/hypergeometricsols does this:

- Take the **differential module** for this equation.
- Compute its **integral basis**.
- Construct integral element Y with **minimal degree** at infinity.
- Then Y satisfies a small equivalent equation:

$$x(8x^2 - 1)(8x^2 + 1) \cdot Y'' + (320x^4 - 1) \cdot Y' + 192x^3 \cdot Y = 0$$

- Quotient method \rightsquigarrow closed form for Y \rightsquigarrow closed form for y .

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Example from the Ising model

$$x(1-11x)(1+4x^2)y'' + (1-22x+8x^2-132x^3)y' + (-3+x-33x^2)y = 0$$

Solutions near $x = 0$:

$$S_1 = 1 + 3x + \frac{37}{2}x^2 + \dots$$

$$S_2 = \ln x + (3 \ln x + 5)x + \left(\frac{37}{2} \ln x + \frac{73}{2}\right)x^2 + \dots$$

Solutions near $x = 1/11$:

$$T_1 = 1 - \frac{77}{25}\left(x - \frac{1}{11}\right) + \dots$$

$$T_2 = \ln\left(\frac{1}{11} - x\right) - \left(\frac{77}{25} \ln\left(\frac{1}{11} - x\right) + \frac{649}{125}\right)\left(x - \frac{1}{11}\right) + \dots$$

Analytic continuation from $x = 0$ to $x = 1/11$ sends S_1 to a linear combination of T_1, T_2 . Which linear combination? Can always find an approximate answer (e.g. by evaluating at intermediate points)

www.math.fsu.edu/~eimamogl/hypergeometricsols

\rightsquigarrow closed form solution \rightsquigarrow exact answer.

Order > 2

If globally bounded equations of order 2 have ${}_2F_1$ -type solutions, what about higher order?

Univariate generalization of ${}_2F_1$: hypergeometric ${}_pF_q$ functions.

Globally bounded order 3 equations need not be ${}_pF_q$ -solvable.

Can construct a **univariate** example from **multivariate** hypergeometric functions (substitution \rightsquigarrow **univariate**).

There are many multivariate hypergeometric functions.

A **particle zoo** of functions?

Fortunately, they have been organized in terms of polytopes:

A-hypergeometric functions

Gelfand, Kapranov, Zelevinsky (1990)

Beukers (recent papers)

Are globally bounded equations solvable in terms of such functions?

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Trying to solve order > 2 equations in terms of such functions leads to many questions, for instance: how they relate to each other? Do we need reducible A -hypergeometric systems?

Example: The Horn G_3 function satisfies a bivariate system of order 3. In the **reducible case** $a = 1 - 2b$ this function

$$G_3(1 - 2b, b | x, y)$$

satisfies the same bivariate differential equations as:

$$(1 + 3y)^{\frac{3}{2}b-1} y^{1-2b} \cdot {}_2F_1 \left(\begin{matrix} \frac{1}{3} - \frac{b}{2}, \frac{2}{3} - \frac{b}{2} \\ \frac{1}{2} \end{matrix} \middle| \frac{(27xy^2 - 9y - 2)^2}{4(1 + 3y)^3} \right)$$

Found similar formulas for other reducible order 3 systems.

Thank you