Graph Complexes and Cutkosky Rules

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joint with Spencer Bloch

Elliptic Conference, Zeuthen, October 23-26, 2017

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- Physical thresholds? Anomalous thresholds?
  What is the systematics?
- Dispersion relations?
  Can we reconstruct a graph from its variations?

Literature

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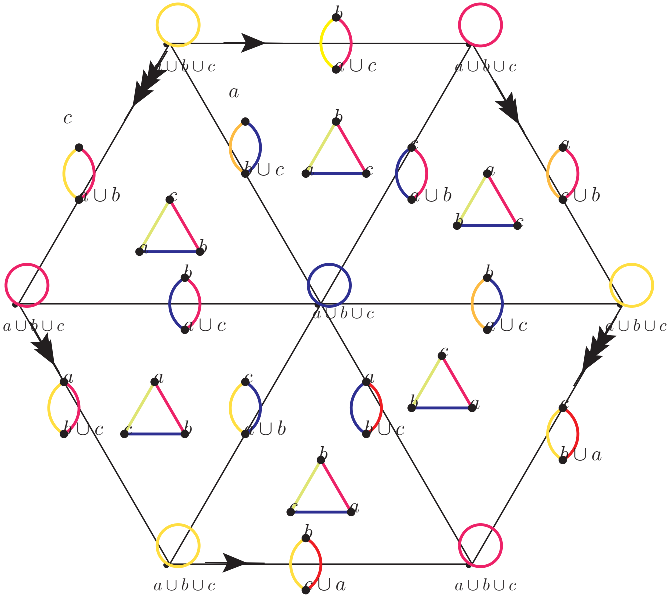
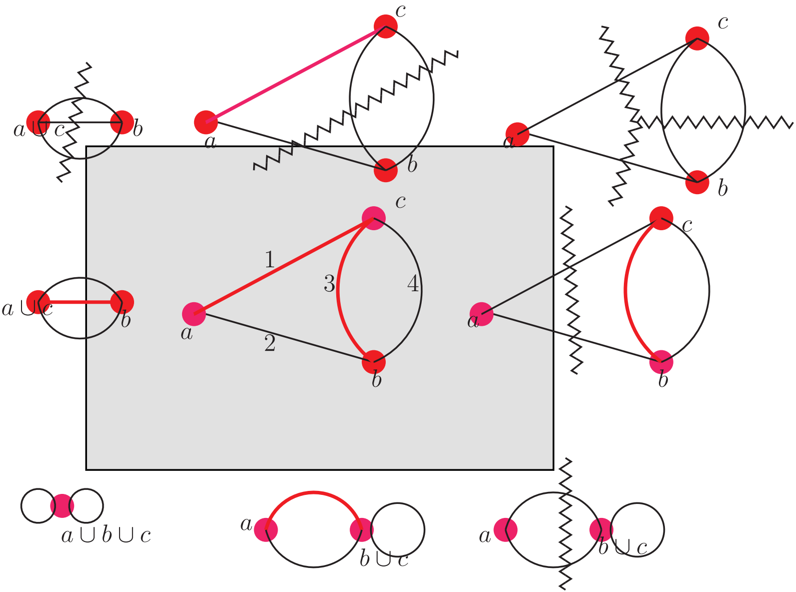
- KAI-UWE BUX, PETER SMILLIE, AND KAREN VOGTMANN,
  ON THE BORDIFICATION OF OUTER SPACE, arXiv:1709.01296.

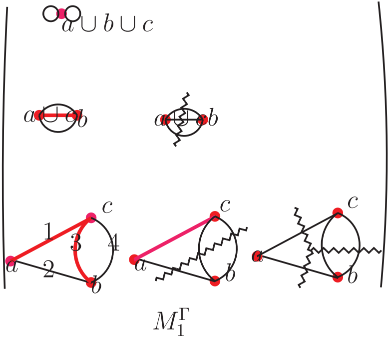
A cell complex for graphs: Outer Space

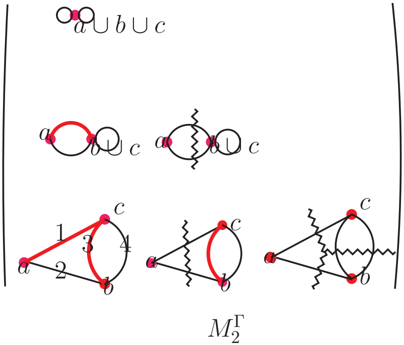


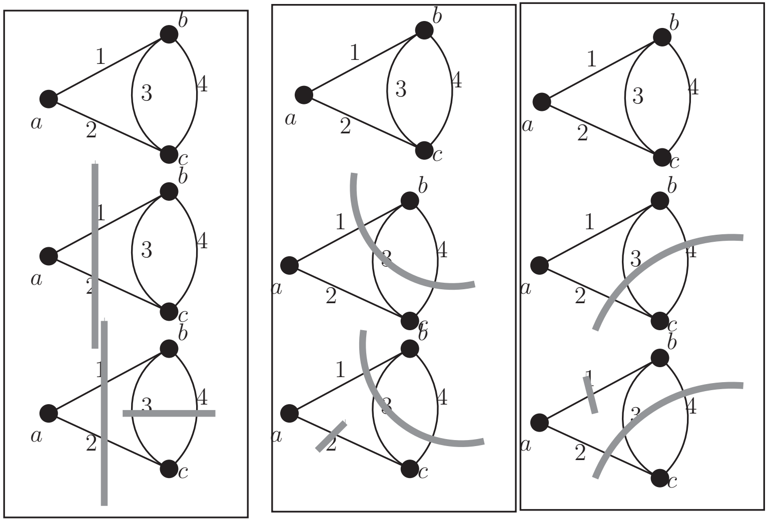
The cubical chain complex



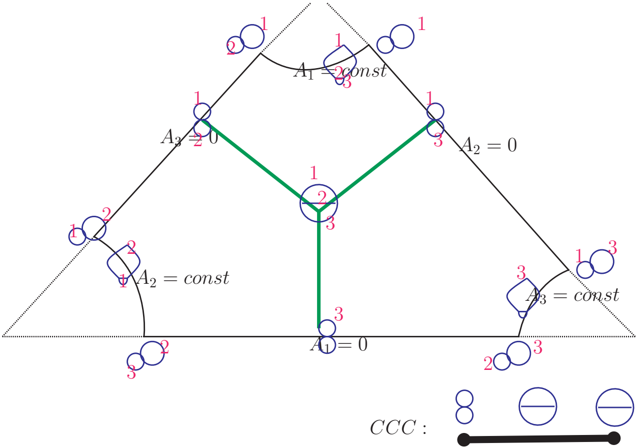
Two matrices, obtained from the two possible orderings of edges in the spanning tree. In total, 5 spanning trees each on two edges $\to$ 10 matrices.



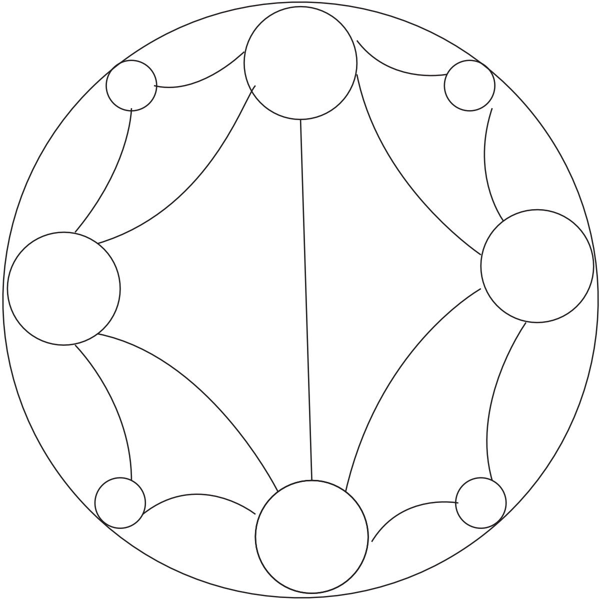



a
b
c
1
2
3
4

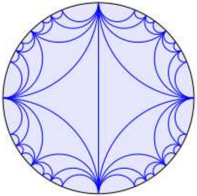
The jewel for the 3-edge banana



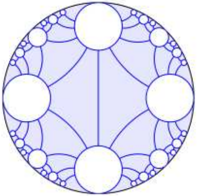
Gluing jewels



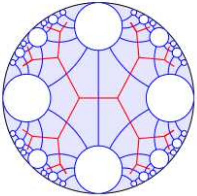
Spaces



Outer space $\mathcal{O}_2$

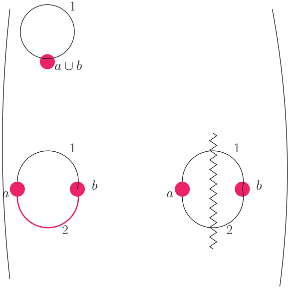


Jeweled subspace $\mathcal{J}_2$



Spine $K_2 \subset \mathcal{J}_2$

Example

$M_1^\gamma =$ 

and the three entries populate a cubical complex which is simply an interval $[0,1]$, with the endpoint at zero associated to $(M_1^\gamma)_{11}$, the endpoint at 1 to $(M_1^\gamma)_{22}$ and the interval $]0,1[$ to $(M_1^\gamma)_{21}$.

$$\Phi_R(s, s_0, m_1^2, m_2^2) = \frac{1}{4\pi^2}\frac{\sqrt{\lambda}}{2s}\ln\frac{x+\sqrt{\lambda}}{x-\sqrt{\lambda}} - \frac{m_1^2-m_2^2}{2s}\ln\frac{m_1^2}{m_2^2} - (s \to s_0),$$

Multivalued Graphs

$$\Phi_R^{\mathrm{MV}}(s, s_0, m_1^2, m_2^2) = \frac{1}{4\pi^2}\frac{\sqrt{\lambda}}{2s}\ln\frac{x+\sqrt{\lambda}}{x-\sqrt{\lambda}} - \frac{m_1^2 - m_2^2}{2s}\ln\frac{m_1^2}{m_2^2} - (s \to s_0)$$

$$+2\pi\imath\mathbb{Z}\frac{\sqrt{\lambda}}{2s},$$

In general, for $|\Gamma| \geq 2$:

$$\Phi_R^{\mathrm{MV}}(\Gamma) = \sum \Phi_R^{\mathrm{MV}}(\gamma) \cdot \Phi_R^{\mathrm{MV}}(\Gamma/\gamma),$$

where $\cdot$ indicates inserting the $\Phi_R^{\mathrm{MV}}(\gamma)$ integrand via Fubini (as an iterated integral), and the core Hopf algebra is $\Delta_c(\Gamma) = \sum \gamma \otimes \Gamma/\gamma$, gneralizing the renormalization Hopf algebra.

Start of induction: from Mixed Hodge Structure of 1-loop graphs (see Bloch & Kreimer)
Furthermore: Consistency with Cutkosky rules, as all cuts come from full cuts and the core Hopf algebra
Gradings: increasing transcendental complexity of a graph by grading à la Vogtmann et al:

$$||\Gamma|| = 2|\Gamma| - v_0,$$

where v_0 is the valence of the basepoint (the vertex at ∞ into which all external edges merge)

Cutkosky's theorem

Theorem (Cutkosky)

Assume the quotient graph G'' has a physical singularity at an external momentum point $p'' \in (\bigoplus_{V''} \mathbb{R}^D)^0$, i.e. the intersection $\bigcap_{e \in E''} Q_e$ of the propagator quadrics associated to edges in E'' has such a singularity at a point lying over p''. Let $p \in (\bigoplus_V \mathbb{R}^D)^0$ be an external momentum point for G lying over p''. Then the variation of the amplitude $I(G)$ around p is given by Cutkosky's formula

$$var(I(G)) = (-2\pi i)^{\#E''} \int \frac{\prod_{e \in E''} \delta^+(\ell_e)}{\prod_{e \in E'} \ell_e}. \tag{1}$$

Parametric Wonderland

- $$\Phi(\Gamma) = \underbrace{\Phi(\Gamma/\gamma)\psi(\gamma)}_{k-1} + \underbrace{\Phi(\Gamma-\gamma)\psi(r_k) - M(\gamma)\psi(\Gamma/\gamma)\psi(\gamma)}_{k} \\ - \underbrace{M(\gamma)\psi(\Gamma-\gamma)\psi(r_k)}_{k+1}$$

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$$- \underbrace{M(\gamma)\psi(\Gamma-\gamma)\psi(r_k)}_{k+1}$$

- $$\phi^r_{x,y}(\Gamma) = \sum_{T_1 \cup T_2} (Q(T_1) \cdot Q(T_2))^r \prod_{e \notin T_1 \cup T_2} A_e,$$
$$(Q(T_1) \cdot Q(T_2))^r = (Q(T_1) \cdot Q(T_2)) - r,$$

if $T_1 \cup T_2$ separates x, y.

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- $$\Phi(\Gamma-\gamma)E^\gamma_k - M(\gamma)\psi(\Gamma/\gamma)E^\gamma_{k-1} = \Phi^u(\Gamma-\gamma),$$
with $u = (\sum_{e\in E_\gamma} m_e)^2$.

From Discriminants to anomalous thresholds

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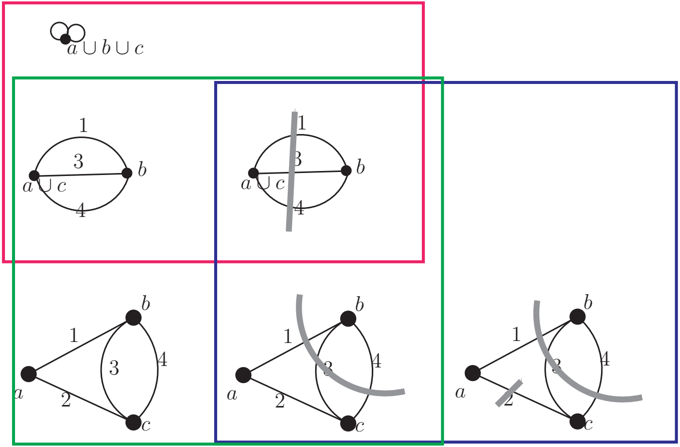
- iii) If for all $T \in T_s^\Gamma$ and for all their forests (Γ, F) we have $s_F > -\infty$, the Feynman integral $\Phi_R(\Gamma)(s)$ is real analytic as a function of s for $s < \min_F\{s_F\}$.

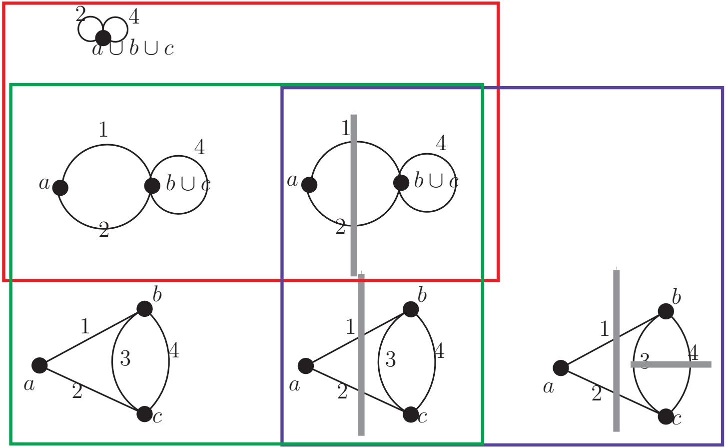
Dispersion

$$\left(\begin{array}{ccccccc}
1 & & 0 & & 0 & & 0\\
\uparrow\pi & & \uparrow\pi & & & & \\
\Upsilon_{\Gamma_2}^{T_2} & \rightleftharpoons_{\mathrm{disp}}^{\mathrm{Var}} & \Upsilon_{\Gamma_2}^{T_2-e_1} & & 0 & & 0\\
\uparrow\pi & & \uparrow\pi & & \uparrow\pi & & \\
\Upsilon_{\Gamma_3}^{T_3} & \rightleftharpoons_{\mathrm{disp}}^{\mathrm{Var}} & \Upsilon_{\Gamma_3}^{T_3-e_1} & \rightleftharpoons_{\mathrm{disp}}^{\mathrm{Var}} & \Upsilon_{\Gamma_3}^{T_3-e_1-e_2} & & 0\\
\uparrow\pi & & \uparrow\pi & & \uparrow\pi & & \uparrow\pi\\
\Upsilon_{\Gamma_4=\Gamma}^{T_4=T} & \rightleftharpoons_{\mathrm{disp}}^{\mathrm{Var}} & \Upsilon_{\Gamma_4}^{T_4-e_1} & \rightleftharpoons_{\mathrm{disp}}^{\mathrm{Var}} & \Upsilon_{\Gamma_4}^{T_4-e_1-e_2} & \rightleftharpoons_{\mathrm{disp}}^{\mathrm{Var}} & \Upsilon_{\Gamma_4}^{T_4-e_1-e_2-e_3}
\end{array}\right).$$

An example for a graph with a length 3 spanning tree. Moving up, we shrink edges, moving right, we put more edges on the mass-shell. Dispersion integrals move left.



Along the diagonal, only normal thesholds appear in dispersion



For the subdiagonals, compute your anomalous thresholds.

Example: The triangle

$$\Phi_\Delta = \overbrace{p_a^2 A_1 A_2 - (m_1^2 A_1 + m_2^2 A_1)(A_1 + A_2)}^{=\Phi_{\Gamma/e_3}} + A_3((p_b^2 - m_3^2 - m_1^2)A_1 + (p_c^2 - m_1^2 - m_3^2$$

so

$$\Phi_\Delta = \Phi_{\Delta/e_3} + A_3 \Phi^{m_3^2}_{\Delta - e_3} - A_3^2 m_3^2 \overbrace{\psi_{\Delta - e_1}}^{=1},$$

as announced ($A_3 = t_\gamma$):

$$X = \Phi_{\Delta/e_3},\ Y = \overbrace{(p_b^2 - m_3^2 - m_1^2)}^{=:l_1} A_1 + \overbrace{(p_c^2 - m_1^2 - m_3^2)}^{=:l_2} A_2,\ Z = m_3^2.$$

We have $Y_0 = m_2 l_1 + m_1 l_2$, and need $Y_0 > 0$ for a Landau singularity.

cont'd

Solving $\Phi(\Delta/e_3) = 0$ for a Landau singularity determines the familiar physical threshold in the $s = p_a^2$ channel, leading for the reduced graph to

$$p_Q : s_0 = (m_2 + m_3)^2,\ p_A : A_1 m_1 = A_2 m_2.$$

We let $D = Y^2 + 4XZ$ be the discriminant. For a Landau singularity we need

$$D = 0.$$

We have

$$\Phi_\Delta = -m_3^2 \left(A_3 - \frac{Y + \sqrt{D}}{2m_3^2} \right) \left(A_3 - \frac{Y - \sqrt{D}}{2m_3^2} \right),$$

where Y, D are functions of A_1, A_2 and $m_1^2, m_2^2, m_3^2, s, p_b^2, p_c^2$.

cont'd

We can write

$$0 = D = Y^2 + 4Z(sA_1A_2 - N),$$

with $N = (A_1 m_1^2 + A_2 m_2^2)(A_1 + A_2)$ s-independent. This gives

$$s(A_1, A_2) = \frac{4ZN - (A_1 l_1 + A_2 l_2)^2}{4ZA_1A_2} =: \frac{A_1}{A_2}\rho_1 + \rho_0 + \frac{A_2}{A_1}\rho_2.$$

Define two Kallen functions $\lambda_1 = \lambda(p_b^2, m_1^2, m_3^2)$ and $\lambda_2 = \lambda(p_c^2, m_2^2, m_3^2)$. Both are real and non-zero off their threshold or pseudo-threshold. Then, for

$$\lambda_1, \lambda_2 > 0,$$

we find the threshold s_1 at

$$s_1 = \frac{4m_3^2(\sqrt{\lambda_1}m_1^2 + \sqrt{\lambda_2}m_2^2)(\sqrt{\lambda_1} + \sqrt{\lambda_2}) - (\sqrt{\lambda_1}l_2 + \sqrt{\lambda_2}l_1)^2}{4m_3^2\sqrt{\lambda_1}\sqrt{\lambda_2}}.$$

cont'd

On the other hand for $r < 0$ and therefore the coefficients of ρ_1, ρ_2 above of different sign we find a minimum

$$s_1 = -\infty, \tag{2}$$

along either $A_1 = 0$ or $A_2 = 0$. Get dispersion from other channels, looking at other spanning trees, that is.

Things are not simpler than they can be, and not more difficult than they must be.