Boltzmann equation for relativistic species and Hot Dark Matter

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Motivation

Hubble constant measurement discrepancy:

Planck CMB data: $67.8 \pm 0.9\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ [arXiv:1502.01589]
Direct measurement: $73.24 \pm 1.74\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ [arXiv:1604.01424]

Possible solution: additional effective number of relativistic degrees of freedom ΔN_{eff}:

$$\rho_{\mathrm{HDM}} = \Delta N_{\mathrm{eff}}\,\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\rho_\gamma$$

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Therefore, it is crucial to estimate ΔN_{eff} with better accuracy in models with HDM component.

Boltzmann equation for relativistic particles

Dolgov & Kainulainen, Nucl. Phys. B **402** (1993) 349 [hep-ph/9211231]

$$E(\partial_t - pH\partial_p)f(p,t) = C_E(p,t) + C_I(p,t)$$

Pseudopotential method ($\bar{\psi}\psi \rightarrow \bar{N}N$, $x = \frac{m}{T}$, $y = \frac{E}{T}$, z):

$$f(p,t) \simeq \left(e^{xy+z} \mp 1\right)^{-1}$$

$$f_N(p,t) \simeq \left(e^{xy_N} \mp 1\right)^{-1}$$

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where:

$$A(z,x) = \frac{g}{2\pi^2} m^3 e^z J_2(z,x)$$

$$B(z,x) \equiv \frac{g}{2\pi^2} m^3 e^z x\, J_3(z,x)\left(H(T) + \frac{1}{T}\frac{dT}{dt}\right)$$

$$J_n(z,x) \equiv \int_0^\infty dy\, y^n \frac{e^{xy}}{(1 \mp e^{xy+z})^2}$$

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$$S_I(z,x) = \frac{m^4}{512\pi^6}\left(e^{2z}-1\right)\int \mathscr{D}\Phi \int_0^{2\pi} d\phi \sum_{\text{spin}} |M(u,v,t,\cos\theta,\phi)|^2$$

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$$S_l(z,x) = \frac{m^4}{512\pi^6}\left(e^{2z}-1\right)\int \mathscr{D}\Phi \int_0^{2\pi} d\phi \sum_{\text{spin}} |M(u,v,t,\cos\theta,\phi)|^2$$

For $|M|^2 = |M(s)|^2$:

$$\mathscr{D}\Phi = \frac{1}{x^4}\int_{2x}^{\infty} dp \int_0^{\sqrt{p^2-4x^2}} dq \, \frac{1}{(1-e^{-p})(e^{p+2z}-1)}$$

$$\times \ln\left[\frac{\cosh\left(\frac{1}{2}(p+q)+z\right)\mp 1}{\cosh\left(\frac{1}{2}(p-q)+z\right)\mp 1}\right] \ln\left[\frac{\cosh\left(\frac{1}{2}\left(p+qV(p,q)\right)\right)\mp 1}{\cosh\left(\frac{1}{2}\left(p-qV(p,q)\right)\right)\mp 1}\right]$$

where $V = \left(1-\frac{4x^2}{p^2-q^2}\right)^{1/2}$, $s = m^2\frac{p^2-q^2}{x^2}$

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Finally:

$$n(x) \equiv n_\psi(x) = g\frac{m^3}{2\pi^2}\int_0^\infty \frac{y^2 dy}{e^{xy+z}\mp 1}$$

Approximations

$$Y'(x) = -\sqrt{\frac{\pi}{45}} \frac{g(x)}{\sqrt{g_s(x)}} \frac{M_{\rm Pl} m}{x^2} \left(Y^2(x) - Y_{\rm eq}^2(x)\right) \langle \sigma v \rangle_{\rm MB} \frac{1}{\zeta^{\mp}}$$

$$Y_{\rm eq}(x) = g \frac{45}{2\pi^4} \frac{\zeta^{\mp}}{g_s(x)} \qquad \zeta^{\mp} \equiv \zeta(3) \begin{cases} 1 & \text{BE } (-) \\ 3/4 & \text{FD } (+) \end{cases}$$

$$\langle \sigma v \rangle_{\rm MB} = \frac{1}{512\pi} \frac{x^5}{m^5} \int_{4m^2}^{\infty} ds \sqrt{s} \sqrt{1 - \frac{4m^2}{s}} \, |M(s)|^2 K_1\left(\frac{x\sqrt{s}}{m}\right)$$

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- Pure MB: $\langle \sigma v \rangle_{\rm MB}$ with $\zeta^{\mp} = 1$ — artificial — MB
- Fractional BE/FD: $\langle \sigma v \rangle_{\rm MB} \times \zeta^{\mp}$ — used — fBE/FD
- Partial BE/FD: $\langle \sigma v \rangle_{\rm p} \times \zeta^{\mp}$ — used — pBE/FD

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$$\times\int_{\sqrt{s}}^{\infty} dE_+\,\frac{e^{-\frac{x}{2m}E_+}}{\sinh\left(\frac{x}{2m}E_+\right)}\ln\left[\frac{\mathrm{fh}\left(\frac{x}{4m}\left(E_+ + \sqrt{E_+^2 - s}\right)\right)}{\mathrm{fh}\left(\frac{x}{4m}\left(E_+ - \sqrt{E_+^2 - s}\right)\right)}\right]$$

Weinberg's Higgs portal model

S. Weinberg, Phys. Rev. Lett. **110** (2013) no.24, 241301 [arXiv:1305.1971]

$$\mathcal{L}_{H,\phi} = (D_\mu H)^\dagger (D^\mu H) + \mu_H^2 H^\dagger H - \lambda_H (H^\dagger H)^2$$
$$+\partial_\mu \phi^* \partial^\mu \phi + \mu_\phi (\phi^* \phi)^2 - \lambda_\phi (\phi^* \phi)^2 - \kappa (H^\dagger H)(\phi^* \phi)$$
$$H_0 = v_H + \frac{\tilde{h} + iG^0}{\sqrt{2}} \qquad \phi = v_\phi + \frac{\rho + i\sigma}{\sqrt{2}}$$

Diagonalization from $(\tilde{h}\ \tilde{\rho})$ to $(h\ \rho)$: $\tan 2\theta = \frac{\kappa\, v_H v_\phi}{\lambda_H v_H^2 - \lambda_\phi v_\phi^2}$

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Convenient set of independent parameters: m_ρ, κ, λ_ϕ

Let's consider HDM annihilation into muons and pions:

$$\sigma\sigma \to \mu^+\mu^-\,, \pi\pi$$

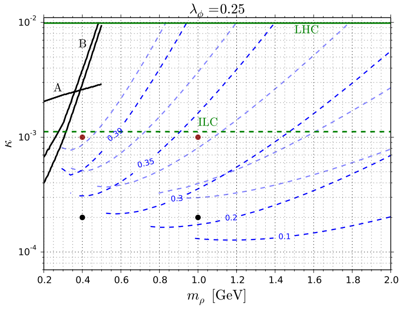
In narrow resonance approximation $(\Gamma_\rho / m_\rho)^2 \ll 1$:

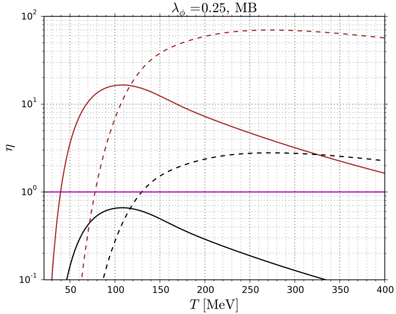
$$|M|^2 = 2\pi\kappa^2 m_\rho^3\, \delta(s - m_\rho^2) \frac{m_\mu^2 (m_\rho^2 - 4m_\mu^2) + \frac{1}{27}(m_\rho^2 + \frac{11}{2} m_\pi^2)^2}{\Gamma_\rho \left[(m_\rho^2 - m_h^2)^2 + \Gamma_h^2 m_h^2\right]}$$

Instantaneous freeze-out approximation

$$\eta(x) = \frac{\Gamma}{H} = \frac{n\langle\sigma v\rangle}{H}\bigg|_{x=x_f} = 1 \quad \Longrightarrow \quad \mathcal{C}\,\frac{x_f^4}{\sqrt{g(x_f)}}\,K_1\left(\frac{x_f\, m_\rho}{m_\mu}\right) \approx 1$$

$$\mathcal{C} \equiv \frac{\kappa^2}{\lambda_\phi}\,\frac{\sqrt{45}\,\zeta(3)}{32\pi^{5/2}}\,\frac{m_\rho^5 M_{\rm Pl}}{m_h^4 m_\mu^2}\left[\left(1-\frac{4m_\mu^2}{m_\rho^2}\right)^{3/2} + \frac{1}{27}\,\frac{m_\rho^2}{m_\mu^2}\left(1+\frac{11}{2}\,\frac{m_\pi^2}{m_\rho^2}\right)^2\left(1-\frac{4m_\mu^2}{m_\rho^2}\right)^{1/2}\right]$$





A, B – astrophysical bounds [arXiv:1706.08340]

Comparison of different statistics & approximations

$$\mathscr{D}\Phi_{\mathrm{fBE}} \approx \mathscr{D}\Phi_{\mathrm{MB}} \times \zeta^{\mp}$$

$$\mathscr{D}\Phi_{\mathrm{fFD}} \approx \mathscr{D}\Phi_{\mathrm{MB}} \times \zeta^{\mp}$$

$$\mathscr{D}\Phi_{\mathrm{pBE}} \approx \mathscr{D}\Phi_{\mathrm{MB}} \times \frac{1}{\zeta^{\mp}} \coth\left(\frac{m_\rho x}{4 m_F}\right)$$

$$\mathscr{D}\Phi_{\mathrm{pFD}} \approx \mathscr{D}\Phi_{\mathrm{MB}} \times \frac{1}{\zeta^{\mp}} \tanh\left(\frac{m_\rho x}{4 m_F}\right)$$

$$\mathscr{D}\Phi_{\mathrm{BEFD}} \approx \mathscr{D}\Phi_{\mathrm{MB}}$$

$$\mathscr{D}\Phi_{\mathrm{FDBE}} \approx \mathscr{D}\Phi_{\mathrm{MB}}$$

$$\mathscr{D}\Phi_{\mathrm{BEBE}} \approx \mathscr{D}\Phi_{\mathrm{MB}} \times \coth^2\left(\frac{m_\rho x}{4 m_F}\right)$$

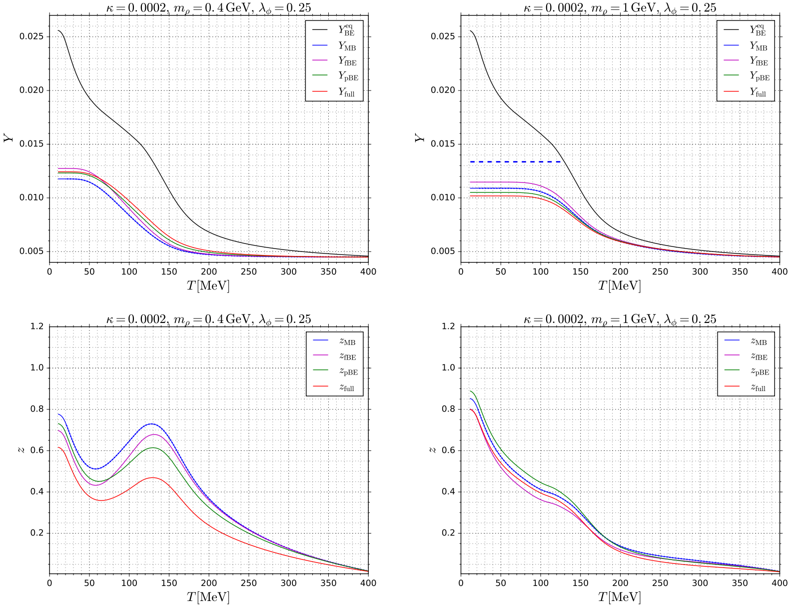
$$\mathscr{D}\Phi_{\mathrm{FDFD}} \approx \mathscr{D}\Phi_{\mathrm{MB}} \times \tanh^2\left(\frac{m_\rho x}{4 m_F}\right)$$

where

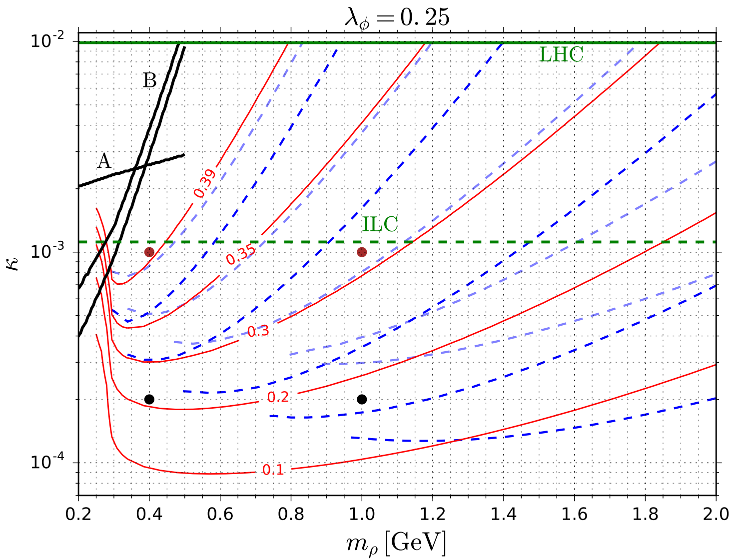
$$\mathscr{D}\Phi_{\mathrm{MB}} = e^{-2z} \frac{|M|^2}{2x} \sqrt{\frac{m_\rho^2}{m_F^2} - 4}\, K_1\left(\frac{x\, m_\rho}{m_F}\right)$$

- Cancellation for mixed statistics $(\mu^+\mu^-)$
- Amplification for BE $(\pi\pi)$
- Suppression for FD

Example: $m_\rho = 0.4$ GeV $\quad$ $m_\rho = 1$ GeV



Numerical scan



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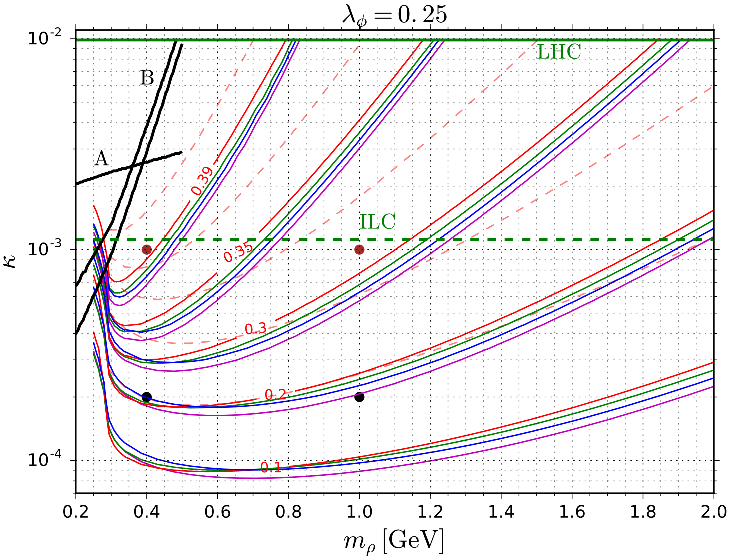
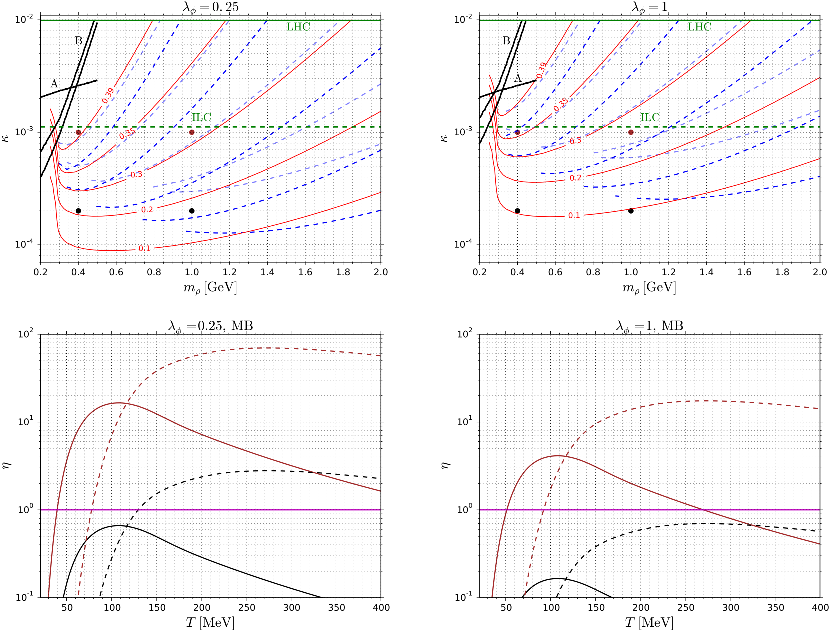


Conclusions

- We compared a few approaches for ΔN_{eff} estimation in the Weinberg's model.
- Instantaneous freeze-out approximation breaks down for small Higgs portal coupling κ.
- In large part of the parameter space pure MB approximation works better than the approach which fractionally includes the BE statistics (fBE).
- Statistics of both incoming and outgoing particles influence the results.
- In order to obtain higher accuracy Boltzmann equation with full statistics should be considered, especially in the light of the CMB-S4 experiment.

Backup slides



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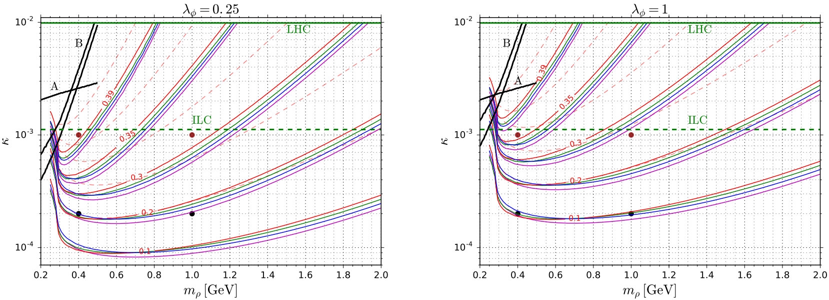

Backup slides

Artificial model with:

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and 4 possible combinations of particles' statistics.