Vacuum energy and Casimir energy

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0 Conventions

metric signature: $(-,+,+,+)$

$$\varkappa = \frac{8\pi G}{c^4} \equiv \frac{8\pi}{m_{Pl}^2} \equiv \frac{1}{M_{Pl}^2}$$

1. 'Bare', 'classical' and 'quantum' cosmological constants (CC)

Einstein-Hilbert action:

$$\boxed{S = \frac{1}{2\varkappa}\int d^4x\sqrt{-g}\,(R - 2\Lambda_B) + S_{matter}[g_{\mu\nu}, \psi]} \quad (1)$$

Λ_B - merely a new parameter of the total action and is not fixed by the structure of the theory $\Rightarrow$ "bare" cosmological constant.

$$[\Lambda_B] \doteq \frac{1}{L^2}$$

Einstein equations of motion: $\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_B g_{\mu\nu} = \varkappa T_{\mu\nu}}$, (2)

where $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}$.

QFT: $\boxed{\langle 0|T_{\mu\nu}|0\rangle = -\rho_{vac}\, g_{\mu\nu}}$ (3), ρ_{vac} = constant energy density of vacuum.

Example: scalar field, $S_\phi = -\int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right]$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right]$$

vacuum = minimum energy $\Rightarrow \partial\phi \overset{!}{=} 0$. $V \overset{!}{=} \min \Rightarrow \boxed{T_{\mu\nu} = -V(\phi_{min})\,g_{\mu\nu}}$

$\boxed{\text{For vacuum } \rho = -p}$

There are 2 sources for vacuum energy (3):

- $V(\phi_{min}) \longrightarrow$ "classical" contribution.
- zero-point fluctuations of the ground state. $\longrightarrow$ "quantum-mechanical" contribution.

Let's assume that zero-point fluctuations gravitate (see below!);

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_B g_{\mu\nu} = \varkappa T^{matter}_{\mu\nu} + \boxed{\varkappa \langle T_{\mu\nu} \rangle}$$

$$\Downarrow$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda_{eff}\, g_{\mu\nu} = \varkappa T^{matter}_{\mu\nu}}, \quad \boxed{\Lambda_{eff} = \Lambda_B + \varkappa \rho_{vac}} \quad (4)$$

1.1. Classical cosmological constant problem.

1.1.1. EW contributions.

$$V(H) = \frac{\mu^2}{2}(H^+H) + \frac{\lambda}{4}(H^+H)^2. \qquad (5).$$

Higgs doublet can be chosen as $H = \begin{pmatrix} 0 \\ \frac{h+v}{\sqrt{2}} \end{pmatrix}$.

1) $\mu^2 > 0 \Rightarrow v = 0 \Rightarrow \boxed{V_{min} = 0} \longleftarrow$ before EWPT

2) $\mu^2 < 0 \Rightarrow v = \sqrt{-\frac{2\mu^2}{\lambda}} \Rightarrow V_{min} = -\frac{\mu^4}{4\lambda}$;

$$m_h^2 = \frac{\lambda v^2}{2};\ \mu^2 = -m_h^2 \Rightarrow \boxed{V_{min} = -\frac{m_h^2 v^2}{8}} \longleftarrow \text{after EWPT}$$

$m_h = 125$ GeV, $v = 246$ GeV $\Rightarrow V_{min} = -1{,}2 \cdot 10^8\ \text{GeV}^4 \simeq -10^{55} \rho_{cri}$

In principle, one can always adjust the vacuum energy by adding $\frac{m_h^2 v^2}{8}$ to $V(H) \Rightarrow$ today $\rho^{EW}_{vac} = 0$, but huge before EWPT

1.1.2. QCD contribution.

$$\underbrace{SU_L(3)\times SU_R(3)}\longrightarrow SU_V(3) \Rightarrow$$ 8 pseudo Goldstone fields: $\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta$

$\hookrightarrow$ approximate symmetry $(m_s\to 0, m_d\to 0, m_u \to 0)$.

$$\langle \bar{u}u\rangle = \langle \bar{d}d\rangle = \langle \bar{s}s\rangle \neq 0$$

$\hookrightarrow \rho_{vac}^{QCD}$ is rather model dependent but generally

$$\rho_{vac}^{QCD}\sim\Lambda_{QCD}^4,\quad \Lambda_{QCD}\sim 200\div 300\ \mathrm{MeV} \Rightarrow \boxed{\rho_{vac}^{QCD}\simeq 10^{-3}\div 10^{-2}\ \mathrm{GeV}^4 \approx \approx 10^{44}\div 10^{45}\rho_{cr}}$$

1.2 The quantum mechanical cosmological problem.

1.2.1. Toy model.

Assumptions: free massive scalar field, flat space-time.

$$V(\varphi)=\frac{m^2\varphi^2}{2}.$$

Klein-Gordon equation: $-\ddot{\varphi}+\Box\varphi-m^2\varphi=0$,

$$\varphi(t,\vec{x})=\frac{1}{(2\pi)^{3/2}}\int\frac{d^3\vec{k}}{\sqrt{2\omega(k)}}\left(c_k e^{-i\omega t+i\vec{k}\vec{x}}+c_k^+ e^{i\omega t-i\vec{k}\vec{x}}\right),\quad \omega(k)=\sqrt{k^2+m^2}.$$

Quantization: $[c_k, c_{k'}^+]=\delta^{(3)}(\vec{k}-\vec{k}')$.

$$T_{00}=\mathcal{H}=\frac{1}{2}\dot{\varphi}^2+\frac{1}{2}\partial_i\varphi\partial_i\varphi+\frac{m^2\varphi^2}{2}.$$

$$\langle\rho\rangle=\langle T_{00}\rangle=\frac{1}{2}\langle 0|\dot{\varphi}^2|0\rangle+\frac{1}{2}\langle 0|\partial_i\varphi\partial_i\varphi|0\rangle+\frac{m^2}{2}\langle 0|\varphi^2|0\rangle=$$

$$=\boxed{\frac{1}{(2\pi)^3}\frac{1}{2}\int d^3\vec{k}\,\omega(k)}\qquad (6).$$

$$\langle p\rangle = \langle 0|\tfrac{1}{3}\perp^{\mu\nu} T_{\mu\nu}|0\rangle, \text{ where } \perp^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \to \text{projector}$$

$$u^\mu = (1,\vec{0}).$$

$$\langle p\rangle = \langle 0|\tfrac{1}{2}\dot\varphi^2 - \tfrac{1}{6}\partial_i\varphi\partial_i\varphi - \tfrac{m^2\varphi^2}{2}|0\rangle = \boxed{\frac{1}{(2\pi)^3}\frac{1}{6}\int d^3\vec{k}\,\frac{k^2}{\omega(k)}} \qquad (7)$$

$$\langle 0|T_{0i}|0\rangle = 0$$

From (4) & (6): $$\boxed{\Lambda_{eff} = \Lambda_B + \frac{\varkappa}{(2\pi)^3}\underbrace{\int d^3\vec{k}\,\frac{\omega(k)}{2}}_{\text{divergent!}}} \qquad (8)$$

<u>No gravity</u>: only differences of energy can be detected $\Rightarrow$

<u>gravity</u>: the vacuum energy weighs and can be measured $\Rightarrow$ we need careful treatment of these divergences

<u>1.2.1.1. Cut-off regularization</u> [1,2,3]

$$\langle \rho\rangle = \frac{1}{(2\pi)^3}\frac{1}{2}\int d^3\vec{k}\,\omega(k) = \frac{1}{4\pi^2}\int_0^\infty dk\,k^2\sqrt{k^2+m^2} \underset{\text{reg.}}{\Rightarrow} \frac{1}{4\pi^2}\int_0^M dk\,k^2\sqrt{k^2+m^2} =$$

$$= \frac{M^4}{16\pi^2}\left[\sqrt{1+\frac{m^2}{M^2}}\left(1+\frac{m^2}{2M^2}\right) - \frac{m^4}{2M^4}\ln\left(\frac{M}{m}+\frac{M}{m}\sqrt{1+\frac{m^2}{M^2}}\right)\right] \qquad (9)$$

$$\langle p\rangle \overset{\text{reg}}{\Rightarrow} \frac{1}{3}\frac{1}{4\pi^2}\int_0^M dk\,\frac{k^4}{\sqrt{k^2+m^2}} = \frac{1}{3}\frac{M^4}{16\pi^2}\left[\sqrt{1+\frac{m^2}{M^2}}\left(1-\frac{3}{2}\frac{m^2}{M^2}\right) + \frac{3}{2}\frac{m^4}{M^4}\ln\left(\frac{M}{m}+\frac{M}{m}\sqrt{1+\frac{m^2}{M^2}}\right)\right] \qquad (10)$$

naively $\sim M^4$

$$M\to\infty \qquad \langle\rho\rangle \to \frac{M^4}{16\pi^2}\left(1+\frac{m^2}{M^2}\right)+\dots;\quad \langle p\rangle \to \frac{1}{3}\frac{M^4}{16\pi^2}\left(1-\frac{m^2}{M^2}\right)+\dots$$

$$\boxed{\frac{\langle\rho\rangle}{\langle p\rangle} \neq -1} \;! \qquad \boxed{\text{but}} \to \quad \langle\rho\rangle^{\log} \to -\frac{m^4}{32\pi^2}\ln\left(\frac{M}{m}+\frac{M}{m}\sqrt{1+\frac{m^2}{M^2}}\right)$$

$$\langle p\rangle^{\log} \to \frac{m^4}{32\pi^2}\ln\left(\frac{M}{m}+\frac{M}{m}\sqrt{1+\frac{m^2}{M^2}}\right)$$

and it is not a coincidence!

$$\boxed{\text{Cut-off regularization breaks Lorentz invariance!}}$$

1.2.1.2 Dimensional regularization [1,2,3]

$$\varphi(t,\vec{x}) = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int \frac{d^{d-1}\vec{k}}{\sqrt{2\omega(k)}} \left(c_k e^{-i\omega t + i\vec{k}\vec{x}} + c_k^{+} e^{i\omega t - i\vec{k}\vec{x}} \right)$$

$$\Downarrow$$

$$\langle \rho \rangle = \frac{\mu^{4-d}}{(2\pi)^{d-1}} \frac{1}{2} \int d^{d-1}\vec{k}\, \omega(k) = \frac{\mu^{4-d}}{(2\pi)^{d-1}} \frac{1}{2} \int_0^\infty dk\, k^{d-2} d^{d-2}\Omega\, \omega(k)$$

Angular integral in d dimensions: $\int d^{d-2}\Omega = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}$.

$$\langle \rho \rangle = \frac{\mu^{4-d}}{(2\pi)^{d-1}} \frac{1}{2} \int_0^\infty dk\, k^{d-2} \cdot \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} (k^2+m^2)^{1/2} = \left[u = \frac{k^2}{m^2} \right] =$$

$$= \frac{\mu^4}{2(4\pi)^{\frac{d-1}{2}}} \frac{\Gamma\left(-\frac{d}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)} \left(\frac{m}{\mu}\right)^d$$

recall factor 1/6 in $d=4$!

The same for pressure: $\langle p \rangle = \frac{\mu^{4-d}}{(2\pi)^{d-1}} \frac{1}{2(d-1)} \int d^{d-1}\vec{k}\, \frac{k^2}{\omega(k)} =$

$$= \frac{\mu^4}{4(4\pi)^{(d-1)/2}} \frac{\Gamma(-d/2)}{\Gamma(1/2)} \left(\frac{m}{\mu}\right)^d = \left[\Gamma(-1/2) = -2\Gamma(1/2) \right] = \frac{-\mu^4}{2(4\pi)^{\frac{d-1}{2}}} \frac{\Gamma\left(-\frac{d}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)} \left(\frac{m}{\mu}\right)^d =$$

$$= -\langle \rho \rangle.$$

Dim. reg. $\Rightarrow$ $\langle \rho \rangle = -\langle p \rangle$.

$d = 4 - 2\epsilon$

$$\boxed{\langle \rho \rangle = -\frac{m^4}{64\pi^2} \left(\frac{1}{\epsilon} + \frac{3}{2} - \gamma_E - \ln\left(\frac{m^2}{4\pi\mu^2}\right) + O(\epsilon) \right)} \quad (11).$$

$\overline{MS}$ scheme: subtract terms proportional to $\frac{1}{\bar{\epsilon}} = \frac{1}{\epsilon} - \gamma_E + \ln 4\pi$

Following ref. [1] we will also subtract $-\frac{m^4}{64\pi^2} \times \frac{3}{2}$ term.

Finally: $\boxed{\langle\rho\rangle_{ren} = \frac{m^4}{64\pi^2}\ln\left(\frac{m^2}{\mu^2}\right)}$ (12) → We will discuss result (12) in the following

1.2.2. The vacuum and Feynman diagrams

1.2.2.1. Free theory

Trace of energy-momentum tensor:

$$\langle T^\mu_\mu\rangle = \langle T\rangle = -\langle\rho\rangle + (d-1)\langle p\rangle$$

We derived expressions for $\langle\rho\rangle$ and $\langle p\rangle$ in the previous section:

$$\langle\rho\rangle = \frac{\mu^{4-d}}{(2\pi)^{d-1}}\frac{1}{2}\int d^{d-1}\vec{k}\,\omega(k)\ ;\quad \langle p\rangle = \frac{\mu^{4-d}}{(2\pi)^{d-1}}\frac{1}{2(d-1)}\int d^{d-1}\vec{k}\,\frac{k^2}{\omega(k)}.$$

Thus, we have

$$\langle T\rangle = -\mu^{4-d}\int\frac{d^{d-1}\vec{k}}{(2\pi)^{d-1}}\,\frac{m^2}{2\omega(k)}$$

Feynman propagator in free theory $(d=4)$:

$$D_F(x-y) = \frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2+m^2-i\epsilon}\,e^{ik(x-y)}$$

Let us calculate $D_F(0)$:

$$D_F(0) = \frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2+m^2} = \left[\omega(k)=\sqrt{\vec{k}^2+m^2}\right] = \frac{i}{(2\pi)^4}\int\frac{dk^0\,d^3\vec{k}}{-(k^0)^2+\omega^2},$$

$$\int_{-\infty}^{+\infty}\frac{dk^0}{-(k^0)^2+\omega^2-i\epsilon} = -\frac{\pi}{i\omega} \Rightarrow D_F(0) = -\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{1}{2\omega}$$

Finally: $\boxed{\langle T\rangle = m^2 D_F(0)}$ (13).

In $d=4$ $\langle T\rangle = -\langle\rho\rangle + 3\langle p\rangle = \left[\langle\rho\rangle = -\langle p\rangle\right] = -4\langle\rho\rangle$

$$\boxed{\langle\rho\rangle = -\frac{m^2}{4}D_F(0)} \qquad (14).$$

Feynman diagram for propagator:

$$\equiv D_F(x_1 - x_2).$$

Hence $D_F(0)$ can be represented as a "bubble diagram":

$$\equiv D_F(0)$$

$$\boxed{\Lambda_{eff} = \Lambda_B - \varkappa \frac{m^2}{4}\ \text{(bubble)}}$$

1.2.2.2. $\lambda\varphi^4$ theory.

$\mathcal{L}_{int} = \frac{\lambda\varphi^4}{4!} \Rightarrow$ new contribution to the energy density of the vacuum

symmetry factor

$$\Delta\rho = \frac{\lambda}{4!}\langle\varphi^4\rangle = \frac{3\lambda}{4!}\langle\varphi^2\rangle^2 = \frac{\lambda}{8}D_F^2(0) =$$

$$= \frac{i}{8\int d^4x}$$

$\boxed{\text{Need to renormalize mass!}}$

1-loop mass renormalization in φ^4: $\underbrace{m^2_{ren}}_{\text{renorm. mass}} = \underbrace{m^2}_{\text{bare mass}} - \underbrace{\frac{\lambda}{2}D_F(0)}$

$$\langle\rho\rangle = -\frac{m^2}{4}D_F(0) + \frac{\lambda}{8}D_F^2(0) = -\frac{m^2_{ren}}{4}D_F(0) - \frac{1}{4}\left[\frac{\lambda}{2}D_F(0)\right]D_F(0) +$$

$$+ \frac{\lambda}{8}D_F^2(0) = -\frac{m^2_{ren}}{4}D_F(0).$$

At one loop the presence of interactions has not modified the expression of the vacuum energy if the final result is expressed in terms of the renormalized mass rather than the bare mass.

Remarks:

(i) The fact that "naive cut-off scheme" led us to quartic divergences $\langle\rho\rangle \sim M^4$ and non-physical behaviour of $\langle\rho\rangle$ ($\langle\rho\rangle \neq -\langle p\rangle$) does not mean by itself that there is no relativistically invariant regularization that leads to $O(M^4)$ terms in $\langle\rho\rangle$. However, eq. (14) rules out this possibility.

(ii) Bubble diagrams can be used in QM for e.g. calculation of shifts of energy levels in case of small interaction. [1]

Example: $$H = \frac{m\dot{x}^2}{2} + \frac{m\omega^2 x^2}{2} + \frac{\lambda m^2 x^4}{4!} = H_0 + H_{int},$$

$$H_0 = \frac{\omega}{2}(cc^+ + c^+c), \text{ where } [c, c^+] = 1,$$

$$H_{int} = \frac{\lambda m^2 x^4}{4!}$$

One can define "propagator" $D(t_1, t_2) = \langle 0 | T[\phi(t_1)\phi(t_2)] | 0\rangle$,

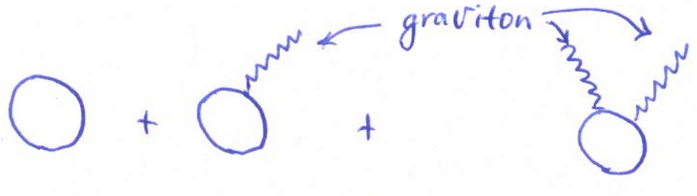
$$\phi = \sqrt{m}\, x.$$

$$D(t_1, t_2) = \frac{1}{2\omega} e^{-i\omega|t_2 - t_1|}.$$

Standard perturbation theory	Calculation based on propagator
$E_n = E_n^0 + \langle n \| H_{int} \| n\rangle$, $E_n - E_n^0 = \Delta E_n$	$\Delta E_0 = \frac{\lambda}{4!}\langle \phi^4\rangle = \frac{\lambda}{8} D^2(0) = \frac{\lambda}{32\omega^2}$.
$\Delta E_0 = \frac{\lambda}{32\omega^2}$.	However,
	$\Delta E_0 = \frac{i}{8\int dt} \infty < +\infty.$ $\Rightarrow$
	$\Rightarrow$ $\boxed{\infty \text{ is divergent even in QM}}$

(iii) In non gravitational physics bubble diagrams always cancel out in the equations describing observable quantities (see e.g. Peskin - Schroeder ch. 4.4.

However, when (classical) gravity is switched on the equivalence principle tells us that all forms of energy gravitate, and the vacuum energy is no exception. $\Rightarrow$ $\boxed{\text{careful renormalization is needed!}}$ [4]



$$+ \ldots \Rrightarrow -V_{vac} \int d^4x \sqrt{-g}.$$

(iv) Result (12) can be rederived with the help of effective potential: $\sim \frac{i}{2} \mathrm{tr}\left[\log\left(-i\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\right)\right] = -\frac{m^4}{(8\pi)^2}\left[-\frac{2}{\epsilon} + \log\left(\frac{m^2}{4\pi\mu^2}\right) + \gamma_E - \frac{3}{2}\right]\int d^4x$

and non-perturbative Gaussian effective potential (which is, in a sense, generalization of the effective potential). $\to$ [1]

For extensive discussion of Gaussian effective potential see.. [1]

(v) Before we considered flat spacetime. In a curved spacetime the curvature only influences the modes with the wavelength $k^{-1} \gtrsim a$, where a is the local curvature radius.

$\boxed{\text{UV divergences}} \Rightarrow \boxed{k^{-1} \ll a} \Rightarrow$ $\boxed{\text{curvature of space-time is negligible.}}$ [2]

For rigorous proof of this fact see [1]

(vi) Contributions of the other SM fields:

$$\boxed{\rho_{vac} = \sum_n (-1)^{2S_n}(2S_n+1)\frac{m_n^4}{64\pi^2}\ln\left(\frac{m_n^2}{\mu^2}\right)} \quad (15)$$

$\boxed{\text{NOTE: Eq. (15) is derived in } \overline{DR} \text{ scheme. } \overline{MS} \text{ leads to additional } \sim m_n^2 \text{ contributions}}$ [5]

1.3 Cosmological constant problem: two sides of the problem.

1.3.1. The value of the cosmological constant

$$\rho_{vac} = \underbrace{\sum_n (-1)^{2S_n}(2S_n+1)\frac{m_n^4}{64\pi^2}\ln\left(\frac{m_n^2}{\mu^2}\right)}_{\boxed{\text{zero-point fluctuations}}} + \underbrace{\rho_B}_{\boxed{\text{bare CC}}} + \underbrace{\rho^{EW}_{vac}}_{\boxed{EW \sim v^4}} + \underbrace{\rho^{QCD}_{vac}}_{\boxed{QCD \sim \Lambda^4_{QCD}}} + \boxed{\ldots\ldots}$$

1.2 $\to$ zero-point fluctuations; 1 $\to$ bare CC; 1.1.1 $\to$ $EW \sim v^4$; 1.1.2 $\to$ $QCD \sim \Lambda^4_{QCD}$; $\boxed{\ldots\ldots} \to$ other possible phase transitions

Remark: Photon does not contribute to ρ_{vac} contrary to what a sharp cut-off calculation would predict.

We use photons coming from the supernovae to determine the CC $\to$ $\boxed{\lambda_\gamma \simeq 500\,\text{nm}} \Rightarrow \boxed{E_\gamma \simeq 2{,}48\,\text{eV}}$

Photons couple to the metric which in turn depends on the CC $\to$ $\boxed{E_{grav} \simeq H_0 \simeq 3{,}7 \cdot 10^{-41}\,\text{GeV}}$

relevant renormalization scale to consider $\sim \mu \sim \sqrt{E_\gamma E_{grav}} \sim 3 \cdot 10^{-25}\,\text{GeV}$ [3]

This leads to $\boxed{\rho_{vac} \simeq -2 \cdot 10^{8}\,\text{GeV}^4 + \rho_B + \rho^{EW}_{vac} + \rho^{QCD}_{vac} + \ldots}$

Experimentally: $\boxed{\rho_{vac} \simeq +10^{-47}\,\text{GeV}^4}$. $\to$ ~54 orders of magnitude difference

1.3.2. Radiative instability (see [4] for details)

Back to $\lambda\varphi^4$ theory: $V^{\varphi, 1L}_{vac} \sim -\frac{m^4}{64\pi^2}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2} + \text{finite}\right)$

counterterm: $\Lambda^{1L} \sim \frac{m^4}{64\pi^2}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{M^2}\right) \to$ depends on an arbitrary substraction scale M

$$\Lambda^{1L}_{ren} \sim \frac{m^4}{64\pi^2}\left[\ln\frac{m^2}{M^2} - \text{finite}\right]$$

$\boxed{\text{measurement } \Lambda^{1L}_{ren} \text{ in cosmology}} \Rightarrow \boxed{\text{fixing } M} \Rightarrow \boxed{\text{need of } 10^{-54} \text{ accuracy cancelation}}$

2 loop V^{ϕ}_{vac} spoils this cancelation completely $\Rightarrow$ need to retune finite contributions in 1 loop counterterm.

2 loop retuned $\Rightarrow$ ~~the~~ cancellation spoiled at 3 loops, etc.

‖ Vacuum energy is uber-sensitive to the details of UV, which we don't know. [4]

$\boxed{2}$ Casimir energy [9, 10, 11]

(In this section we use the following notation for metric $\eta_{\mu\nu} = (1,-1,-1,-1)$)

Toy model: (1+1)-dimensional massless scalar field, confined between two "plates". The distance between "plates" is a.

$$\begin{cases} \partial_\mu \partial_\mu \varphi = 0 \\ \varphi(0,t) = \varphi(a,t) = 0 \end{cases}$$

We seek solution in the form $\varphi(x,t) = f(x)\cdot\psi(t)$, where $f(0) = f(a) = 0$

We obtain $f_n(x) = \sin\left(\frac{\pi n}{a}x\right)$, $n = 1, 2, 3, \dots$, $\boxed{k_n = \frac{\pi n}{a}}$

General solution for $\varphi(x,t) = \sum_{n=1}^{+\infty} \sin\left(\frac{\pi n}{a}x\right)\left(A_n^+ e^{ik_n t} + A_n^- e^{-ik_n t}\right)$.

$$\mathcal{H} = \int_0^a dx\left(\frac{\dot\varphi^2}{2} + \frac{(\partial_x\varphi)^2}{2}\right) = \dots = \frac{1}{2}\sum_{n=1}^{\infty} k_n\left(a_n^+ a_n^- + a_n^- a_n^+\right), \quad a_n^{\pm} = \sqrt{a k_n}\, A_n^{\pm}$$

Quantization:

$$[a_n^+, a_m^+] = 0 \qquad a_n^- |0\rangle = 0$$

$$[a_n^-, a_m^-] = 0 \qquad \langle 0| a_n^+ = 0$$

$$[a_m^-, a_n^+] = \delta_{mn}$$

$$\langle 0| T_{00}^{(2)} |0\rangle = \langle 0| \frac{\dot{\varphi}^2}{2} + \frac{(\partial_x \varphi)^2}{2} |0\rangle = \frac{1}{2a} \sum_{n=1}^{\infty} k_n = \boxed{\frac{\pi}{2a^2} \sum_{n=1}^{\infty} n} \qquad (16)$$

In the absence of plates:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{dk_1}{\sqrt{2k_0}} \left(a^+(k_1) e^{ikx} + a^-(k_1) e^{-ikx}\right)\Big|_{k_0 = +|k_1|}.$$

Calculation similar to (16) gives:

$$\langle 0| T_{00}^{(1)} |0\rangle = \int_0^{+\infty} \frac{dk_0}{2\pi} k_0 . \qquad (17)$$

Change in the vacuum energy density associated with the presence of plates:

$$\langle \Delta T_{00} \rangle = \frac{\pi}{2a^2} \sum_{n=1}^{\infty} n - \int_0^{+\infty} \frac{dk_0}{2\pi} k_0 = \left[u = \frac{k_0 a}{\pi}\right] = \frac{\pi}{2a^2} \left(\sum_{n=1}^{\infty} n - \int_0^{+\infty} du\, u\right) \qquad (18)$$

To calculate the difference between an infinite sum and an infinite integral in (18) we introduce regulator $f\left(\frac{x}{a\Lambda}\right)$, where $[\Lambda] = \frac{1}{L}$

$$\sum_{n=1}^{\infty} n f\left(\frac{n}{a\Lambda}\right) - \int_0^{+\infty} du\, u\, f\left(\frac{u}{a\Lambda}\right) = \lim_{M\to\infty} \left[\sum_{n=1}^{M} n f\left(\frac{n}{a\Lambda}\right) - \int_0^{+M} du\, u\, f\left(\frac{u}{a\Lambda}\right)\right] =$$

$$= \left[F(x) = x f\left(\frac{x}{a\Lambda}\right)\right] = \underbrace{\left\{\frac{F(0)+F(M)}{2} + \frac{F'(M)-F'(0)}{12} + \dots \underbrace{B_j}_{\text{Bernoulli numbers}} \frac{F^{(j-1)}(M)-F^{(j-1)}(0)}{j!} + \dots\right\}}_{\lim_{M\to\infty}}$$

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$$F^{(j)}(x) = \sum_{k=0}^{j} C_j^k \, x^{(k)} \left(\frac{1}{a\Lambda}\right)^{j-1} f^{(j-k)}(x) = \left(\frac{1}{a\Lambda}\right)^{j-1} \left[\frac{x}{a\Lambda} f^{(j)}(x) + j\, f^{(j-1)}(x)\right].$$

Assume that $f(x)$ dies sufficiently fast:

$$\boxed{\lim_{x\to\infty} x f^{(j)}(x) = 0}$$

[Physical meaning: UV modes go right through the box].

$$\boxed{\begin{array}{l} F(M)\to 0 \\ M\to\infty \end{array}}$$

$F^{(j)}(M) \to 0$ when $M\to\infty$,

$F(0) = 0$

$F^{(j)}(0) = \left(\frac{1}{a\Lambda}\right)^{j-1} j\, f^{(j-1)}(0)$.

Hence we get:

$$\sum_{n=1}^{\infty} n\, f\left(\frac{n}{a\Lambda}\right) - \int_0^{+\infty} du\, u\, f\left(\frac{u}{a\Lambda}\right) = -\frac{f(0)}{12} + O\left(\frac{1}{a\Lambda}\right).$$

Another assumption: $\boxed{f(0)=1}$ [regulator does not affect the spectrum in the IR]

$$\sum_{n=1}^{\infty} n\, f\left(\frac{n}{a\Lambda}\right) - \int_0^{+\infty} du\, u\, f\left(\frac{u}{a\Lambda}\right) = -\frac{1}{12} + O\left(\frac{1}{a\Lambda}\right),$$

When setting $\Lambda\to\infty$: $\displaystyle\sum_{n=1}^{\infty} n - \int_0^{+\infty} du\, u = -\frac{1}{12}$,

$\langle \Delta T_{00}\rangle_{reg} = -\dfrac{\pi}{24a^2}$, $\Rightarrow$ Shift in the vacuum energy when the plates are present compared to the situation when they are absent:

$$\boxed{E(a) = -\frac{\pi}{24a}}$$

Corresponding force: $\boxed{\mathcal{F}(a) = -\dfrac{\partial E(a)}{\partial a} = -\dfrac{\pi}{24a^2}}$.

Observation of Casimir effect $\to$ [12]

 Another examples of Casimir force: [10]

- d-dimensional plates (massless scalar field has d+1 spatial dimensions)

$$\mathcal{E}(a) = -\frac{1}{2^{d+2}\pi^{d/2+1}}\frac{1}{a^{d+1}}\Gamma\left(1+\frac{d}{2}\right)\zeta(2+d)$$

- EM field: between conducting plates:
(3+1) $\mathcal{E}(a) = -\frac{\pi^2}{240a^4}$.

- Fermionic Casimir force

$$\mathcal{F}_f = -\frac{7\pi^2}{1920a^4}.$$

References

[1] J. Martin, Comptes Rendus Physique **13**, 566 (2012) doi:10.1016/j.crhy.2012.04.008 [arXiv:1205.3365 [astro-ph.CO]].

[2] E. K. Akhmedov, hep-th/0204048.

[3] J. F. Koksma and T. Prokopec, arXiv:1105.6296 [gr-qc].

[4] A. Padilla, arXiv:1502.05296 [hep-th].

[5] S. P. Martin, Adv. Ser. Direct. High Energy Phys. **21**, 1 (2010) [Adv. Ser. Direct. High Energy Phys. **18**, 1 (1998)] doi:10.1142/9789812839657_0001, 10.1142/9789814307505_0001 [hep-ph/9709356].

[6] S. M. Carroll, Living Rev. Rel. **4**, 1 (2001) doi:10.12942/lrr-2001-1 [astro-ph/0004075].

[7] R. Bousso, Pontif. Acad. Sci. Scr. Varia **119**, 129 (2011) [arXiv:1203.0307 [astro-ph.CO]].

[8] J. Polchinski, hep-th/0603249.

[9] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model.* Cambridge University Press, 2014.

[10] K.A.Milton. *The Casimir Effect. Physical Manifestations of Zero-Point Energy.* World Scientific, 2001.

[11] H. B. G. Casimir, Indag. Math. **10**, 261 (1948) [Kon. Ned. Akad. Wetensch. Proc. **51**, 793 (1948)] [Front. Phys. **65**, 342 (1987)] [Kon. Ned. Akad. Wetensch. Proc. **100N3-4**, 61 (1997)].

[12] S. K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997) Erratum: [Phys. Rev. Lett. **81**, 5475 (1998)]. doi:10.1103/PhysRevLett.78.5