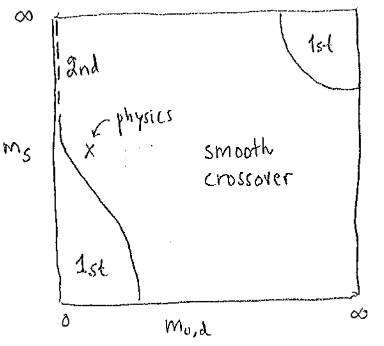
Thermal field theory and cosmological phase transitions

(M. Laine // Cargèse 2018 // http://www.laine.itp.unibe.ch/basics.pdf)

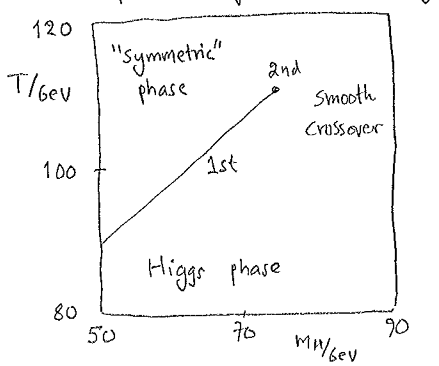
1. Introduction

Both QCD and the electroweak theory display a "broken symmetry" in vacuum, which gets "restored" at high temperatures, even if only "smoothly":

"Columbia plot" for QCD



phase diagram of EW theory



In extensions of the electroweak theory, things might go otherwise.

Given a model:
- (i) is there an actual phase transition?
- (ii) what are its "equilibrium properties" (T_c, L, σ)?
- (iii) how does it proceed in real time?
- (iv) does it leave remnants (gravitational waves, baryogenesis)?

2. Basics of quantum statistical physics

Partition function: $Z = \mathrm{Tr}\,(e^{-\beta \hat{H}})$, $\beta \equiv \frac{1}{T}$, $\hat{H}$ = Hamiltonian.

Free energy: $Z = e^{-\beta F} \iff F = -T \ln Z$.

Free energy density: $f = \frac{F}{V}$, V = volume $\to \infty$.

Let us compute these for a harmonic oscillator: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2} = \hbar\omega\,(\hat{a}^\dagger \hat{a} + \frac{1}{2})$.

Denote $\epsilon \equiv \hbar\omega$.

$$\Rightarrow Z = \sum_{n=0}^{\infty} \langle n | e^{-\beta\hat{H}} | n \rangle = \sum_{n=0}^{\infty} e^{-\beta\epsilon(n+\frac{1}{2})} = \frac{e^{-\beta\epsilon/2}}{1-e^{-\beta\epsilon}}$$

$$\Rightarrow F = \frac{\epsilon}{2} + T\ln(1-e^{-\beta\epsilon}).$$

We also need a "propagator":

$$\langle \hat{x}^2 \rangle \propto 2\frac{\partial F}{\partial \epsilon^2} = \frac{1}{\epsilon}\frac{\partial F}{\partial \epsilon} = \frac{1}{\epsilon}\left[\frac{1}{2} + \frac{e^{-\beta\epsilon}}{1-e^{-\beta\epsilon}}\right] = \frac{1}{\epsilon}\left[\frac{1}{2} + \underbrace{\frac{1}{e^{\beta\epsilon}-1}}_{\equiv n_B(\epsilon)}\right]$$

Scalar field theory is a collection of oscillators: $\varepsilon \to \varepsilon_k \equiv \sqrt{k^2+m^2}$.

$$Z_\phi = \prod_{\vec k} \exp\left\{-\frac{1}{T}\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right]\right\},$$

$$f_\phi = \lim_{V\to\infty}\frac{F_\phi}{V} = \lim_{V\to\infty}\frac{1}{V}\sum_{\vec k}\left[\frac{\varepsilon_k}{2} + T\ln\left(1-e^{-\beta\varepsilon_k}\right)\right] = \int_{\vec k}\left[\frac{\varepsilon_k}{2} + \underbrace{T\ln\left(1-e^{-\beta\varepsilon_k}\right)}_{<0}\right].$$

(vs. $F = E - TS$).

Here the thermal part $f_T(m) \equiv T\int_{\vec k} \ln\left(1-e^{-\beta\varepsilon_k}\right)$ is exponentially convergent.
It has a formal high-temperature expansion which is non-analytic in m^2:

$$f_T(m) = -\frac{\pi^2 T^4}{90} + \frac{m^2T^2}{24} - \frac{m^3 T}{12\pi} - \frac{m^4}{32\pi^2}\left[\ln\left(\frac{m}{4\pi T}\right) + \gamma_E - \frac{3}{4}\right] + \mathcal{O}\left(\frac{m^6}{\pi^4 T^2}\right) \quad ; m \equiv \sqrt{m^2}.$$

Example: Higgs condensate at high T

$$\langle\phi^\dagger\phi\rangle_T = 2\int_{\vec k}\frac{1}{\varepsilon_k}\left[\frac{1}{2} + n_B(\varepsilon_k)\right] \quad ; \quad \frac{d}{d\varepsilon_k^2} = \frac{d}{dm^2}$$

$$= (\text{vacuum part}) + 4\frac{d}{dm^2}f_T(m) = \frac{T^2}{6} + \mathcal{O}(m).$$

Thus, at $T \gg v \simeq 246$ GeV, zero-temperature masses $m_W^2 \sim \frac{g^2 v^2}{4}$ may get replaced with thermal masses $\sim g^2T^2$ (more later).

Issues of convergence

In perturbation theory, the most important domain is where propagators are largest:

$$\frac{1}{\varepsilon_k}\left[\frac{1}{2} + \frac{1}{e^{\varepsilon_k/T}-1}\right] \approx \frac{1}{\varepsilon_k}\left[\frac{1}{2} + \frac{1}{\varepsilon_k/T + \varepsilon_k^2/2T^2 + \ldots}\right] = \frac{T}{\varepsilon_k^2} + \mathcal{O}\left(\frac{1}{T}\right).$$

The large term from $\varepsilon_k \ll T$ originates from Bose-Einstein-enhancement.
There is a useful alternative interpretation:

$$\frac{1}{\varepsilon_k}\left[\frac{1}{2} + \frac{1}{e^{\varepsilon_k/T}-1}\right] \overset{!}{=} T\sum_{\omega_n}\frac{1}{\omega_n^2+\varepsilon_k^2}, \quad \omega_n \equiv 2\pi T n, \quad n \in \mathbb{Z}.$$

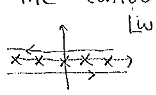
Then the large term is associated with the "Matsubara zero mode" $\omega_n = 0$.
Dimensionless loop expansion parameter (after integration over $\vec k$):

$$\frac{g^2 T}{\pi m} \sim \begin{cases} \frac{g^2}{\pi^2} & \text{if } m \gtrsim \pi T \\ \frac{g}{\pi} & \text{if } m \sim gT \\ 1 & \text{if } m \sim \frac{g^2 T}{\pi} \end{cases}$$

"Linde problem" / "IR problem"

Exercises: (i) verify the above sum by considering the contour integral

$$\int \frac{d\omega}{2\pi i}\frac{1}{\omega^2+\varepsilon^2}\, i n_B(i\omega)$$



(ii) show that the fermion propagator $\frac{1}{\varepsilon}\left[\frac{1}{2} - \frac{1}{e^{\beta\varepsilon}+1}\right]$ leads to no IR problems.

3. Phase transition in scalar field theory

The Matsubara formalism corresponds to an "imaginary-time" / "Euclidean" path integral:

$$Z = \int_{b.c.} \mathcal{D}[A^a_\mu, \phi, \psi, \bar\psi]\, e^{-S_E} ,$$

$$S_E \equiv \int_0^\beta d\tau \int_V d^3\vec{x}\; L_E ,$$

$$L_E \equiv -\mathcal{L}_M(t \to -i\tau) .$$

Here boundary conditions (b.c.) are periodic for bosons and antiperiodic for fermions.

Check: $\phi(\beta,\vec{x}) = \phi(0,\vec{x}) \Rightarrow e^{i\omega_n\beta} = 1 \Rightarrow \omega_n = 2\pi T n$ ok!

Apply this to a real scalar field with $V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$.

$$\mathcal{L}_M = \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi)$$

$$\Rightarrow L_E = \frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) .$$

Effective potential

We return to a finite volume for a moment. Let $\bar\phi$ be the mode with $\omega_n = 0, \vec{k} = \vec{0}$, and ϕ' the modes with $K = (\omega_n, \vec{k}) \neq 0$. Note that $\int_0^\beta d\tau \int_V d^3\vec{x}\, \phi' = 0$! We can write:

$$Z = \int_{-\infty}^{\infty} d\bar\phi \int \mathcal{D}\phi'\; e^{-S_E[\phi = \bar\phi + \phi']}$$

$$\equiv \int_{-\infty}^{\infty} d\bar\phi\; e^{-\frac{V}{T} V_{eff}(\bar\phi)} \approx \int_{-\infty}^{\infty} d\bar\phi\; e^{-\frac{V}{T}\left[V_{eff}(\bar\phi_{min}) + \frac{1}{2}V''_{eff}(\bar\phi_{min})(\bar\phi - \bar\phi_{min})^2 + \ldots\right]}$$

$$\Rightarrow f_\phi = V_{eff}(\bar\phi_{min}) + \mathcal{O}\left(\frac{\ln V}{V}\right) .$$

Insert $\phi = \bar\phi + \phi'$ in L_E:

$$\begin{array}{llllll}
\frac{1}{2}(\partial_\mu\phi)^2 & \to & & & \frac{1}{2}(\partial_\mu\phi')^2 & \\
-\frac{1}{2}m^2\phi^2 & \to & -\frac{1}{2}m^2\bar\phi^2 & -m^2\bar\phi\phi' & -\frac{1}{2}m^2\phi'^2 & \\
+\frac{1}{4}\lambda\phi^4 & \to & +\frac{1}{4}\lambda\bar\phi^4 & +\lambda\bar\phi^3\phi' & +\frac{3}{2}\lambda\bar\phi^2\phi'^2 & +\lambda\bar\phi\phi'^3 + \frac{1}{4}\lambda\phi'^4 \\
 & & \underbrace{}_{\text{independent of } \tau,\vec{x}} & \underbrace{}_{\text{vanishes because } \int d\tau\int d^3\vec{x}\,\phi' = 0} & \underbrace{}_{\text{like free theory but with } m^2_{eff} \equiv -m^2 + 3\lambda\bar\phi^2} & \underbrace{}_{\text{interactions}}
\end{array}$$

So, in summary:

* tree-level potential: $V^{(0)}_{eff}(\bar\phi) = -\frac{1}{2}m^2\bar\phi^2 + \frac{1}{4}\lambda\bar\phi^4$
* 1-loop potential: $V^{(1)}_{eff}(\bar\phi) = f_\phi(m_{eff}) = \int_k \left[\frac{E_k}{2} + T\ln\left(1 - e^{-\beta E_k}\right)\right]_{E_k = \sqrt{k^2 + m^2_{eff}}}$
* higher order corrections.

What can be said about the properties of the transition?

Let us insert the high-temperature expansion from p. 2 and see what kind of an effect $V_{eff}^{(1)}$ has:

* leading term $-\frac{\pi^2 T^4}{90}$ $\Rightarrow$ independent of $\bar{\phi}$ $\Rightarrow$ no effect

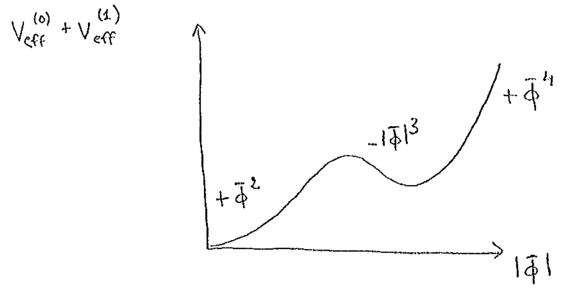
* NLO term $\frac{m_{eff}^2 T^2}{24} = \frac{-m^2 + 3\lambda\bar{\phi}^2}{24} T^2$ $\Rightarrow$ thermal mass correction:

$$V_{eff}^{(0)} + V_{eff}^{(1)} \approx [\bar{\phi}\text{-indep.}] + \frac{1}{2}\left(-m^2 + \frac{\lambda T^2}{4}\right)\bar{\phi}^2 + \frac{1}{4}\lambda\bar{\phi}^4$$

Thus symmetry seems to be restored at $T_c \sim \frac{2m}{\sqrt{\lambda}}$.

* NNLO term $-\frac{m_{eff}^3 T}{12\pi}$; let us for simplicity set $m^2 = 0$

$$\Rightarrow \quad V_{eff}^{(0)} + V_{eff}^{(1)} \approx [\bar{\phi}\text{-indep.}] + \frac{\lambda}{8}T^2\bar{\phi}^2 - \frac{T}{12\pi}(3\lambda)^{3/2}|\bar{\phi}|^3 + \frac{1}{4}\lambda\bar{\phi}^4$$



This looks like a "fluctuation induced" / "radiatively generated" first order phase transition.

However at the end we have to ask whether this prediction is reliable?

The "broken minimum" is where $\frac{T}{\pi}\lambda^{3/2}|\bar{\phi}|^3 \sim \lambda\bar{\phi}^4 \Rightarrow |\bar{\phi}| \sim \lambda^{1/2}\frac{T}{\pi}$.

Thus the dimensionless (loop) expansion parameter (p. 2) is

$$\frac{\lambda T}{\pi m_{eff}} \sim \frac{\lambda T}{\pi\sqrt{\lambda\bar{\phi}^2}} \sim \frac{\lambda^{1/2} T}{\pi|\bar{\phi}|} \sim 1 .$$

Therefore the prediction of a first-order transition is not reliable; in fact in O(N) scalar field theory the transition is of 2nd order.

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