**Cargese 2018 — Particle production during and after inflation**

**Homework 1**

Consider a perfect fluid with energy density $\rho$ and pressure $p$, in a flat, isotropic, and homogeneous Universe with scale factor $a$. All these quantities evolve in time, according to the Einstein equations

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{\rho}{3M_p^2} \\ 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 &= -\frac{p}{M_p^2} \end{aligned} \tag{1}$$

where dot denotes time differentiation, and $M_p$ is the reduced Planck mass (related to Newton constant by $8\pi G = \frac{1}{M_p^2}$).

(i) Manipulate these equations, so to obtain the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{2}$$

(ii) Denote by $w \equiv \frac{p}{\rho}$ the constant equation of state of the fluid. Use the continuity equation to obtain $\rho(a)$. (hint: write this equation as a differential, in terms of $d\rho$ and $da$; separate the $\rho$ and the $a$ dependence, and integrate; use the initial condition $\rho_{\rm in}$ at the initial value of the scale factor $a_{\rm in} = 1$)

(iii) Insert the solution you just found in the first of (1) and find $a(t)$. (hint: to simplify the algebra in the case of $w \neq -1$, choose the value of the initial time $t_{\rm in}$ so that $a = 0$ at $t = 0$).

(iv) Under which condition on $w$ is the expansion accelerated ?

(v) Find the relation between the Hubble rate $H \equiv \frac{\dot{a}}{a}$ and time $t$.

(vi) By definition, a cosmological constant has a constat energy density. What is the corresponding value of $w$ ? The energy density of non-relativistic matter rescales instead as $\rho \propto \frac{1}{\text{volume}}$. What is the corresponding value of $w$ ? The energy density of radiation instead rescales by an additional $\frac{1}{a}$ with respect to that of matter. (why is it the case ?) What is the corresponding value of $w$ ?

(vii) Write the explicit solution for $a(t)$ for a universe filled with a cosmological constant, a universe filled with non-relativistic matter, and a universe filled with radiation.

**Solutions to Homework 1**

(i) We take the time derivative of the first of (1)

$$2\frac{\dot{a}\ddot{a}}{a^2} - 2\frac{\dot{a}^3}{a^3} = \frac{\dot{\rho}}{3M_p^2} \tag{3}$$

which we rewrite as

$$\dot{\rho} = 6M_p^2\frac{\dot{a}}{a}\left[\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right] \tag{4}$$

Take the second equation in (1) minus three times the first equation in (1)

$$2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 = -\frac{p}{M_p^2} - \frac{\rho}{M_p^2} \tag{5}$$

From the last two equations we just wrote, we have

$$\dot{\rho} = 3M_p^2\frac{\dot{a}}{a} \times \left[-\frac{p}{M_p^2} - \frac{\rho}{M_p^2}\right] \tag{6}$$

which immediately gives the continuity equation indicated in the text.

(ii) We have

$$\frac{d\rho}{dt} + 3\frac{1}{a}\frac{da}{dt}\left(1+w\right)\rho = 0 \tag{7}$$

and we consider the differential

$$d\rho + 3\left(1+w\right)\frac{da}{a}\,\rho = 0 \tag{8}$$

We separate variables

$$\frac{d\rho}{\rho} = -3\left(1+w\right)\frac{da}{a} \tag{9}$$

and integrate

$$\int_{\rho_{\rm in}}^{\rho}\frac{d\rho}{\rho} = -3\left(1+w\right)\int_1^a\frac{da}{a} \tag{10}$$

This is integrated to give

$$\ln\frac{\rho}{\rho_{\rm in}} = -3\left(1+w\right)\,\ln a \tag{11}$$

and, taking the exponential of this,

$$\rho = \rho_{\rm in}\, a^{-3(1+w)} \tag{12}$$

(iii) We have

$$\frac{\dot{a}}{a} = \frac{\rho^{1/2}}{\sqrt{3} M_p} = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p}\, a^{-\frac{3(1+w)}{2}} \tag{13}$$

which we again rewrite in separate form

$$\frac{da}{a} \times a^{\frac{3(1+w)}{2}} = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p}\, dt \tag{14}$$

We need to distinguish two cases

$$\begin{aligned} w = -1 \quad &\rightarrow \quad \int_1^a \frac{da}{a} = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p} \int_{t_{\rm in}}^t dt \\ &\rightarrow \quad \ln\, a = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p}\, (t - t_{\rm in}) \\ &\rightarrow \quad a = \exp\left[\frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p}\, (t - t_{\rm in})\right] \end{aligned} \tag{15}$$

and

$$\begin{aligned} w \neq -1 \quad &\rightarrow \quad \int_1^a da\, a^{\frac{3(1+w)}{2} - 1} = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p} \int_{t_{\rm in}}^t dt \\ &\rightarrow \quad \frac{2}{3\,(1+w)} \left[a^{\frac{3(1+w)}{2}} - 1\right] = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p}\, (t - t_{\rm in}) \end{aligned} \tag{16}$$

Imposing $a = 0$ at $t = 0$ amounts in

$$\frac{2}{3\,(1+w)}\, [0 - 1] = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p}\, (0 - t_{\rm in}) \tag{17}$$

namely, the constant factors cancel, and

$$\begin{aligned} w \neq -1 \quad &\rightarrow \quad a^{\frac{3(1+w)}{2}} = \frac{3\,(1+w)}{2}\, \frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p}\, t \\ &\rightarrow \quad a = \left[\frac{3\,(1+w)}{2}\, \frac{\rho_{\rm in}^{1/2}}{\sqrt{3} M_p}\right]^{\frac{2}{3(1+w)}} \; t^{\frac{2}{3(1+w)}} \\ &\rightarrow \quad a = C\, t^{\frac{2}{3(1+w)}} \end{aligned} \tag{18}$$

where we collectively denoted all the constant factors as $C$ for brevity.

(iv) The expansion is accelerated when the exponent is greater than one. Namely

$$\frac{2}{3\,(1+w)} > 1 \quad \Rightarrow \quad 1+w < \frac{2}{3} \quad \Rightarrow \quad w < -\frac{1}{3} \tag{19}$$

(we also note that the expansion is accelerated for the specific case of $w = -1$).

(v) We have

$$\begin{aligned} w = -1 \qquad & \rightarrow a = \exp\left[\frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\,\left(t - t_{\rm in}\right)\right] \\ & \rightarrow H = \frac{\dot{a}}{a} = \frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p} = {\rm constant} \end{aligned} \tag{20}$$

and

$$\begin{aligned} w \neq -1 \qquad & \rightarrow a = C\,t^{\frac{2}{3(1+w)}} \\ & \rightarrow H = \frac{\dot{a}}{a} = \frac{2}{3(1+w)t} \end{aligned} \tag{21}$$

(vi) Let us go back to

$$\rho = \rho_{\rm in}\,a^{-3(1+w)} \tag{22}$$

We see that a cosmological constant is associated to $w = -1$. For non-relativistic matter, dilution as one over volume means instead $\rho_{\rm matter} \propto a^{-3}$. This gives $w = 0$. The energy density in radiation is diluted by an additional $\frac{1}{a}$ factor, that accounts for the loss of energy (due to redshift) of each individual photon. Therefore $\rho_\gamma \propto a^{-4}$. This is associated to $w = \frac{1}{3}$.

(vii) We have

$$\begin{aligned} & {\rm cosmological\ constant}, \quad w = -1 \quad , \quad a = \exp\left[\frac{\rho_{\rm in}^{1/2}}{\sqrt{3}M_p}\,\left(t - t_{\rm in}\right)\right] \\ & {\rm non-relativistic\ matter}, \quad w = 0 \quad , \quad a = C\,t^{2/3} \\ & {\rm radiation}, \quad w = \frac{1}{3} \quad , \quad a = C\,t^{1/2} \end{aligned} \tag{23}$$