Cargese 2018 — Particle production during and after inflation

In the next pages you will find the text of three homeworks, connected to the three lectures on "Particle production during and after inflation". The homeworks are optional, and they will not be graded. You are recommended to solve them, since they contains details of calculations that we will not have time to cover during the lectures. Please solve homework 1 before the first lecture, homework 2 before the second lecture, and homework 3 before the third lecture. This will make the lectures easier to follow. Solutions of each homework will be provided after the corresponding lecture.

Best wishes, Marco Peloso

Homework 1

Consider a perfect fluid with energy density ρ and pressure p, in a flat, isotropic, and homogeneous Universe with scale factor a. All these quantities evolve in time, according to the Einstein equations

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{\rho}{3M_p^2} \\ 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 &= -\frac{p}{M_p^2} \end{aligned} \tag{1}$$

where dot denotes time differentiation, and M_p is the reduced Planck mass (related to Newton constant by $8\pi G = \frac{1}{M_p^2}$).

(i) Manipulate these equations, so to obtain the continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{2}$$

(ii) Denote by $w \equiv \frac{p}{\rho}$ the constant equation of state of the fluid. Use the continuity equation to obtain $\rho(a)$. (hint: write this equation as a differential, in terms of $d\rho$ and da; separate the ρ and the a dependence, and integrate; use the initial condition ρ_{in} at the initial value of the scale factor $a_{\text{in}} = 1$)

(iii) Insert the solution you just found in the first of (1) and find $a(t)$. (hint: to simplify the algebra in the case of $w \neq -1$, choose the value of the initial time t_{in} so that $a = 0$ at $t = 0$).

(iv) Under which condition on w is the expansion accelerated ?

(v) Find the relation between the Hubble rate $H \equiv \frac{\dot{a}}{a}$ and time t.

(vi) By definition, a cosmological constant has a constat energy density. What is the corresponding value of w ? The energy density of non-relativistic matter rescales instead as $\rho \propto \frac{1}{\text{volume}}$. What is the corresponding value of w ? The energy density of radiation instead rescales by an additional $\frac{1}{a}$ with respect to that of matter. (why is it the case ?) What is the corresponding value of w ?

(vii) Write the explicit solution for $a(t)$ for a universe filled with a cosmological constant, a universe filled with non-relativistic matter, and a universe filled with radiation.

Homework 2

Consider a universe filled with a massive inflaton that decays into radiation with decay rate Γ_ϕ. Denote by ρ_ϕ and ρ_γ the energy densities of the inflaton and of the radiation, respectively, and by $H \equiv \frac{\dot{a}}{a}$ the Hubble rate (a is the scale factor of the Universe, and dot denotes time differentiation).

The system is governed by the equations

$$\begin{aligned} &\dot{\rho}_\phi + (3H + \Gamma_\phi)\,\rho_\phi = 0 \\ &\dot{\rho}_\gamma + 4H\rho_\gamma = \Gamma_\phi \rho_\phi \\ &\rho_\phi + \rho_\gamma = 3M_p^2 H^2 \end{aligned} \tag{3}$$

(i) Verify that, for $\Gamma_\phi = 0$, the solutions $\rho_\phi\,(a)$ and $\rho_\gamma\,(a)$ agree with what we found in homework 1, namely $\rho_\phi \propto a^{-3}$ and $\rho_\gamma \propto a^{-4}$.

(ii) We are interested in the evolution of ρ_γ at very early times. At these times $\rho_\gamma \ll \rho_\phi$, and $\Gamma_\phi \ll H$. Use these approximations in the first and third equations of the above system, and find the approximate solution for $\rho_\phi\,(a)$ and $H\,(a)$. Take the initial condition $\rho_\phi = \bar{\rho}$ at $a = 1$.

Show that the second equation in the above system can be rewritten as

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{\Gamma_\phi}{aH}\,\rho_\phi \tag{4}$$

Insert the solutions $\rho_\phi\,(a)$ and $H\,(a)$ into this equation. Solve this equation to obtain the early time solution for $\rho_\gamma\,(a)$. Take the initial condition $\rho_\gamma = 0$ (hint: it might be useful to consider the differential equation for the combination $a^4\rho_\gamma$).

(iii) Find the maximum value of $\rho_\gamma\,(a)$, and denote it by $\rho_{\rm max}$.

(iv) A common approximation done in studying this problem is to assume that the inflaton instantaneously decays at $\Gamma_\phi = H$. Use the third equation above to find the energy density of radiation at the decay obtained with this assumption. Denote it as $\rho_{\rm inst}$.

(v) Discuss the ratio $\frac{\rho_{\rm max}}{\rho_{\rm inst}}$, showing that it is parametrically given by the ratio between the initial Hubble rate and the inflaton decay rate. Notice that this quantity can be several orders of magnitude greater than one.

(vi) Given what you just found, why is the approximation of instantaneous inflaton decay so popular ?

Homework 3

A cosmological perturbation $X(t)$ obeys a second order differential equation in presence of a source $j(t)$

$$\ddot{X} + f(t)\,\dot{X} + g(t)\,X = j(t) \tag{5}$$

where f and g are some known functions, and dot denotes time derivative.

(i) Consider the Green's function, obeying

$$\left[\frac{d^2}{dt^2} + f\,\frac{d}{dt} + g\right] G(t,\, t') = \delta(t - t') \tag{6}$$

where δ is the Dirac delta-function. Show that the quantity

$$X(t) \equiv \int_{-\infty}^{+\infty} dt'\, G(t,\, t')\, j(t') \quad \text{solves (5)} \tag{7}$$

(ii) Causality indicates to look at the retarded Green's function

$$G(t,\, t') = \tilde{G}(t,\, t')\;\theta(t - t') \tag{8}$$

where θ is the Heaviside step function, and $\tilde{G}$ a regular function (notice that, once inserted into (7), it indicates that only $j(t')$ evaluated at $t' < t$ can affect the solution $X(t)$). Show that, provided that

$$\left[\frac{d^2}{dt^2} + f\,\frac{d}{dt} + g\right] \tilde{G}(t,\, t') = 0\,, \quad \tilde{G}(t,\, t) = 0 \;\;, \quad \frac{d}{dt}\tilde{G}(t,\, t')\Big|_{t'=t} = 1 \tag{9}$$

the combination (8) satisfies (6) (hint: notice that the first derivative of (8) has two terms; one vanishes, due to the second property in (9). When you compute the second derivative, do not differentiate the vanishing term !)

(iii) Construct the Green's function for the simple harmonic oscillator

$$f(t) = 0 \;\;, \quad g(t) = \omega^2 \tag{10}$$

To satisfy the first condition in (9) consider a function of the form

$$\tilde{G}(t,\, t') = h_1(t)\;\alpha(t') + h_2(t)\;\beta(t') \tag{11}$$

where h_1 and h_2 solve the homogeneous equation. Find the functions $\alpha(t')$ and $\beta(t')$ that allow to satisfy the other two conditions.

(iv) Use the Green's function that you just constructed to find the non-homogeneous solution (7) of the equation (5) for a source that switches from zero for $t < 0$ to the constant value J at $t > 0$.