# Cargèse Lectures on the Hierarchy Problem (Personal Notes)

Raffaele Tito D'Agnolo[1]

[1] *SLAC*

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## 1 Introduction

Naturalness hints for new physics.

Massless particle: Vector 2 d.o.f's, Fermion 2 d.o.f.'s, Scalar 1 d.o.f.
Massive particle: Vector 3 d.o.f's, Fermion 4 d.o.f.'s, Scalar 1 d.o.f.

A naturally small parameter is always associated to a symmetry that becomes exact when $\lambda$ goes to zero. Example with the fermion? Stress difference between fine-tuning of the Higgs and other fine-tunings (technical naturalness).

Example 1. Pions mass difference.

$$m_{\pi^+}^2 - m_{\pi^0}^2 \simeq \frac{3\alpha\Lambda^2}{4\pi} \approx (35.5 \text{ MeV})^2 \tag{1.1}$$

Hence $\Lambda \lesssim 850$ MeV. Experimentally $m_\rho = 770$ MeV. To get the estimate, start from the chiral Lagrangian

$$\Lambda_{\rm QCD}^2 f_\pi^2 \frac{\pi^\dagger \overleftrightarrow{\partial}_\mu \pi}{f_\pi^2 \Lambda_{\rm QCD}} \frac{eA^\mu}{\Lambda_{\rm QCD}} = eA^\mu \pi^\dagger \overleftrightarrow{\partial}_\mu \pi \,, \tag{1.2}$$

then write down the simplest one-loop diagram that contributes to the mass difference

$$e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2 - m_\pi^2)^2} \sim \frac{\alpha}{4\pi}\Lambda^2 \,. \tag{1.3}$$

Exercise 1. Estimate $(m_{\pi^+}^2 - m_{\pi^0}^2)$.
Example 2. Kaons mass difference.

$$\frac{m_{K_L^0} - m_{K_S^0}}{m_{K_L^0}} = \frac{G_F^2 f_K^2}{6\pi^2} \cos^2\theta_c \sin^2\theta_c \Lambda^2, \quad f_K \approx 114 \text{ MeV} \,. \tag{1.4}$$

The mass difference comes from the mixing between $K^0$ and $\overline{K}^0$:

$$\begin{pmatrix} \overline{K}^0 & K^0 \end{pmatrix} \begin{pmatrix} m_K^2 & \delta m_K^2 \\ \delta m_K^2 & m_K^2 \end{pmatrix} \begin{pmatrix} \overline{K}^0 \\ K^0 \end{pmatrix} \tag{1.5}$$

$$m_L - m_S = \sqrt{m_K^2 + \delta m_K^2} - \sqrt{m_K^2 - \delta m_K^2} \approx \frac{\delta m_K^2}{m_K} = \frac{1}{m_K} \langle \overline{K}^0 | V | K^0 \rangle \tag{1.6}$$

The potential $V$ can be obtained from an effective Lagrangian computed from box diagrams. We start with

$$\mathcal{L}_{SM} \supset -\frac{g}{\sqrt{2}} W^\mu \left[ \bar{u}\gamma_\mu \frac{(1-\gamma_5)}{2} (d\cos\theta_c + s\sin\theta_c) + \bar{c}\gamma_\mu \frac{(1-\gamma_5)}{2} (-d\sin\theta_c + s\cos\theta_c) \right], \tag{1.7}$$

and "integrate out" up and charm quarks. The dependence on $\Lambda^2$, which is actually a dependence on $m_c^2$ is due to the unitary cancellation of terms that are not proportional to quark mass differences (**check**).

Exercise 1. Estimate $(m_L - m_S)$ parametrically.
Hence $\Lambda \lesssim 2$ GeV. Experimentally $m_c = 1.2$ GeV.

After this same introductory part as at ICTP (electron self-energy and Kepler's solar system).

# 2 Effective Field Theory

Start with ICTP lecture and after the part on surprising and not surprising Lagrangians do the exercise of integrating out high momentum modes in the Wilsonian picture. Stress the fact that you can already say that there is a hierarchy problem without doing any calculation. No symmetry protects $m^2$ and hence you expect it to be proportional to the largest scale in your theory $M^2$ with an $\mathcal{O}(1)$ coefficient.

The first step is to impose a hard cut-off in Euclidean signature, $k^2 \leq M^2$ $(t \to -it_E)$ in Minkowski space is not enough. Then we can start with an action

$$S_{\rm E,toy} = \int d^dx \left( \frac{(\partial_E\phi)^2}{2} + \frac{m^2\phi^2}{2} + \overline{\Psi} i \not{\partial}_E \Psi + M_\Psi \overline{\Psi}\Psi + y\phi\overline{\Psi}\Psi \right) \tag{2.1}$$

and separate high and low momentum modes (I have written only the high momentum modes of $\Psi$)

$$\begin{aligned} S_{E,\rm toy} &= \int d^dx \sum_{i=L,H} \left( \frac{(\partial_E\phi_i)^2}{2} + \frac{m^2\phi_i^2}{2} \right) + \overline{\Psi} i \not{\partial}_E \Psi - M_\Psi \overline{\Psi}\Psi \\ &- y \int d^dx \left( \phi_H \overline{\Psi}\Psi + \phi_L \overline{\Psi}\Psi \right) . \end{aligned} \tag{2.2}$$

Aside on integrating out and Feynman diagrams. To make contact with perturbation theory we do this in Lorentzian signature. Integrating out high momentum modes means to do the functional integral

$$\int \mathcal{D}\phi_H e^{iS(\phi_H,\phi_L)} \tag{2.3}$$

In general we do not know how to do it. However we know how to do Gaussian integrals of the type

$$\int \mathcal{D}\phi e^{i\int d^dx\left[\frac{1}{2}\phi(x)(-\partial^2-m^2+i\epsilon)\phi(x)+J(x)\phi(x)\right]} \tag{2.4}$$

The $i\epsilon$ selects the time-ordered contour of integration in the path integral. We first complete the square

$$\phi'(x)=\phi(x)-i\int d^dy D_F(x-y)J(y)\,, \quad D_F(x-y)=(-\partial^2-m^2+i\epsilon)^{-1} \tag{2.5}$$

obtaining

$$\int d^dx\left[\frac{1}{2}\phi'(x)(-\partial^2-m^2+i\epsilon)\phi'(x)-\frac{1}{2}\int d^dy J(x)D_F(x-y)J(y)\right] \tag{2.6}$$

The Jacobian in the functional integral is one since the transformation is just a constant shift. The first piece is now a Gaussian integral that we know how to do it is $\propto \det(-\partial^2-m^2)$. The second piece is independent of $\phi$.

This is useful because, we might never be able to integrate out $\Psi$ exactly, but in perturbation theory we only have to do integrals of the type

$$\int \mathcal{D}\Psi e^{iS_0}(\overline{\Psi}\Psi)^n y^n \times ... \tag{2.7}$$

from expanding the exponent of (2.2) in small powers of $y$. The crucial point is that $S_0$ is Gaussian. Then we can do these integrals by noticing that

$$\int \mathcal{D}\phi\phi(z)\phi(y)...e^{iS_0(\phi)}=\frac{\delta}{\delta J(z)}\frac{\delta}{\delta J(y)}...\int \mathcal{D}\phi\phi(z)\phi(y)...e^{iS_0(\phi)+i\int d^dx\phi(x)J(x)}\Big|_{J=0} \tag{2.8}$$

and we know how to do the integral on the right-hand side. Taking derivatives of (2.6) wrt $J$ gives you products of propagators. From here emerges the usefulness of Feynman diagrams which are just book-keeping devices for the propagators. The choice of putting $i\epsilon$ in the functional integral is regulating it in such a way that we are considering time-ordered products of free fields (i.e. the Gaussian-free Lagrangian is describing the physical propagation of the particles).

After this integrate out the fermion in the usual straightforward way. Definition of $\delta m^2(p^2)$

$$\frac{i}{p^2-m^2-\delta m^2(p^2)}\,. \tag{2.9}$$

Then integrating out a single momentum shell gives

$$\begin{aligned} -i\delta m^2(0) &= -4y^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^2+M_\Psi^2}{\left(k^2-M_\Psi^2\right)^2} = 4iy^2 \int \frac{d\Omega_4}{(2\pi)^4} \int_M^{M+dM} dk_E k_E^3 \frac{k_E^2-M_\Psi^2}{(k_E^2+M_\Psi^2)^2} \\ &= i\frac{y^2}{2\pi^2}\left(M - \frac{3M_\Psi^2}{M} + ...\right) dM \end{aligned} \tag{2.10}$$

In the limit $dM/M_\Psi \ll 1$ and $M_\Psi/M \ll 1$. All our calculations here are carried out in the approximation $M \gg M_\Psi \gg m_\phi$. I had started with $m_\phi^2(M)$, now at $m_\phi \ll M$ I have

$$m_\phi^2(m_\phi) = -M^2 + m_\phi^2(M) + ... \tag{2.11}$$

Introduce definition of tuning $\Delta \sim \frac{m_h^2}{M^2}$. Aside. Note that if I had integrated out a scalar at one-loop in $\lambda\phi^4$ I would have obtained the opposite sign for both terms (this is physical)

$$i\delta m_{\phi^4}^2(0) = -i\frac{\lambda}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} = -i^3(-1)\frac{\lambda}{2}\int dk_E \frac{k_E^3}{k_E^2+m^2} = -i\frac{\lambda}{4\pi^2}\left(M - \frac{m^2}{M} + ...\right) dM\,. \tag{2.12}$$

Discuss the flow and only mention these other two calculations as an aside. Instead if I do the integral in the full theory with a hard momentum cut-off I get

$$-i\delta m_\Lambda^2(0) = i\frac{y^2}{2\pi^2}\left(\frac{\Lambda^2}{2} + M_\psi^2 - \frac{3M_\Psi^2}{2}\log\frac{\Lambda^2}{M_\psi^2} + ...\right) \tag{2.13}$$

In dimensional regularization we have

$$\begin{aligned} -i\delta m_{DR}^2(0) &= -4y^2\mu^\epsilon \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{k^2+M_\Psi^2}{\left(k^2-M_\Psi^2\right)^2} \\ &= \frac{iy^2}{2\pi^2}\left(-\frac{M_\Psi^2}{\epsilon} + M_\Psi^2 - \frac{3M_\Psi^2}{2}\log\frac{\mu^2}{M_\Psi^2} + \mathcal{O}(\epsilon)\right) \end{aligned} \tag{2.14}$$

Comment on $\Lambda^2$ parametrizing our short-wavelength ignorance and maybe not being physical (but maybe yes!) versus the logarithm getting equal contributions at each energy scale, having a physical IR divergence and being related to unitarity. Remark on the fact that the $M$ term is conceptually different from $\Lambda^2$ and it is parametrizing a real flow from the UV, since it is implicitly assuming that $M$ is a real scale. This makes $M$ different also from $1/\epsilon$ for the same reasons. Even if we assume that there is no real threshold at $M$ there is still fine tuning from $M_\Psi \gg m$.

Is the Higgs sensitive only to particles coupling to it or to any threshold? The answer is to any threshold in QFT. We can show this more effectively in position space [] and using dimensional regularization (since the cut-off breaks conformal invariance)

$$\begin{aligned} i\delta m^2(0) &= -y^2\mu^{4-d}\int d^dx\langle 0|T\{(\overline{\Psi}\Psi)(x)(\overline{\Psi}\Psi)(0)\}|0\rangle \\ &= -4y^2\mu^{4-d}\int d^dx e^{i(p-q)x}\int \frac{d^dk}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{k\cdot q+M_\Psi^2}{(k^2-M_\Psi^2+i\epsilon)(q^2-M_\Psi^2+i\epsilon)} \\ &= -4y^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{k^2+M_\Psi^2}{(k^2-M_\Psi^2)^2}\,. \end{aligned} \tag{2.15}$$

Imagine to have two fixed points (i.e. two CFTs) one in the IR which is approximately free (the SM) and one in the UV that we do not know. We can always parametrize

$$\langle 0|T\{(\overline{\Psi}\Psi)(x)(\overline{\Psi}\Psi)(0)\}|0\rangle = \frac{1}{x^{2(d-1)}}f(x^2M^2)\,, \tag{2.16}$$

where $M$ is the scale associated to the transition between IR and UV fixed points. Now

$$i\delta m^2(0) = -iM^2\frac{y^2\pi^{d/2}}{\Gamma(d/2)}\left(\frac{\mu^2}{M^2}\right)^{2-d/2}\int_0^\infty dy\frac{f(y)}{y^{d/2}}\,. \tag{2.17}$$

Let us consider an abrupt ($f_1$) and a smooth transition ($f_2$)

$$f_1(y) = \Theta(y-1)+\frac{\Theta(1-y)}{y^{\gamma_{UV}}}\,,\quad f_2(y) = \left(\frac{1}{1+y^{n\gamma_{UV}}}\right)^{1/n} \tag{2.18}$$

In both cases we are sensitive to $M^2$

$$\delta m_1^2(0) = -M^2\pi^2y^2\frac{\gamma_{UV}}{\gamma_{UV}-1}\,,\quad \delta m_2^2(0) = -M^2\pi^2y^2\frac{\Gamma\left(\frac{1}{n}+\frac{1}{n\gamma_{UV}}\right)\Gamma\left(\frac{1}{n}-\frac{1}{n\gamma_{UV}}\right)}{\Gamma(1/n)}\,, \tag{2.19}$$

but note that I have thrown away a UV infinite piece incompatible with conformal invariance! If $n$ is too small conformal invariance is broken in the UV and we get extra infinite pieces other than the one subtracted in arXiv:1308.0025

Exercise 3. Read arXiv:1308.0025 and get confused.

At this point SM Lagrangian and discussion of SM HP as in the ICTP lecture.

# 3 Pheno of the Little Hierarchy

Proton decay

$$\mathcal{L}\supset\frac{u^cu^cd^ce^c}{M^2}+\frac{QQQL}{M^2}+\ldots\quad \Gamma\sim\frac{m_p^5}{M^4} \tag{3.1}$$

| Operator | Bounds on $\Lambda$ in TeV ($c_{ij}=1$) | | Bounds on $c_{ij}$ ($\Lambda = 1$ TeV) | | Observables |
|---|---|---|---|---|---|
| | Re | Im | Re | Im | |
| $(\bar{s}_L\gamma^\mu d_L)^2$ | $9.8 \times 10^2$ | $1.6 \times 10^4$ | $9.0 \times 10^{-7}$ | $3.4 \times 10^{-9}$ | $\Delta m_K$; $\epsilon_K$ |
| $(\bar{s}_R\, d_L)(\bar{s}_L d_R)$ | $1.8 \times 10^4$ | $3.2 \times 10^5$ | $6.9 \times 10^{-9}$ | $2.6 \times 10^{-11}$ | $\Delta m_K$; $\epsilon_K$ |

Figure 1: Bounds on $\overline{K}^0 - K^0$ mixing

$$\begin{aligned}
\frac{\tau_p}{\mathrm{Br}(p \to e^+\pi^0)} &\gtrsim 1.7 \times 10^{34} \text{ years} \quad \text{SuperKamiokande} \\
\frac{\tau_p}{\mathrm{Br}(p \to \text{invisible})} &\gtrsim 2 \times 10^{29} \text{ years} \quad \text{SNO} \\
M &\gtrsim 3 \times 10^{16} \text{ GeV} \quad \text{SuperKamiokande} \\
M &\gtrsim 1.5 \times 10^{15} \text{ GeV} \quad \text{SNO}
\end{aligned} \tag{3.2}$$

Neutron Oscillations (mass matrix as for the case of the Kaons, oscillation probability $P(t) \sim \sin^2(\delta m t)$)

$$\mathcal{L} \supset \frac{(u^c d^c d^c)^2}{M^5} \qquad \tau_{n\to\bar{n}} = \delta m \sim \frac{m_n^6}{M^5} \tag{3.3}$$

$$\tau_{n\to\bar{n}} > 0.86 \times 10^8 \text{ s} \quad M \gtrsim 3 \times 10^6 \text{ GeV} \quad \text{ILL reactor} \tag{3.4}$$

the bound comes from the search for neutrons annihilating on a target.
Flavor, see Fig. 1 for $\overline{K}^0 - K^0$ mixing. Lepton Flavor

$$\mathcal{L} \supset \frac{m_\mu}{M^2} \bar{\mu}_L \sigma_{\mu\nu} e_R F^{\mu\nu}\,, \quad \Gamma \sim \frac{m_\mu^5}{M^4} \tag{3.5}$$

$$\mathrm{Br}(\mu \to e\gamma) < 4 \times 10^{-13} \quad M \gtrsim 3 \times 10^6 \text{ GeV} \quad \text{MEG} \tag{3.6}$$

# 4 Messing with Gravity

Lessons of the day: it's hard to mess with gravity, if you don't have a symmetry the HP always pops out somewhere else

Aside on the EFT of gravity

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G_N} R - 2\Lambda_{\mathrm{CC}} + \mathcal{L}_{\mathrm{matter}} \right) \tag{4.1}$$

$$R \sim \text{const} + \partial^2 g, \quad \frac{1}{G_N} \equiv M_{\mathrm{Pl}}^{D-2} \tag{4.2}$$

In 4D this definition of $M_{\rm Pl}$ corresponds to $M_{\rm Pl} \approx 10^{19}$ GeV. Often people do include the extra factor of $1/(16\pi)$. Aside on factors of 2

$$\frac{1}{16\pi G_N} R \to G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \to \nabla^2\Phi = 4\pi G_N \rho \to \Phi = -\frac{G_N m_1 m_2}{r} \,. \tag{4.3}$$

Higher dimensional terms in the action

$$S = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\left(R - 32\pi G_N \Lambda_{\rm CC} + 16\pi G_N \mathcal{L}_{\rm matter} + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + c_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \ldots\right) \tag{4.4}$$

Let us explore this EFT in its regime of validity (i.e. low energy and low curvature)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{4.5}$$

then the action becomes

$$S \sim M_{\rm Pl}^2 \int d^4x \left[\partial h \partial h + h\partial h \partial h + a T_{\mu\nu} h^{\mu\nu} \ldots + \frac{1}{M^2}\left(\partial^2 h \partial^2 h + h \partial^2 h \partial^2 h + \ldots\right)\right] \tag{4.6}$$

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{\rm matter}\sqrt{-g}}{\delta g^{\mu\nu}} \tag{4.7}$$

The higher dimensional terms odd in $h$ come from higher curvature corrections $R \sim {\rm const} + \partial^2 h$. Canonically normalizing the kinetic term

$$S \sim \int d^4x \left[\partial h \partial h + \frac{1}{M_{\rm Pl}} h\partial h \partial h + \frac{a}{M_{\rm Pl}} T_{\mu\nu} h^{\mu\nu} + \ldots + \frac{1}{M^2}\left(\partial^2 h \partial^2 h + \frac{1}{M_{\rm Pl}} h \partial^2 h \partial^2 h + \ldots\right)\right] \tag{4.8}$$

It's hard to define a running coupling in the EFT of gravity for reasons that I'm not going to discuss here. Heuristically you can just notice that you are starting with a dim 6 operator. If you close the loop (draw), you are going to get multiple dim 8 operators with different numerical coefficients (and possibly signs) $R^2, R_{\mu\nu}R^{\mu\nu}$. Which one are you going to pick? Read 1111.2875 for more details. However at one loop scattering processes receive corrections of order $\delta A_{2\to 2} \sim (N G_N E^2)/(16\pi^2)$, from the action that I've written above (after gauge fixing) where $N$ is the number of particles in the loop. So it is natural to expect something to happen at

$$E \sim \frac{4\pi M_{\rm Pl}}{\sqrt{N}} \,. \tag{4.9}$$

So the easiest way to solve the hierarchy problem is to imagine (I'm using $M_{\rm Pl} = 10^{19}$ GeV).

$$N \sim \frac{M_{\rm Pl}^2}{v^2} \approx 10^{33} \,. \tag{4.10}$$

Of course you have all the problems discussed above with higher dimensional operators and direct LHC searches. See Gia Dvali's papers.

$R =$ typical size of the extra dimension. In some sense large extra dimensions are an example of this. $d = 4 + n$

$$F(r) \sim \begin{cases} \frac{m_1 m_2}{M^{n+2} r^{n+2}} , & r \ll R \\ \frac{m_1 m_2}{M^{n+2} R^n r^2} , & r \gg R \end{cases} \tag{4.11}$$

This is just an application of Gauss' theorem.

$$M_{\rm Pl}^2 = M^{n+2} R^n \, , \quad R = 10^{\frac{30}{n} - 17} \text{ cm} \left( \frac{\text{TeV}}{M} \right)^{1 + \frac{2}{n}} \, . \tag{4.12}$$

The weak scale is known at $E \sim m_{EW}$ so the SM fields must be stuck on a 4D brane. On the contrary we don't know gravity that well below mm. Only gravity propagates in the extra dimension, so we can't stick a finger in it!

If $M \sim$ TeV we have solved the hierarchy problem, but to do so we need $R$ to be large compared to $M_{\rm Pl}^{-1}$. The new hierarchy problem is $R \gg M_{\rm Pl}^{-1}$. Before seeing this in more detail let's see where the connection with large $N$ comes from. It is already manifest that $N$ in the previous theories is playing the role of the volume in this case. Consider one extra dimension compactified on a circle. Here I follow Rattazzi's Cargese lectures

$$g_{MN} = \begin{pmatrix} \eta_{\mu\nu} + h_{\mu\nu} & h_{\mu 5} \\ h_{\mu 5} & h_{55} \end{pmatrix} \, . \tag{4.13}$$

The action of diffs is

$$h_{MN} \to h_{MN} + \partial_M \epsilon_N + \partial_N \epsilon_M \, , \tag{4.14}$$

since the extra dimension is compact $p_5 \sim n/R$. So $\delta h_{55} = 2 \partial_5 \epsilon_5 \propto \sum_n n \epsilon_5^{(n)}$. So we can eliminate all $n \neq 0$ components of $h_{55}$ and $h_{\mu 5}$. We are left with a scalar $\phi \equiv h_{55}^{(0)}$, a four-vector $A_\mu \equiv h_{5\mu}^{(0)}$ and a tower of KK gravitons $h_{\mu\nu}^{(n)}$. To see this use the periodicity of the spatial coordinate in the extra dimension to write

$$h_{\mu\nu}(x, x_5) = \sum_{n=-\infty}^{n=+\infty} h_{\mu\nu}^{(n)}(x) e^{\frac{i n x_5}{R}} \, , \tag{4.15}$$

Integrating the Einstein-Hilbert action over $x_5$ we are left with

$$S = 2\pi R M^3 \int d^4 x \left( h^{\mu\nu} \Box h_{\mu\nu} - h^\mu_\mu \Box h^\nu_\nu + 2 h_{\mu\nu} \partial^\mu \partial^\nu h^\rho_\rho - 2 h_{\mu\nu} \partial^\mu \partial^\rho h^\nu_\rho + \frac{n^2}{4R^2} \left[ h^\mu_\mu h^\nu_\nu - h^{\mu\nu} h_{\mu\nu} \right] \right) + \ldots \tag{4.16}$$

Note that we are always pairing a $n$ and a $-n$ component and a sum is implied. Note also that I've changed the definition of the dimensionful scale $M$ associated with $G_N$. Aside on factors of 2

$$\int_{-L}^{L} dx_5 e^{i(n+m)\frac{x_5}{R}} = \frac{2R \sin((n+m)L/R)}{m+n} \tag{4.17}$$

From the action we can conclude that

$$M_{\rm Pl}^2 = 2\pi R M^3\,, \quad \left(\Box + \frac{n^2}{R^2}\right) h_{\mu\nu}^{(n)} = 0 \tag{4.18}$$

Exercise 3. Derive the EOM from the previous action and also the previous action from Einstein-Hilbert. So how many gravitons do we have? When we hit the scale $M$ we have to UV complete gravity also in the extra dimension therefore we can have at most

$$\frac{N^2}{R^2} \sim M^2 \quad N \sim \left(\frac{M_{\rm Pl}}{M}\right)^{\frac{2}{n}} \tag{4.19}$$

gravitons in our EFT. For $n = 1$ and $M \sim$ TeV (phenomenologically excluded because of modifications of gravity of solar system scales) we recover our large $N$ from before. This is interesting, things can go wrong way before $M$ in the case of large extra dimensions, but they can be cured (for example by $n \geq 2$).

How about the new hierarchy problem $R \gg M_{\rm Pl}^{-1}$? The radius of curvature in the presence of energy density or a CC in the bulk $\Lambda_n$ is (derived below, change order of this derivation?)

$$L \sim \sqrt{\frac{M^{n+2}}{\Lambda_n}}\,, \tag{4.20}$$

if we don't want our space to split in separate inflating patches of size $L$ or collapse into black holes we need

$$L \gtrsim R \to \Lambda_n \lesssim M^{4+n}\left(\frac{M}{M_{\rm Pl}}\right)^{4/n} \tag{4.21}$$

Smaller than its natural value $M^{4+n}$. So we need to tune $\Lambda_n$ and possibly keep it stable with supersymmetry. More in general we would like to stabilize the radii of the extra dimensions. How do we do it? A potential for them arises from the Einstein-Hilbert Lagrangian

$$\int d^{4+n}x\sqrt{-g}\Lambda_n \sim \int d^4x\sqrt{-\bar{g}}\Lambda_n R^n\,. \tag{4.22}$$

In the presence of curvature $\kappa$ in the extra dimensions

$$M^{2+n}\int d^{4+n}x\sqrt{-g}R \sim -\int d^4x\sqrt{-\bar{g}}\kappa M^{2+n}R^{n-2}\,. \tag{4.23}$$

Summing these two terms we can find a stable potential with a minimum $R_* \sim \sqrt{M^{2+n}/\Lambda_n}$. This roughly proves (4.20).

To reproduce our observed 4D universe we need the effective (long distance) 4D CC to approximately vanish

$$\sum_i f_i^4 + R^n\Lambda_n \approx 0\,, \tag{4.24}$$

where $f$ are brane tensions. They are nothing mysterious, just the equivalent of a CC on the 4D brane. Their natural value is $f^4 \approx M^4$. If there are many of them $N_w$

$$\Lambda_n \lesssim N_w M^{4+n} \left(\frac{M}{M_{\mathrm{Pl}}}\right)^{4/n} , \tag{4.25}$$

so the extra dimension can be large for the same reason that people are large (they carry large baryon number). However we are still tuning.

Lesson: Things that look simpler (large $N$) are often "incomplete". Analogy with SUSY and anthropics.

More craziness: gravity without a scale.

$$S = \int d^4x\sqrt{-g}\left[\frac{R^2}{6f_0^2} + \frac{\frac{1}{3}R^2 - R_{\mu\nu}^2}{f_2^2} - \xi_S|S|^2 R + \mathcal{L}_{\mathrm{matter}}\right] , \quad \xi_S\langle S\rangle^2 = \frac{M_{\mathrm{Pl}}^2}{16\pi} \tag{4.26}$$

The other higher order terms are pure derivatives or can be redefined away. The second term is the square of the Weyl or conformal tensor obtained by subtracting all traces from the Riemann tensor. Schematically this gives an EOM of the type

$$\Box h + \frac{1}{M^2}\Box^2 h = 0 \to \frac{1}{M^2p^2 - p^4} = \frac{1}{M^2}\left(\frac{1}{p^2} - \frac{1}{p^2 - M^2}\right) \tag{4.27}$$

Problem 1: There is a ghost
Problem 2: There is a ghost
Problem 3: Landau pole for the Yukawa coupling. You have to modify the SM at the TeV scale. Problem 3 can be solved with some model building gymnastics. The ghost is hard to accommodate. Maybe ghost condensation? Make analogy with Higgs

$$\phi = Mt + \tilde{\phi} , \tag{4.28}$$

however breaks Lorentz invariance quite spectacularly.

Very speculative: soft behavior of gravity at high scales. Illustrative of the point that saying that gravity doesn't give you a threshold is a solution! You would have to UV complete gravity, not to mention get rid of $U(1)_Y$ and see all the rest of new physics couple very weakly to the Higgs.

We have not really addressed the issue of what happens at $M_{\mathrm{Pl}}$. At the moment we have no idea, but there are a few basic facts to keep in mind. In gravity local diffeomorphisms are a gauge symmetry and correlation functions are not good observables, but note that this is only a non-perturbative problem! We can look at the $S$ matrix or at correlation functions along a worldline $x^\mu(\tau)$. Although the number $x^\mu(\tau)$ is arbitrary, it unambigously identifies a point on the spacetime manifold and we can consider $\langle 0|O(x^\mu(\tau_1))...O(x^\mu(\tau_n))|0\rangle$. The $S$ matrix is defined at infinity where gauge symmetries are not redundancies anymore, they change states in the Hilbert space to different states.

How can we see the hierarchy problem in terms of these observables? Nobody really knows, but there is one example in 2D where $M_{\rm Pl}^2$ enters the $S$ matrix only through a phase, not affecting the pole structure of $S$.

$$\hat{S}_n(p_i) = e^{i\frac{1}{M_{\rm Pl}^2}\sum_{i<j}\epsilon_{\alpha\beta}p_i^\alpha p_j^\beta} \,. \tag{4.29}$$

Attractive feature: asymptotic fragility (absence of local off-shell observables). Unattractive feature: gravity in 2D does not have propagating massless spin-2 degree of freedom and this looks very much like just eikonal scattering (large impact parameter $b$)

$$b \gg \frac{E}{M_{\rm Pl}^2} \to e^{-i\frac{s}{4M_{\rm Pl}^2}\log b/R_{\rm IR}} \tag{4.30}$$

The U(1) landau pole problem remains.

RS. Consider adding one extra dimension with metric $\phi \sim -\phi, \phi \sim 2\pi\phi$,

$$ds^2 = e^{-2kr_c|\phi|}dx_\mu dx^\mu + r_c^2 d\phi^2 \tag{4.31}$$

The fluctuation around this classical solution are

$$r_c \to r_c + T(x) \quad \eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}(x) \equiv \bar{g}_{\mu\nu}(x) \tag{4.32}$$

Gravity is in the bulk. The SM is on a brane at $\phi = \pi$

$$\int d^4x d\phi \delta(\phi - \pi)\mathcal{L}_{SM} \,, \tag{4.33}$$

In the 4D effective theory the Planck mass is

$$M^3 \int d^4x \int_{-\pi}^{\pi} d\phi e^{-2kr_c|\phi|} r_c \sqrt{-\bar{g}} R_4 \to M_{\rm Pl}^2 = \frac{M^3}{k}\left(1 - e^{-2kr_c\pi}\right) \approx \frac{M^3}{k} \,. \tag{4.34}$$

On the SM brane we have

$$\int d^4x \sqrt{-\bar{g}} e^{-4kr_c\pi} \left[\bar{e}^{4kr_c\pi} g^{\mu\nu} \left(D_\mu H\right)^\dagger D^\nu H + m_{H0}^2|H|^2 + ...\right] \tag{4.35}$$

After rescaling the kinetic term

$$m_H^2 = e^{-2kr_c\pi} m_{H,0}^2 \tag{4.36}$$

Note that a covariant action satisfies

$$S(\phi, m) = S(\phi', \frac{m}{w}), \tag{4.37}$$

where $\phi'$ is Weyl rescaled $g \to w^{-2}g$, $H \to wH$, $\psi \to w^{3/2}\psi$, ... Only ratios matter. We can in fact see that this is the same as large ED by assuming that the fundamental mass scale is at a

TeV and by rescaling everything by $e^{-kr_c\pi}$ getting a blue-shifted Planck mass. Here the volume of the ED is made large by the exponential factor. So now you should wonder about stabilizing $r_c$, this is the new hierarchy problem!

The exponential is just a convenient artifact, but I might have chosen different coordinates

$$z = \frac{e^{-2kr_c y}}{k} \qquad ds^2 = \frac{1}{k^2 z^2}\left(dx_\mu dx^\mu + dz^2\right) \tag{4.38}$$

Here it is clear that we need a large ED in some sense. How to stabilize it? See Goldberger-Wise paper before these become lectures on EDs.

# 5 The oldest solutions

Technicolor just as at ICTP.

SUSY just as at ICTP maybe increased by the notes and with something more on the flavor problem.

Star coincidence. The surface temperature of stars $T_s$ is about the ionization temperature $T_I$ of molecules because

$$\alpha^{12}\frac{m_e^4}{m_N^4} \approx G_N m_N^2 \,. \tag{5.1}$$

The parametric reasons are $T_I \sim \alpha^2 m_e$, $T_s^4 \sim \alpha^2\sqrt{\alpha_G}$, $(T_s/T_I)^4 \sim \frac{1}{\alpha^6}$. $T_s$ is dominated by gravity and thermodynamic pressure, hence nuclei.

Fusion of hydrogen to helium. When four nucleons make ${}^4_2\mathrm{He}$ 0.7% of their mass is converted to energy. If this number was smaller we would have only hydrogen otherwise there would be no hydrogen.

Triple-$\alpha$ process. When a star runs out of Hydrogen it collapses until its core temperature reaches 10 keV. Then

$$ {}^4_2\mathrm{He} + {}^4_2\mathrm{He} \to {}^8_4\mathrm{Be} \tag{5.2}$$

$$ {}^4_2\mathrm{He} + {}^8_4\mathrm{Be} \to {}^{12}_6\mathrm{C} + 2\gamma \tag{5.3}$$

$$ {}^4_2\mathrm{He} + {}^{12}_6\mathrm{C} \to {}^{16}_8\mathrm{O} + \gamma \tag{5.4}$$

We need the excited state of Carbon on the right hand side to be between 7.3 and 7.9 MeV to produce sufficient carbon for life to exist, and must be further "fine-tuned" to between 7.596 MeV and 7.716 MeV to produce the amount observed in nature. There is an excited state of oxygen which, if it were slightly higher, would provide a resonance and speed up the reaction. In that case insufficient carbon would exist in nature; it would almost all have converted to oxygen. Hoyle used these facts to predict the existence of the ${}^{12}_6\mathrm{C}$ excited state. The ground state of Carbon is at 7.3367 MeV, below the ${}^4_2\mathrm{He} + {}^8_4\mathrm{Be}$ energy.

Higgs vev. Dependence of nuclear parameters when $m_H^2 < 0$.

$$m_n - m_p = (m_d - m_u) + \Delta m_{\mathrm{em}} \approx 3\ \mathrm{MeV}\frac{v}{v_{\mathrm{us}}} + \Delta m_{\mathrm{em}} \tag{5.5}$$

For $v \lesssim$ few hundred, $m_{d,u} < \Lambda_{QCD}$ and we can leave $\Delta m_{\rm em} = -1.7$ MeV fixed at the value that it has in our universe.

$$\Lambda_{QCD} \sim \Lambda_{QCD,\rm us}\frac{v^\xi}{v_{\rm us}^\xi} \quad \xi \approx 0.3 \text{ for } 10^{-2} < \frac{v}{v_{\rm us}} < 10^4 \tag{5.6}$$

$$m_{3/2} - m_{1/2} \approx 300 \text{ MeV}\frac{v^\xi}{v_{\rm us}^\xi}\,. \tag{5.7}$$

Long range nucleon potential = single pion exchange. $m_\pi^2 \sim f_\pi(m_u + m_d)$. $m_\pi \sim m_{\pi,us}\sqrt{v/v_{\rm us}}$. If $v$ decreases at some point Hydrogen becomes unstable, but other nuclei still exist since $m_p - m_n$ never gets above 1.7 MeV. So this kind of universes might support life. On the contrary if $v$ becomes too big, the nuclear binding energy decreases from $m_\pi$ increasing and $m_n - m_p$ increases indefinitely. At some point $v/v_{\rm us} \gtrsim 5$ no elements form. In our universe the nuclear binding energy is negative, i.e. the mass of a nucleus is less than the mass of its constituents by an amount given by the nuclear force minus the EM repulsion, when $m_n - m_p$ exceeds the binding energy the nucleus decays rapidly if it ever forms.

$${}^{\rm A}_{\rm Z}{\rm X} \to {}_{\rm Z+1}^{\rm A}{\rm X} + e^- + \bar\nu_e\,, \quad m({}^{\rm A}_{\rm Z}{\rm X}) = m_N({}^{\rm A}_{\rm Z}{\rm X}) + Zm_e - \sum_{i=1}^{Z} B_{i,e} \tag{5.8}$$

$$\Gamma \sim G_F^2 Q^5 \quad Q \approx m({}^{\rm A}_{\rm Z}{\rm X}) - m({}_{\rm Z+1}^{\rm A}{\rm X}) - m_e \approx m_N({}^{\rm A}_{\rm Z}{\rm X}) - m_N({}_{\rm Z+1}^{\rm A}{\rm X}) = (m_n - m_p) - B_N \tag{5.9}$$

The difference in electron binding energy is very small for high $Z$ atoms. $B_N$ is the difference of the nuclear binding energies. Note that $-B_N$ is always negative because replacing a neutron with a proton increases the electrostatic repulsion. When $Q > 0$ the decay is allowed and it grows rapidly with $Q$. If you scan also the Yukawa's other phenomena might select $v$ (supernova explosions to spread heavy elements, not too much ${}^4$He and too little hydrogen.

In $m_H^2 > 0$ universes baryons are washed-out through sphalerons to neutrinos unless an asymmetry is produced after the EW phase transition, molecules do not form until much later (when the microwave background cools below $\epsilon\alpha^2 m_e \sim \epsilon\alpha^2 y_e \frac{\Lambda_{QCD}^3}{m_H^2}$, $\epsilon \approx 10^{-3}$). Biochemical energy. Atom

$$V(r) = \frac{p_e^2}{2m_e} - \frac{\alpha}{r} = \frac{1}{2m_e r^2} - \frac{\alpha}{r} \quad \text{minimum} \quad r = \frac{1}{\alpha m_e} \tag{5.10}$$

Typical kinetic energy $p_e^2/2m_e \sim \alpha^2 m_e$. You can roughly understand the $\epsilon$ suppression factor from the fact that molecules are bigger ($r$ in the previous equation is larger). Can you? Check parametrics.

# 6 The newest solutions

In the case of $N$naturalness we imagine that multiple copies of the SM exist and that they have different values of the Higgs mass. The point $m_H^2 = 0$ is not special in any way, so we have both sectors with $m_H^2 > 0$ and sectors with $m_H^2 < 0$. We take a uniform distribution for $m_H^2$, so if the theory has $N$ sectors and a cut-off $M$, the lightest Higgs is at $m_H \approx M/\sqrt{N}$. We identify this sector with the SM that we observe and imagine that all the other sectors are coupled to us only through gravity. Obviously in this setup it is expected to have sectors with a Higgs mass that appears unnaturally small and arises from a cancellation. We just need to have enough sectors, given a cut-off $M$. However even a relatively low cut-off $M \approx 10$ TeV, requires a large number of new sectors $N \approx 10^4$ to get at least one with the observed Higgs mass.

Aside. To make contact with the paper I can explain the parameter $r$

$$\left(m_H^2\right)_i = -\frac{M^2}{N}\left(2\,i + r\right), \qquad -\frac{N}{2} \leq i \leq \frac{N}{2}, \tag{6.1}$$

where $i = 0 =$ "us" is the lightest sector with a non-zero vev: $\left(m_H^2\right)_{\text{us}} = -r \times M^2/N \simeq -(88\text{ GeV})^2$ is the Higgs mass parameter inferred from observations. The parameter $r$ can be seen as a proxy for fine-tuning,[1] since it provides a way to explore how well the naive relation between the cutoff and the mass scale of our sector works in a detailed analysis. Specifically, $r = 1$ corresponds to uniform spacing, while $r < 1$ models to an accidentally larger splitting between our sector and the next one. A simple physical picture for this setup is that the new sectors are localized to branes which are displaced from one another in an extra dimension. In this scenario, the lack of direct coupling is clear, and the variation of the mass parameters can be explained geometrically: the $m_H^2$ parameters may be controlled by the profile of a quasi-localized field shining into the bulk.

Sectors with $m_H^2 > 0$ look very different from us. The $W$ and the $Z$ get a mass from QCD of $\mathcal{O}(\Lambda_{QCD})$, quarks and leptons get a tiny mass from

$$\frac{y_t y_\psi (\bar{t}t)}{m_H^2}\bar{\psi}\psi \sim y_t y_\psi \frac{\Lambda_{QCD}^3}{m_H^2}\bar{\psi}\psi\,. \tag{6.2}$$

The photon is massless everywhere and neutrinos are nearly massless everywhere, but their mass grows in the $m_H^2 < 0$ sectors.

It seems that we have already explained the size of the Higgs mass with this "brute force" approach, however there is still one experimental fact that we have not taken into account. Why is most of the energy density contained in the sector with the smallest negative $m_H^2$? The observed value of $\Delta N_{\text{eff}}$ (all the energy density gravitationally coupled to us normalized to that contained in one SM neutrino) has an upper bound of approximately 0.5 at the epoch of recombination [**?**].

$$\Delta N_{\text{eff}} = \frac{\rho - \rho_{SM}}{\rho_\nu}\,. \tag{6.3}$$

We can not simply give it special couplings to the inflaton or to whatever reheats the Universe, otherwise we would not have really solved the problem. We would still need to explain why the smallest negative $m_H^2$ sector is also the one that couples to the inflaton. $N$naturalness explains the smallness of the observed Higgs mass only if all the sectors are treated democratically.

[1] There are a variety of other ways one might choose to implement a measure of fine-tuning in this model. For example, one could assume the distribution of Higgs mass squared parameters is random with some (arbitrary) prior, and then ask statistical questions regarding how often the resulting theory is compatible with observations.

To obtain the observed value of $\Delta N_{\rm eff}$ we have to imagine that at some point the energy density was dominated by a gauge-singlet field, the *reheaton*. For illustrative purposes I take it to be a scalar $\phi$. Then we can couple $\phi$ to all the Higgs bosons with the most relevant coupling that we can write down

$$a\sum_i \phi|H_i|^2 \tag{6.4}$$

and let $\phi$ decays reheat the SM and all other sectors. If $m_\phi \lesssim m_{H_i}, \forall i$ we can compute the decays in the EFT where we have integrated out all the Higgs bosons. The leading operators that we need to consider are[2]

$$\frac{a}{m_{h_i}} y_\psi \phi\bar{\psi}\psi\,, \quad \text{if } m_{H_i}^2 < 0 \tag{6.5}$$

$$\frac{a}{m_{H_i}^2}\phi F^2\,, \quad \text{if } m_{H_i}^2 > 0\,. \tag{6.6}$$

Here $F$ is the field strength of any $SU(2)_L \times U(1)_Y$ gauge boson and this operator is allowed because only QCD is breaking the electroweak symmetry in sectors with $m_{H_i}^2 > 0$, where the Higgs boson does not have a vev. So $m_W, m_Z \sim \Lambda_{QCD} \ll m_{H_i}$. As we did in the previous section, we distinguish between $m_{h_i}$ the physical Higgs mass and the coefficient of $|H_i|^2$ in the Lagrangian, $m_{H_i}$. They coincide only for sectors with $m_{H_i}^2 > 0$.

From the operators above it is clear that even with equal couplings to all sectors the reaheaton decays preferentially to the lightest one with $m_{H_i}^2 < 0$ since

$$\Gamma_{m_{H_i}^2<0} \sim \frac{a^2 m_\phi}{m_{h_i}^2} \tag{6.7}$$

$$\Gamma_{m_{H_i}^2>0} \sim \frac{a^2 m_\phi^3}{m_{H_i}^4}\,. \tag{6.8}$$

$$\rho_i = \rho \mathrm{BR}_{\phi\to \mathrm{i}} = \rho\frac{\Gamma_i}{\Gamma} \quad \text{naively} \quad \Delta N_{\rm eff} \approx \sum_i \frac{\Gamma_i}{\Gamma} \sim \sum_i \frac{1}{i} \sim \log N \tag{6.9}$$

$$\Delta N_{\rm eff} \approx \sum_i \frac{\Gamma_i}{\Gamma} \sim \sum_{i=1}^{N_b} \frac{1}{2\,i+1} + \frac{y_c^2}{y_b^2}\sum_{i=N_b+1}^{N_c}\frac{1}{2\,i+1} \simeq \frac{1}{2}\left(\log 2N_b + \frac{y_c^2}{y_b^2}\log\frac{N_c}{N_b}\right)\,, \tag{6.10}$$

$$N_{b,c} = \left(\frac{m_\phi^2}{8\,m_{b,c}^2} - \frac{1}{2}\right) \tag{6.11}$$

This is not quite enough to meet experimental constraints, but it is the parametric argument underlying the experimental feasibility of $N$naturalness. For more details I refer to the original paper [?]. Experimental consequence $\Delta N_{\rm eff} \gtrsim 0.03$, neutrinos in the other sectors. There are all

[2] As an exercise check this explicitly. What other operators that can lead to $\phi$ decays are present in the $m_{H_i}^2 > 0$ sectors up to dimension five?

sorts of other interesting constraints. Let me mention one: massive stable particles from other $m_H^2 < 0$ sectors can overclose the Universe. Electrons are especially troublesome. In the SM they remain in equilibrium until their symmetric abundance is totally negligible, but in the other sectors their masses are $\sqrt{i}$ larger and subsequently their annihilation cross-sections decrease as $1/i$. In sectors where they thermalize ($n_e^i \sim T_{F,i}^2/(\langle\sigma_e v\rangle_i M_{\rm Pl})(T_0^3/T_{F,i}^3)$)

$$\begin{aligned}\Omega_e^\phi\, h^2 &= \sum_{i=1}^{N_{\rm th}} \frac{m_e^i\, n_e^i}{\rho_c^0} \simeq \frac{\left(m_e^{\rm us}\, T_0^{\rm us}\right)^3}{\rho_c^0} \frac{N_{\rm th}^{5/2}}{M_{\rm pl}\, v_{\rm us}\, \alpha^2} \\ &\lesssim 0.1 \times \Omega_{\rm DM}\, h^2 \quad\Longrightarrow\quad N_\phi \lesssim 10^5\,, \end{aligned} \tag{6.12}$$

Note that we have been pessimistic with this estimate, taking $T_F \approx v_{\rm us}$ applicable to the heaviest sector (however not too pessimistic since it dominates). To overcome this problem we can introduce new particles at the weak scale

$$\begin{aligned}&\mathcal{L}_{L_4} \supset \mathcal{L}_{\rm mix} + \mathcal{L}_Y + \mathcal{L}_M\,, \\ &\mathcal{L}_{\rm mix} = -\lambda\, S^c \sum_i \left(L_4\, H\right)_i - \mu_E \sum_i \left(e^c\, E_4\right)_i\,, \\ &\mathcal{L}_Y = -\sum_i \Big[Y_E\big(H^\dagger\, L_4\, E_4^c\big)_i + Y_E^c\big(H\, L_4^c\, E_4\big)_i \\ &\qquad\qquad + Y_N\big(H\, L_4\, N_4^c\big)_i + Y_N^c\big(H^\dagger\, L_4^c\, N_4\big)_i\Big]\,, \\ &\mathcal{L}_M = -\sum_i \Big[M_E\big(E_4^c\, E_4\big)_i + M_L\big(L_4^c\, L_4\big)_i \\ &\qquad\qquad + M_N\big(N_4^c\, N_4\big)_i\Big] - m_S\, S\, S^c\,,\end{aligned} \tag{6.13}$$

One way to decay to the SM is 1) $L$ is a doublet so $S$ has a $\lambda v/M_L$ coupling to the $W$ 2) L goes to $E^c$ (Yv) 3) $E^c$ goes to $E$ ($M_E$) 4) $E$ goes to $e^c$ ($\mu_e$), parametrically

$$\Gamma_{S\to i} \sim \left(\frac{\lambda v_i^2 \mu_E}{M_i^4}\frac{m_S^4}{m_{W,i}^2}\right)^2 \sim \frac{\mu_E m_S^8}{m_{H,i}^8} \tag{6.14}$$

**Relaxion** As at ICTP. Aside on slow-roll ($\phi_I$ inflaton, $\phi$ relaxion)

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{6.15}$$

$$H_I^2 \sim G_N\rho = G_N\left(\frac{1}{2}\dot\phi_I^2 + V_I(\phi_I)\right) \quad V \gg \dot\phi^2 \quad V_I \gg \dot\phi_I^2 \tag{6.16}$$

$$\frac{\delta S_\phi}{\delta\phi} = \frac{1}{\sqrt{-g}}\partial^\mu\left(\sqrt{-g}\partial_\mu\phi\right) + V' = 0 \quad \ddot\phi + 3H\dot\phi + V' = 0 \quad \dot\phi \approx \frac{V'}{3H} \tag{6.17}$$

Exercise 4. Solve the equation of motion for $\phi$ in various cases. Extra constraints on the relaxion not mentioned at ICTP

$$H_I > \frac{M^2}{M_{\rm Pl}} \tag{6.18}$$

The relaxion is not the inflaton (hard to construct an explicit potential that reproduces observational constraints on inflation).

$$H_I^3 < V' \quad H_I < (gM^2)^{1/3} \tag{6.19}$$

$e$-folds by Gustavo. Even if you are clever and get $g$ to be larger you still have to deal with

$$\dot{\phi}^2 \lesssim \text{barrier height} \to \frac{gM^2}{H_I} \lesssim f_\pi m_\pi \to \frac{H_I}{g} \gtrsim \frac{M^2}{f_\pi m_\pi} \quad N \gtrsim \left(\frac{M^2}{f_\pi m_\pi}\right)^2 \tag{6.20}$$

This is stronger than the second slow roll constraint

$$\ddot{\phi} \ll \dot{\phi} H_I \quad \dot{\phi} \approx \frac{V'}{H_I} \quad \ddot{\phi} \approx \frac{V''\dot{\phi}}{H_I} \tag{6.21}$$

The slow roll conditions $g < H_I$ and $g < H_I^2 M_{\rm Pl}/M^2$ are trivially satisfied.

To solve the strong CP problem. It is actually the inflaton through the term $k\sigma^2\phi^2$ to stop the rolling. $\sigma$ is the inflaton and is roughly constant during inflation. After inflation the barrier drops by $\theta \approx 10^{-10}$. We need the original barrier slope to be $\theta$ times smaller of the new one $gM^2 \sim \theta k\sigma^2 M^2/g$, so $g \sim \theta\sqrt{k}\sigma$. So now we need $N \gtrsim (H_I^2/g^2)\theta$ $e$-folds and $g$ a factor of $\theta$ smaller. After reheating the relaxion does not roll by much even if you go higher than the barriers with the reheating temperature

$$\frac{\Delta\phi}{f} \sim \frac{\dot{\phi}}{Hf} \sim \frac{V'}{H^2 f} \sim \theta \frac{f_\pi^2 m_\pi^2}{T^4}\frac{M_{\rm Pl}^2}{f^2}\,. \tag{6.22}$$

Is this valid only for $T \sim \Lambda_{QCD}$? Otherwise it seems suspicious. Alternative: add new vector-like fermions at the weak scale and new confining force. Note however that low scale inflation is challenging and tuning might hide there.

$$\Delta_s^2 = \Delta_{\mathcal{R}}^2 \approx \frac{H_I^2}{M_{\rm Pl}^2}\frac{1}{-d\log H_I/dN}\bigg|_{k=aH} \approx 10^{-9}\,, \quad n_s - 1 = \frac{d\log\Delta_s^2}{d\log k} \tag{6.23}$$

$$\Delta_s^2 \sim \left(\frac{\delta\rho}{\rho}\right)^2 \quad -d\log H_I/dN = \epsilon \tag{6.24}$$

$n_s$ in the slow-roll approximation is linearly proportional to $\epsilon$ and $\eta$, both very small numbers, but observationally it is $n_s \approx 0.968(6)$ from Planck.