Cargese 2018 — Particle production during and after inflation

Homework 2

Consider a universe filled with a massive inflaton that decays into radiation with decay rate Γ_ϕ. Denote by ρ_ϕ and ρ_γ the energy densities of the inflaton and of the radiation, respectively, and by $H \equiv \frac{\dot{a}}{a}$ the Hubble rate (a is the scale factor of the Universe, and dot denotes time differentiation).

The system is governed by the equations

$$\begin{aligned} &\dot{\rho}_\phi + (3H + \Gamma_\phi)\,\rho_\phi = 0 \\ &\dot{\rho}_\gamma + 4H\rho_\gamma = \Gamma_\phi \rho_\phi \\ &\rho_\phi + \rho_\gamma = 3M_p^2 H^2 \end{aligned} \tag{1}$$

(i) Verify that, for $\Gamma_\phi = 0$, the solutions $\rho_\phi(a)$ and $\rho_\gamma(a)$ agree with what we found in homework 1, namely $\rho_\phi \propto a^{-3}$ and $\rho_\gamma \propto a^{-4}$.

(ii) We are interested in the evolution of ρ_γ at very early times. At these times $\rho_\gamma \ll \rho_\phi$, and $\Gamma_\phi \ll H$. Use these approximations in the first and third equations of the above system, and find the approximate solution for $\rho_\phi(a)$ and $H(a)$. Take the initial condition $\rho_\phi = \bar{\rho}$ at $a = 1$.

Show that the second equation in the above system can be rewritten as

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{\Gamma_\phi}{aH}\,\rho_\phi \tag{2}$$

Insert the solutions $\rho_\phi(a)$ and $H(a)$ into this equation. Solve this equation to obtain the early time solution for $\rho_\gamma(a)$. Take the initial condition $\rho_\gamma = 0$ (hint: it might be useful to consider the differential equation for the combination $a^4\rho_\gamma$).

(iii) Find the maximum value of $\rho_\gamma(a)$, and denote it by $\rho_{\rm max}$.

(iv) A common approximation done in studying this problem is to assume that the inflaton instantaneously decays at $\Gamma_\phi = H$. Use the third equation above to find the energy density of radiation at the decay obtained with this assumption. Denote it as $\rho_{\rm inst}$.

(v) Discuss the ratio $\frac{\rho_{\rm max}}{\rho_{\rm inst}}$, showing that it is parametrically given by the ratio between the initial Hubble rate and the inflaton decay rate. Notice that this quantity can be several orders of magnitude greater than one.

(vi) Given what you just found, why is the approximation of instantaneous inflaton decay so popular ?

Solutions to Homework 2

(i) For $\Gamma_\phi = 0$, we have

$$\frac{d\rho_\phi}{dt} + 3\frac{1}{a}\frac{da}{dt}\rho_\phi = 0 \tag{3}$$

We consider the differential, and separate variables,

$$\frac{d\rho_\phi}{\rho_\phi} = -3\frac{da}{a} \tag{4}$$

which we integrate

$$\int_{\rho_{\phi,\text{in}}}^{\rho_\phi} \frac{d\phi}{\phi} = -3\int_{a_\text{in}}^{a} \frac{da}{a} \tag{5}$$

to find

$$\ln\frac{\rho_\phi}{\rho_{\text{phi,in}}} = -3\,\ln\frac{a}{a_\text{in}} \tag{6}$$

which we exponentiate so to write

$$\rho_\phi = \rho_{\phi,\text{in}}\left(\frac{a_\text{in}}{a}\right)^3 \tag{7}$$

The computation is identical for radiation, with 3 replaced by 4

$$\rho_\gamma = \rho_{\gamma,\text{in}}\left(\frac{a_\text{in}}{a}\right)^4 \tag{8}$$

(ii) Assuming $\Gamma_\phi \ll H$ and $\rho_\gamma \ll \rho_\phi$, the first and third equation rewrite

$$\begin{aligned}\dot{\rho_\phi} + 3H\rho_\phi &= 0\\ \rho_\phi &= 3H^2M_p^2\end{aligned} \tag{9}$$

From the first equation we obtained eq. (7), which we rewrite (using the initial values specified in the text) as

$$\rho_\phi = \bar{\rho}\,a^{-3} \tag{10}$$

The other equation then gives

$$H = \frac{\rho_\phi^{1/2}}{\sqrt{3}M_p} = \frac{\bar{\rho}^{1/2}}{\sqrt{3}M_p}\,\frac{1}{a^{3/2}} \tag{11}$$

We now need to solve

$$\frac{d\rho_\gamma}{dt} + 4H\,\rho_\gamma = \Gamma_\phi\,\rho_\phi \tag{12}$$

We divide by H to write

$$\frac{d\rho_\gamma}{dt} a\,\frac{dt}{da} + 4\rho_\gamma = \frac{\Gamma_\phi}{H}\,\rho_\phi \tag{13}$$

or

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{\Gamma_\phi}{aH}\,\rho_\phi \tag{14}$$

or

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{\Gamma_\phi}{a}\,\frac{\sqrt{3}M_p\,a^{3/2}}{\bar{\rho}^{1/2}}\,\frac{\bar{\rho}}{a^3} \tag{15}$$

or

$$\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma = \frac{C}{a^{5/2}} \quad , \quad C \equiv \sqrt{3}\Gamma_\phi M_p \bar{\rho}^{1/2} \tag{16}$$

As suggested in the text, we note that

$$\frac{d\left(a^4\rho_\gamma\right)}{da} = a^4\,\frac{d\rho_\gamma}{da} + 4\,a^3\,\rho_\gamma = a^4\left(\frac{d\rho_\gamma}{da} + \frac{4}{a}\rho_\gamma\right) \tag{17}$$

Therefore

$$\frac{d\left(a^4\rho_\gamma\right)}{da} = C\,a^{3/2} \tag{18}$$

We integrate it to write

$$\int_0^{a^4\rho_\gamma} d\left(a^4\,\rho_\gamma\right) = C\,\int_1^a a^{3/2}da \tag{19}$$

which gives

$$a^4\rho_\gamma = C\frac{2}{5}a^{5/2}\Big|_1^a \quad \Rightarrow \quad \rho_\gamma = \frac{2\,C}{5}\,\frac{a^{5/2}-1}{a^4} \tag{20}$$

(iii) We need to find the maximum of the function

$$f\left(a\right) \equiv \frac{a^{5/2}-1}{a^4} \tag{21}$$

for $a > 1$. We see that this function starts at zero (at the initial value $a = 1$ of the scale factor), then it grows, due to the inflaton decay, and then it decreases, since the dilution from the expansion is stronger than the production from the inflaton. We have

$$\frac{df}{da} = \frac{d}{da}\left(a^{-3/2} - a^{-4}\right) = -\frac{3}{2}a^{-5/2} + 4a^{-5} = 0 \tag{22}$$

giving

$$a^{5/2} = \frac{8}{3} \quad \Rightarrow \quad a = \left(\frac{8}{3}\right)^{2/5} \tag{23}$$

so that

$$f_{\rm max} = \left(\frac{8}{3}\right)^{-8/5}\left(\frac{8}{3} - 1\right) \simeq 0.35 \tag{24}$$

Therefore

$$\rho_{\rm max} \simeq \frac{2}{5}\sqrt{3}\Gamma_\phi M_p \bar{\rho}^{1/2}\, 0.35 \simeq 0.24\Gamma_\phi M_p \bar{\rho}^{1/2} \tag{25}$$

(iv) In this case we have

$$\Gamma_\phi = H = \frac{\rho_{\rm inst}^{1/2}}{\sqrt{3}M_p} \quad \Rightarrow \quad \rho_{\rm inst} = 3M_p^2\Gamma_\phi^2 \tag{26}$$

(v) The ratio is

$$\frac{\rho_{\rm max}}{\rho_{\rm inst}} \simeq \frac{0.24\bar{\rho}^{1/2}}{3M_p\Gamma_\phi} = \frac{0.24 \times \sqrt{3}M_p H_{\rm initial}}{3M_p\Gamma_\phi} \simeq 0.14\,\frac{H_{\rm initial}}{\Gamma_\phi} \tag{27}$$

(vi) Typically we are interested in the abundance of a particle X, defined by the ratio between the number density of this particle and that of the thermal bath,

$$Y_X \equiv \frac{N_X}{N_\gamma} \tag{28}$$

Particles are produced by the thermal bath formed immediately, with this high $T_{\rm rh}$. However, these particles are diluted by the radiation that keeps being produced by the decaying inflaton. This dilution effect becomes less and less relevant as ρ_ϕ decreases below ρ_γ, which happens when $H \simeq \Gamma_\phi$.

In most cases, the dilution effect makes the particles produced at the very early times negligible at late times (although their number density, in absolute units, was very large at those initial times). This is for instance what happens to gravitinos. Exception to this is the production of very heavy particles (with mass $\gg T_{\rm inst}$), or of particles whose production cross section is strongly sensitive to the temperature, for instance, particles related to the thermal bath by a heavy mediator, with a production cross section $\langle \sigma v \rangle \sim T^n/M^{n+2}$, and $n \geq 6$. [1]

[1] Se for instance https://arxiv.org/pdf/hep-ph/9809453.pdf and https://arxiv.org/pdf/1709.01549.pdf