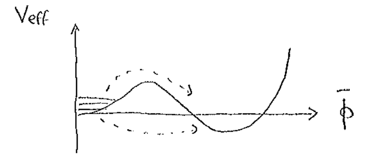
6. Bubble nucleation and expansion

We now assume that the system has a first order transition, and consider how it proceeds in real time. Idea:



Let us use boundary conditions at spatial infinity, $\lim_{|\vec x|\to\infty} \phi(\vec x) = 0$, to define metastable energy eigenstates, and look at their time evolution:

$$|\phi(t)\rangle = e^{-iEt}|\phi(0)\rangle = e^{-i[\mathrm{Re}E + i\,\mathrm{Im}E]t}|\phi(0)\rangle$$

$$\Rightarrow \quad \langle \phi(t)|\phi(t)\rangle = e^{2\,\mathrm{Im}E\,t}\langle\phi(0)|\phi(0)\rangle$$

$$\Rightarrow \quad \Gamma(E) = -2\,\mathrm{Im}E \, .$$

In a thermal ensemble, we might similarly expect $\langle\Gamma\rangle \simeq -2\,\mathrm{Im}F$.
Look at the imaginary-time path integral:

$$F = -T\ln \int_{\rm b.c.} \mathcal{D}\phi \, e^{-S_E[\phi]} \, .$$

Let us assume that there are <u>two</u> saddle points: $\phi \equiv 0$ and $\phi = \hat\phi(\tau,\vec x)$:

$$\frac{\delta S_E}{\delta\phi}\bigg|_{\phi=\hat\phi} = 0 \; ; \quad \hat\phi(0,\vec x) = \hat\phi(\beta,\vec x) \; ; \quad \lim_{|\vec x|\to\infty}\hat\phi(\tau,\vec x) = 0 \, .$$

The <u>fluctuation operator</u> around $\hat\phi$ could have a negative eigenmode:

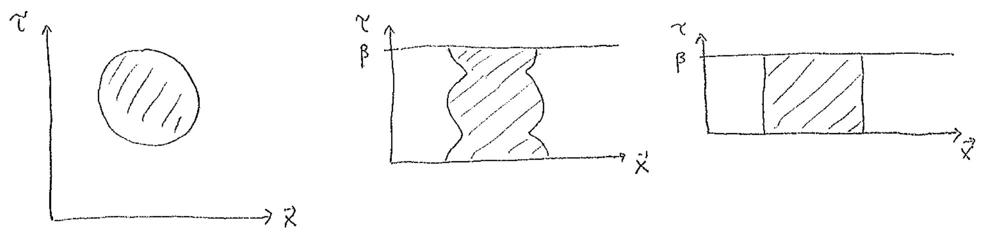
$$\frac{\delta^2 S_E}{\delta\phi^2}\bigg|_{\phi=\hat\phi} f_-(\tau,\vec x) = -\lambda_-^2 f_-(\tau,\vec x) \, .$$

Then

$$F \simeq -T\ln\Big\{ \underbrace{Z[\phi=0]}_{\text{"large"}} + \underbrace{e^{-S_E[\hat\phi]}}_{\text{"small"}} \underbrace{\int df_- \, e^{\frac12\lambda_-^2 f_-^2}}_{\simeq\, i\sqrt{\frac{2\pi}{\lambda_-^2}}} \int \prod_{n\ge0} df_n \, e^{-\frac12\lambda_n^2 f_n^2} \Big\}$$

$$\Rightarrow \quad \Gamma \simeq \frac{T}{Z[\phi=0]} \, \frac{e^{-S_E[\hat\phi]}}{\left|\det\left(\frac{\delta^2 S_E[\hat\phi]}{\delta\phi^2}\right)\right|^{\frac12}} \, .$$

It turns out that the precise computation (or even definition) of the "prefactor" is delicate. Here we focus on $S_E[\hat\phi]$, often sufficient for practical purposes.
Schematically:



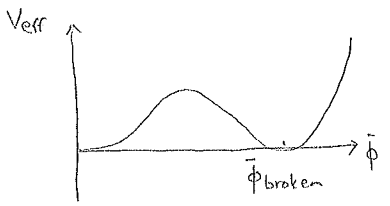
$T = 0$
$\Rightarrow$ 4d symmetry
$\Rightarrow$ "instanton" for quantum tunnelling

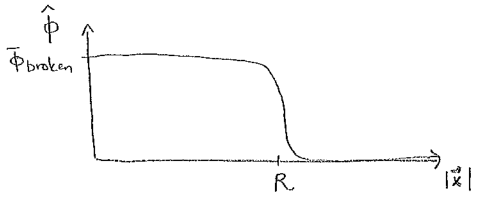
$T \neq 0$
$\Rightarrow$ (3+1)d symmetry
$\Rightarrow$ "caloron" for thermally modified quantum tunnelling

$T \gg 0$
$\Rightarrow$ 3d symmetry
$\Rightarrow$ $S_E[\hat\phi] = \beta\int d^3\vec x \, L_E^{(n=0)}$
$\Rightarrow$ classical thermal tunnelling

Classical tunnelling action in the thin-wall limit

Consider the situation just below T_c:





Euclidean action: $\quad L_E^{(n=0)} = \frac{1}{2}(\partial_i \hat\phi)^2 + V_{eff}(\hat\phi)$.

Equation of motion, assuming 3d spherical symmetry ($r \equiv |\vec{x}|$):

$$\frac{d^2\hat\phi}{dr^2} + \frac{2}{r}\frac{d\hat\phi}{dr} = V_{eff}'(\hat\phi) \quad ; \quad \hat\phi'(0) = 0 \quad ; \quad \hat\phi(\infty) = 0 \, .$$

Let us first inspect the region $r \simeq R$, where the term $2\hat\phi'/R$ is small:

$$\frac{d^2\hat\phi}{dr^2} \simeq V_{eff}'(\hat\phi) \quad \Rightarrow \quad \frac{1}{2}\left(\frac{d\hat\phi}{dr}\right)^2 \simeq V_{eff}(\hat\phi) - V_{eff}(\bar\phi_{broken}) \, .$$

multiply both sides by $\hat\phi'$ and integrate

We can now write the nucleation action as

$$S_E[\hat\phi] = \int d^3x \left\{ \frac{1}{2}\hat\phi'^2 + V_{eff}(\hat\phi) \right\}$$

$$\simeq 4\pi \left\{ \int_0^{R-\delta} dr\, r^2 \left[V_{eff}(\bar\phi_{broken}) \right] + \int_{R-\delta}^{R+\delta} dr\, r^2 \left[\frac{1}{2}\hat\phi'^2 + V_{eff}(\hat\phi) \right] + \int_{R+\delta}^{\infty} dr\, r^2 \left[0 \right] \right\}$$

$$\simeq 4\pi \left\{ \int_0^{R+\delta} dr\, r^2 \left[V_{eff}(\bar\phi_{broken}) \right] + \underbrace{\int_{R-\delta}^{R+\delta} dr\, r^2 \left[\frac{1}{2}\hat\phi'^2 + V_{eff}(\hat\phi) - V_{eff}(\bar\phi_{broken}) \right]} \right\}$$

$$\simeq R^2 \int_{R-\delta}^{R+\delta} dr \left(\frac{d\hat\phi}{dr}\right)^2 = R^2 \int_0^{\bar\phi_{broken}} d\hat\phi \, \frac{d\hat\phi}{dr}$$

$$\simeq R^2 \int_0^{\bar\phi_{broken}} d\hat\phi \sqrt{2\left[V_{eff}(\hat\phi) - V_{eff}(\bar\phi_{broken}) \right]}$$

Let $\sigma \equiv \int_0^{\bar\phi_{broken}} d\hat\phi \sqrt{2[\cdots]}$ be the surface tension.

Moreover the free energy density can be written as

$$V_{eff}(\bar\phi_{broken}) = -L\left(1 - \frac{T}{T_c}\right) , \qquad L \equiv \text{latent heat} .$$

Then $\quad S_E[\hat\phi] \simeq -\frac{4\pi}{3}R^3 L\left(1-\frac{T}{T_c}\right) + 4\pi R^2 \sigma$.

Now extremize with respect to R $\quad \Rightarrow \quad R = \frac{8\pi\sigma}{4\pi L(1-\frac{T}{T_c})} = \frac{2\sigma}{L(1-\frac{T}{T_c})}$.

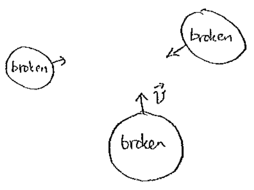
Insert back into $S_E[\hat\phi]$ $\quad \Rightarrow \quad S_E[\hat\phi] = 4\pi\left(-\frac{8}{3}+4\right)\frac{\sigma^3}{L^2(1-\frac{T}{T_c})^2} = \frac{16\pi}{3}\frac{\sigma^3}{L^2(1-\frac{T}{T_c})^2}$.

Summary: $\quad S_E[\hat\phi] \to \infty$ for $T \to T_c$

$\Rightarrow$ nucleation can only happen after some supercooling .

Nucleation dynamics

Overall picture:



Denote the nucleation probability per time and volume by $p = \frac{\Gamma}{V} = p_0\, e^{-\hat{S}_E}$.

Let us expand $\hat{S}_E(t)$ around an effective "nucleation time" t_n:

$$\hat{S}_E(t) \approx \hat{S}_E(t_n) + \hat{S}_E'(t_n)(t-t_n)\,, \qquad \hat{S}_E'(t_n) < 0 \quad (\text{cf. p. 10})$$

Let v be an effective velocity at which further nucleations are stopped. (could be that of a shock wave, $v \simeq c_s$). Then t_n is determined from

$$1 \simeq \int_{-\infty}^{t_n} dt\, \frac{4\pi v^3 (t_n - t)^3}{3}\, p_0\, e^{-\hat{S}_E(t)}$$

$$\approx \frac{4\pi v^3 p_0}{3}\, e^{-\hat{S}_E(t_n)} \int_{-\infty}^{t_n} dt\,(t_n-t)^3\, e^{-|\hat{S}_E'(t_n)|(t_n-t)}$$

$$\boxed{x = t_n - t \qquad \int_0^\infty dx\, x^3\, e^{-|\hat{S}_E'(t_n)|x} = \frac{3!}{|\hat{S}_E'(t_n)|^4}} \; = \; \frac{8\pi v^3 p_0}{|\hat{S}_E'(t_n)|^4}\, e^{-\hat{S}_E(t_n)} \qquad (*)$$

We can also estimate the number density of the bubbles:

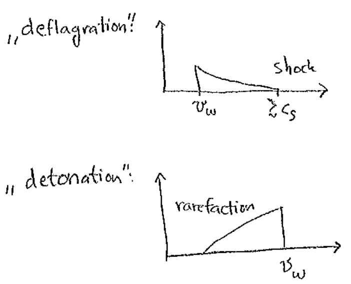
$$\frac{N}{V} \simeq \int_{-\infty}^{t_n} dt\, p_0\, e^{-\hat{S}_E(t)} \approx \frac{p_0}{|\hat{S}_E'(t_n)|}\, e^{-\hat{S}_E(t_n)} \overset{(*)}{\approx} \frac{|\hat{S}_E'(t_n)|^3}{8\pi v^3}\,.$$

Distance scale: $\ell \equiv \left(\frac{N}{V}\right)^{-1/3} \sim \frac{v}{|\hat{S}_E'(t_n)|}$.

Time scale: $\Delta t \equiv \frac{\ell}{v} \sim \frac{1}{|\hat{S}_E'(t_n)|}$.

Therefore an important parameter is $|\hat{S}_E'(t_n)| \overset{!}{=} 3c_s^2\, T_n H_n\, \hat{S}_E'(T_n)$. $\quad$ Hubble rate at $T=T_n$

Summary: the main characteristics of the transition dynamics are



* energy released: $\alpha \equiv \frac{L}{e(T_n)}$.
* inverse duration: $\beta \equiv |\hat{S}_E'(t_n)|$.
* nucleation temperature T_n.
* heating velocity for nucleation (v) vs. wall velocity (v_w).

Exercise: Basic thermodynamic functions are energy density (e), pressure (p) which equals minus free energy density ($p=-f$), entropy density (s). These are related by $s = dp/dT$ and $e = Ts - p$. Show that the latent heat defined on p. 10 equals the discontinuity of e across a first order transition. Moreover, denoting by $c_s^2 = \frac{p'(T)}{e'(T)}$ the speed of sound squared, make use of Einstein equations to verify that $\frac{dT}{dt} = -3c_s^2\, T H$, where H is the Hubble rate $\left(= \sqrt{\frac{8\pi e}{3 m_{Pl}^2}}\right)$.

7. Gravitational waves from phase transition dynamics

Though we cannot go to details, let us work through a simple "linear response" relation for how weakly interacting particles ($\equiv$ gravitons) are produced from a statistical system. Let

$$\hat{H} = \hat{H}_{plasma} + \hat{H}_{gravitons} + \hat{H}_{int} \,, \quad \hat{H}_{int} = \int_{\vec{x}} \varepsilon\, \hat{h}\, \hat{J} \,, \quad \varepsilon \ll 1$$

"graviton" operator containing plasma fields

On-shell field: $\hat{h} = \int \frac{d^3\vec{k}}{\sqrt{(2\pi)^3 2\epsilon_k}} \left(\hat{a}_k e^{-iK\cdot X} + \hat{a}_k^{+} e^{iK\cdot X} \right)$.

States: $|\vec{k}\rangle = \hat{a}_k^{+} |0\rangle$; $|I\rangle \equiv |i\rangle \otimes |0\rangle$; $|F\rangle \equiv |f\rangle \otimes |\vec{k}\rangle$.

Transition matrix element: $T_{FI} = \langle F | \int_0^t dt' \, \hat{H}_{int}(t') | I \rangle$.

Phase space rate: $$\frac{\dot{f}(\vec{k})}{(2\pi)^3} = \lim_{t, V \to \infty} \sum_{f,i} \frac{e^{-\beta E_i}}{Z} \frac{|T_{FI}|^2}{tV} \,.$$

Inserting the on-shell field operator we find $\left(\langle \vec{k} | \hat{a}_q^{+} | \vec{0} \rangle = \delta^{(3)}(\vec{k} - \vec{q}) \right)$

$$T_{FI} = \varepsilon \int_{X'} \frac{e^{iK\cdot X'}}{\sqrt{(2\pi)^3 2\epsilon_k}} \langle f | \hat{J}(X') | i \rangle \,.$$

$$\Rightarrow |T_{FI}|^2 = \frac{\varepsilon^2}{(2\pi)^3 2\epsilon_k} \int_{X', Y'} e^{iK\cdot(X'-Y')} \langle f | \hat{J}(X') | i \rangle \langle i | \hat{J}(Y') | f \rangle \,.$$

Now sum over i with the Boltzmann weight, and make use of translational invariance to cancel tV. Also, denote $X' \to X$.

$$\Rightarrow \dot{f}(\vec{k}) = \frac{\varepsilon^2}{2\epsilon_k} \int_X e^{iK\cdot X} \langle \hat{J}(0)\, \hat{J}(X) \rangle \,.$$

For gravitational waves, we replace $\hat{J} \to \hat{T}^{\mu\nu}$, $\varepsilon \to \frac{\sqrt{8\pi}}{m_{pl}}$; sum over transverse-traceless polarizations (TT); and weigh by $\epsilon_k = k$ to get energy density.

$$\Rightarrow \frac{d e_{GW}}{dt\, d^3\vec{k}} = \frac{2}{\pi^2 m_{pl}^2} \int_{(t,\vec{x})} e^{ik(t-z)} \langle \hat{T}^{xy}(0,\vec{0})\, \hat{T}^{xy}(t,\vec{x}) \rangle \,.$$

Here, for a plasma with a scalar field, (employing "gravity-signature" $-{+}{+}{+}$)

$$T^{\mu\nu} = w\, u^\mu u^\nu + p\, g^{\mu\nu} + \phi^{,\mu}\phi^{,\nu} - \frac{g^{\mu\nu}\, \phi_{,\alpha}\phi^{,\alpha}}{2} \,,$$

$$p = p_0(T) - V_{eff}(\phi, T) \,, \quad w \equiv T \partial_T p \,.$$

Normally numerical simulations are needed for a reliable result!