Multi-gluon one-loop amplitudes using tensor integrals

Andreas van Hameren

IFJ-PAN, Kraków

Mini-workshop on fixed order multi-leg automatic NLO calculations

Universität Wuppertal, 02-06-2009

Supported in part by the EU RTN European Programme, MRTN-CT-2006-035505 (HEPTOOLS, Tools and Precision Calculations for Physics Discoveries at Colliders) and by the Polish Ministry of Scientific Research and Information Technology grant No 153/6 PR UE/2007/7 2007-2010.

Motivation

- LHC is a proton-proton collider, and the physical events to be studied are collision events;
- The events are related to physical quantities in a statistical manner via distributions;
- physics at LHC demands precise qualitative knowledge about signals and backgrounds;
- Monte Carlo programs are a preferred tools to crystallize such knowledge;
- multi-leg hard processes need to be included in these.
- NLO corrections have to be included
  - to reduce scale dependence;
  - to get better description of shapes of distributions;
- several groups of researchers are dealing with the problem of calculating multi-leg processes at NLO.

Ingredients for the calculations

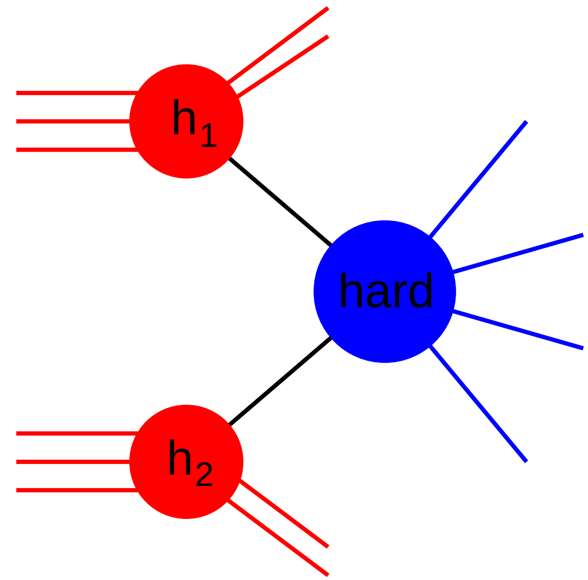
The mathematical framework of calculations in elementary particle physics is Quantum Field Theory. Two important ingredients in the calculations related to LHC physics are:

<u>Factorization</u>

$$d\sigma(\,h_1(p_1)h_2(p_2) \to X\,) =$$

$$\sum_{k,l}\int dx_1\,dx_2\,f_{1,k}(x_1,\mu_F)f_{2,l}(x_2,\mu_F)$$

$$\times\, d\sigma_{\text{hard}}(\,\phi_k(x_1p_1)\phi_l(x_2p_2) \to X;\,\mu_F\,)$$



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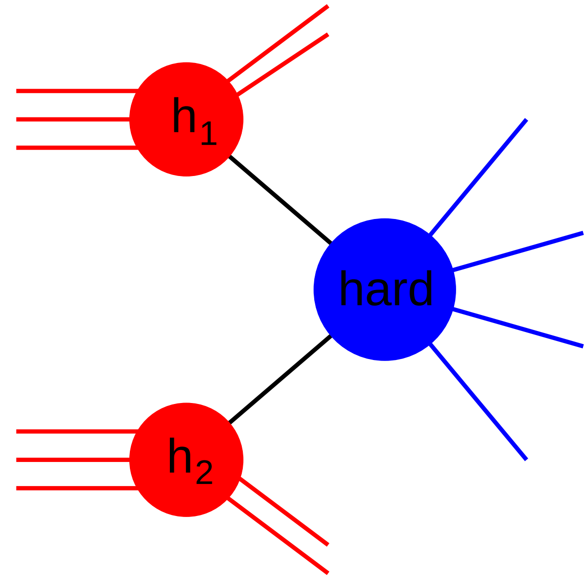

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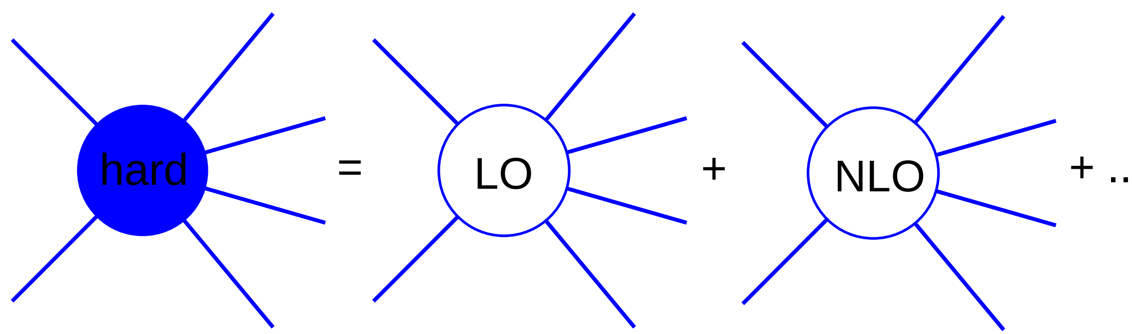
<u>Perturbation theory</u>

$$d\sigma_{\text{hard}} =$$

$$d\sigma^{(0)}_{\text{hard}} + \alpha d\sigma^{(1)}_{\text{hard}} + \ldots$$



Ingredients for the calculations

NLO cross sections

- one order higher in perturbation theory: one more loop or one more leg (squared);
- IR-divergence of integral over phase space for which the extra leg is unobserved cancels against IR-divergence of loop integral KLN.

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$$\langle O\rangle^{\mathrm{LO}} = \int d\Phi_n \, |\mathcal{M}_n^{(0)}|^2 \, O_n^{\mathrm{LO}}$$

$$\langle O\rangle^{\mathrm{NLO}} = \int d\Phi_n \left[2\mathfrak{R}\big(\mathcal{M}_n^{(0)}\mathcal{M}_n^{(1)}\big) + \mathcal{C}_n + \int d\Phi_1 \, \mathcal{A}_{n+1} \right] O_n^{\mathrm{LO}}$$
$$+ \int d\Phi_{n+1} \left[\, |\mathcal{M}_{n+1}^{(0)}|^2 \, O_{n+1}^{\mathrm{NLO}} - \mathcal{A}_{n+1} O_n^{\mathrm{LO}} \, \right]$$

IR-divergencies can be conveniently canceld with the help of subtraction, eg. dipole subtraction Catani, Seymour '97

Ingredients for the calculations

Phase space integration

$$\langle O \rangle = \int d\Phi_n(P;\{p\}_n)\,|\mathcal{M}_n(\{p\}_n)|^2\,O_n(\{p\}_n)$$

In practice, PS integration has to be, and *can* be, done by Monte Carlo.

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Color treatment

$$\langle O \rangle = \int d\Phi_n(P;\{p\}_n)\sum_{\{\lambda\}_n}\sum_{\{a\}_n}|\mathcal{M}_n(\{p\}_n,\{\lambda\}_n,\{a\}_n)|^2\,O_n(\{p\}_n)$$

$$\mathcal{M}_n(\{p\}_n,\{\lambda\}_n,\{a\}_n) = \sum_{\text{perm}}\mathcal{C}(\{a\}_n)\,\mathcal{A}_n(\{p\}_n,\{\lambda\}_n)$$

Amplitude calculation

LSZ-formula: amplitude = connected Green function with external propagators replaced by spinors/polarization vectors.

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Dyson-Schwinger equation (=field theory): for the connected Green functions (for scalar ϕ^3-theory)

$$-i(p^2 - m^2)G_{n+1}(p, p_1, \ldots, p_n) =$$

$$g \int dp_b \delta(p - p_a - p_b) \left[\sum_{\{j\}} G_{k+1}(p_a, p_{j_1}, \ldots, p_{j_k})\, G_{n-k+1}(p_b, p_{j_{k+1}}, \ldots, p_{j_n}) + \frac{1}{2}\, G_{n+2}(p_a, p_b, p_1, \ldots, p_n) \right]$$

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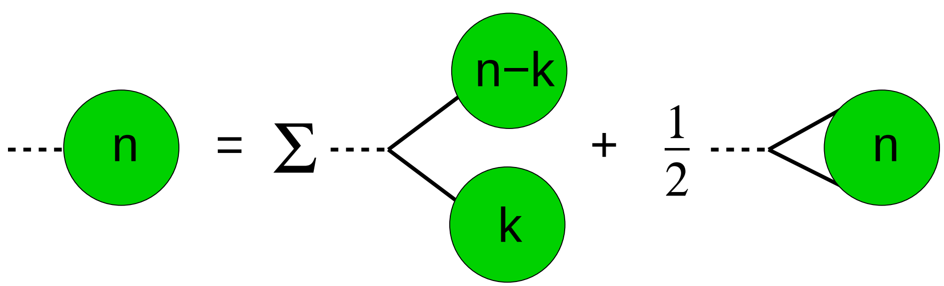
$$\left. + \frac{1}{2} G_{n+2}(p_a, p_b, p_1, \ldots, p_n) \right]$$

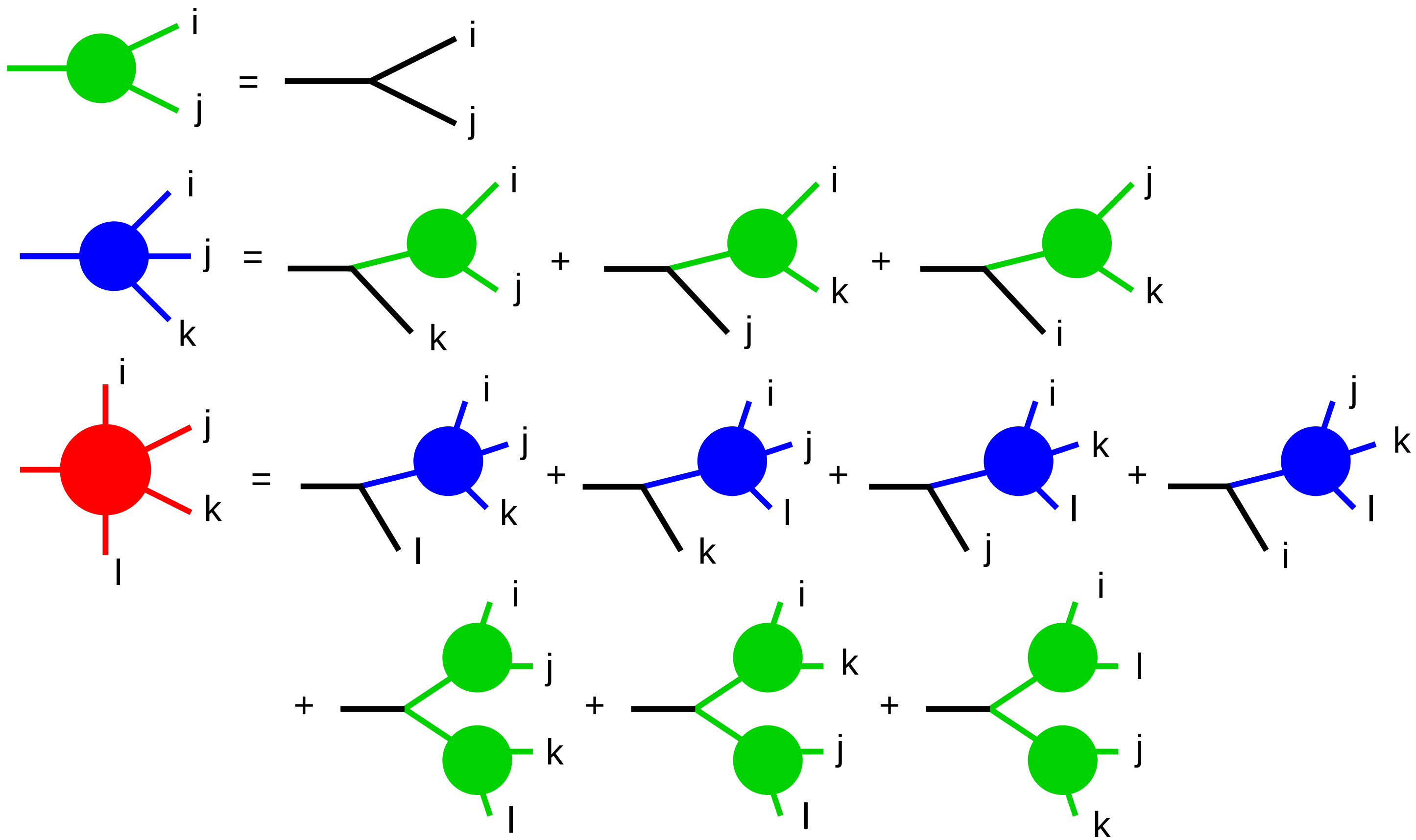
Replace external propagators 1 to n by spinors/polarization vectors $\rightarrow$ off-shell currents.



Analytic solution:
sum of Feynman graphs.

Calculation of tree-level amplitudes

Dyson-Schwinger approach: calculate off-shell currents instead of graphs.

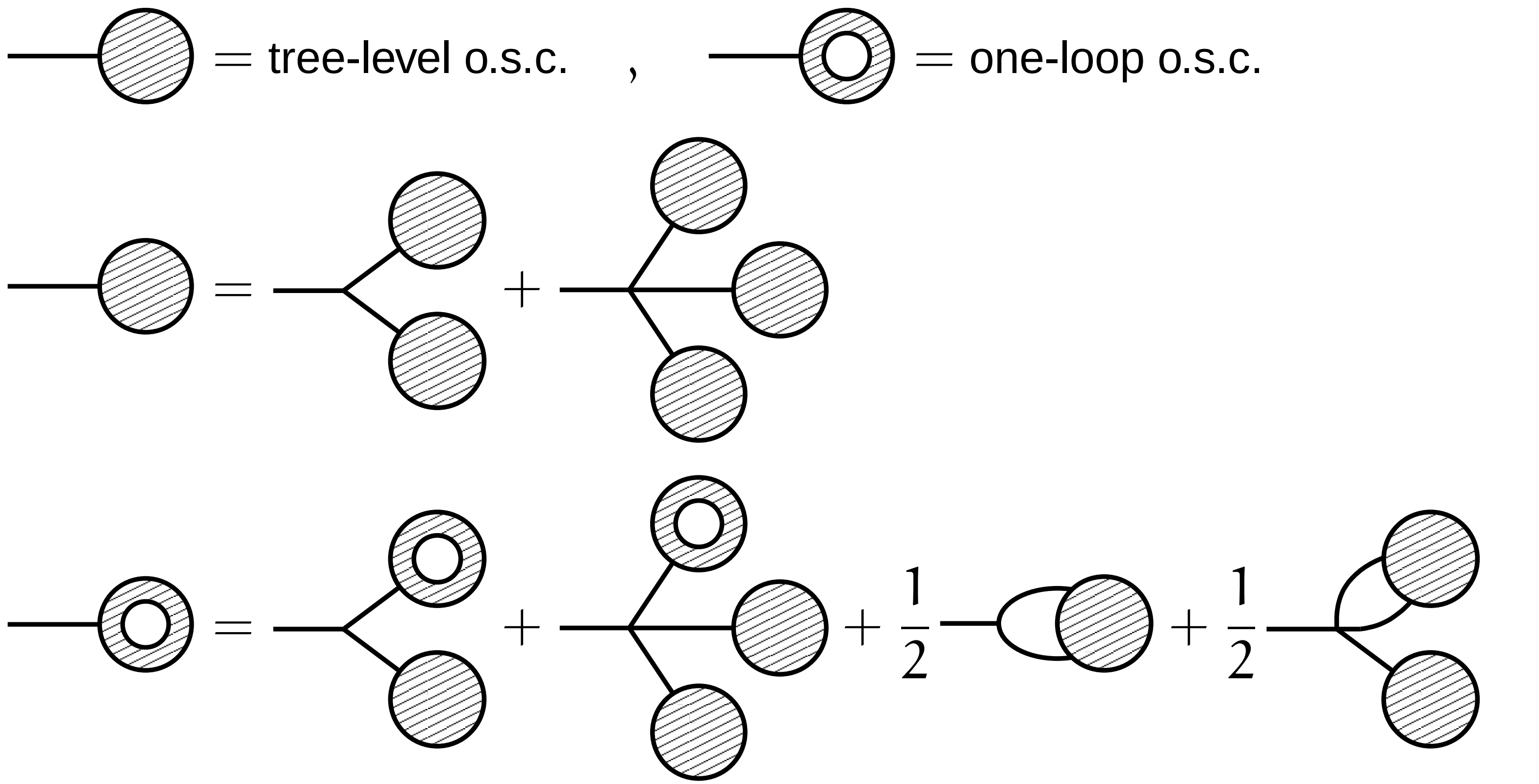


Berends, Giele ’89; Caravaglios, Moretti ’95

- Efficient: $O(n!)$ for graphs to $O(3^n)$, $n =$ number of exteral legs.
- Straightforward to automatize.

Calculation of one-loop amplitudes

Dyson-Schwinger to 1 loop equation (for $\phi^3 + \phi^4$) formally:



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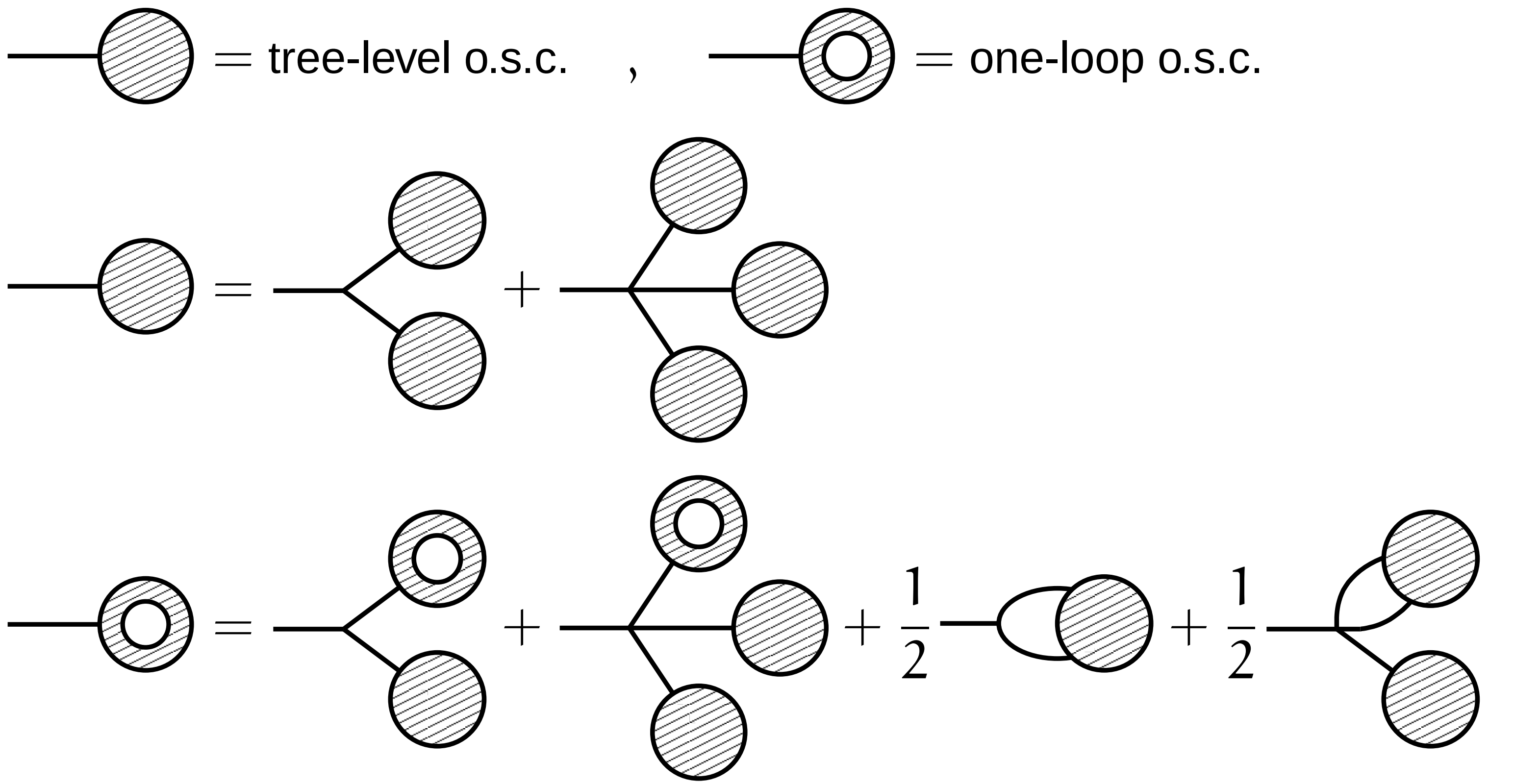


Q: How to deal with the integration over the loop momentum?

A: Expand o.s.c.s in terms of a basis of objects for which the integration has been performed.

One-loop basis

- Eventually, the basis is the set of one-loop scalar 1-, 2-, 3- and 4-point functions.

$$\mathcal{T}_{n,0} = \int \frac{d^D q}{i\pi^{D/2}} \frac{1}{\prod_{j=1}^{n}[(q+p_j)^2 - m_j^2]} \quad , \quad n = 1, 2, 3, 4$$

- In the *unitarity-approach* the coefficients for this basis are determined directly.
- Traditionally, amplitudes are expanded in terms composite one-loop objects, whose determination is universal.
- Most common are tensor integrals (or their coefficient functions when expanded in Lorentz-covariant objects). I use

$$\mathcal{T}_{n,r}^{\nu_1\nu_2\cdots\nu_r} = \int \frac{d^D q}{i\pi^{D/2}} \frac{q_4^{\nu_1} q_4^{\nu_2} \cdots q_4^{\nu_r}}{\prod_{j=1}^{n}[(q+p_j)^2 - m_j^2]}$$

Needs calculation $\mathcal{R}_2$-term Draggiotis, Garzelli, Papadopoulos, Pittau '09.

Computational complexity

Consider ordered amplitudes:

- Number of scalar 1-,2-,3-,4-point scalar functions is

$$\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + \binom{n}{1} = \frac{14n + 11n^2 - 2n^3 + n^4}{24} = \mathcal{O}(n^4) \ .$$

- Number of coefficients to be determined in the *unitarity approach* is the same. The number of operations to be performed increases the computational complexity, but it stays polynomial $\mathcal{O}(n^9)$
Giele, Zanderighi '08.

- Number of tensor integrals is

$$\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{l+3}{l} = \frac{384 + 538n + 203n^2 + 26n^3 + n^4}{384}\,2^n = \mathcal{O}(n^4 2^n)$$

- By using recursivity, the number of operations to be performed does not increase the computational complexity.

C.c. tensor integrals

Using

$$2(p_j - p_n) \cdot q = [(q + p_j)^2 - m_j^2] - [(q + p_n)^2 - m_n^2] + f_{j,n}$$

where $f_{j,n} = m_j^2 - p_j^2 - m_n^2 + p_n^2$, we have

$$2(p_j - p_n)_\nu \mathcal{T}_{n,r}^{\nu,\{\mu\}_{r-1}} = \mathcal{T}_{n-1,r-1}^{\{\mu\}_{r-1}}(j) - \mathcal{T}_{n-1,r-1}^{\{\mu\}_{r-1}}(n) + f_{j,n}\,\mathcal{T}_{n,r-1}^{\{\mu\}_{r-1}} \ .$$

Choosing 4 different vectors p_j appearing in the denominators, we get 4 relations, enough to determine the 4 integrals $\mathcal{T}_{n,r}^{\nu,\{\mu\}_{r-1}}, \nu = 0, 1, 2, 3$

- For high n, tensor integrals can be determined using a constant number of lower integrals (eg. 3 on average above).
- Calculation of the tensor integrals does not increase the asymptotic computational complexity of $\mathcal{O}(n^4 2^n)$.

Inversion of 4×4-matrices can be avoided del Aguila, Pittau '04.

Symmetrization tensor coefficients

The coefficients $\mathcal{G}_{n,r}^{\nu_1\nu_2\cdots\nu_r}$ to be contracted with the tensor integrals in order to obtain the one-loop o.s.c.s satisfy recursive equations

$$\mathcal{G}_{n,r}^{\nu_1\nu_2\cdots\nu_r} = \mathcal{G}_{n-1,r}^{\nu_1\nu_2\cdots\nu_r} * A_n + \mathcal{G}_{n-1,r-1}^{\nu_1\nu_2\cdots\nu_{r-1}} * B_n^{\nu_r}$$

where A_n and $B_n^{\nu_r}$ consist of tree-level o.s.c.s.

- Calculation of these coefficients does not increase the computational complexity......if the relation is symmetrized.

Introduce symmetrized coefficients

$$S_{l_0,l_1,l_2,l_3}^{n,r} = \mathcal{G}_{n,r}^{\{\nu_1\nu_2\cdots\nu_r\}} \quad , \quad l_0 + l_1 + l_2 + l_3 = r$$

which can be calculated directly following

$$\begin{aligned} S_{l_0,l_1,l_2,l_3}^{n,r} = S_{l_0,l_1,l_2,l_3}^{n-1,r} * A_n + S_{l_0-1,l_1,l_2,l_3}^{n-1,r} * B_n^0 + S_{l_0,l_1-1,l_2,l_3}^{n-1,r} * B_n^1 \\ + S_{l_0,l_1,l_2-1,l_3}^{n-1,r} * B_n^2 + S_{l_0,l_1,l_2,l_3-1}^{n-1,r} * B_n^3 \end{aligned}$$

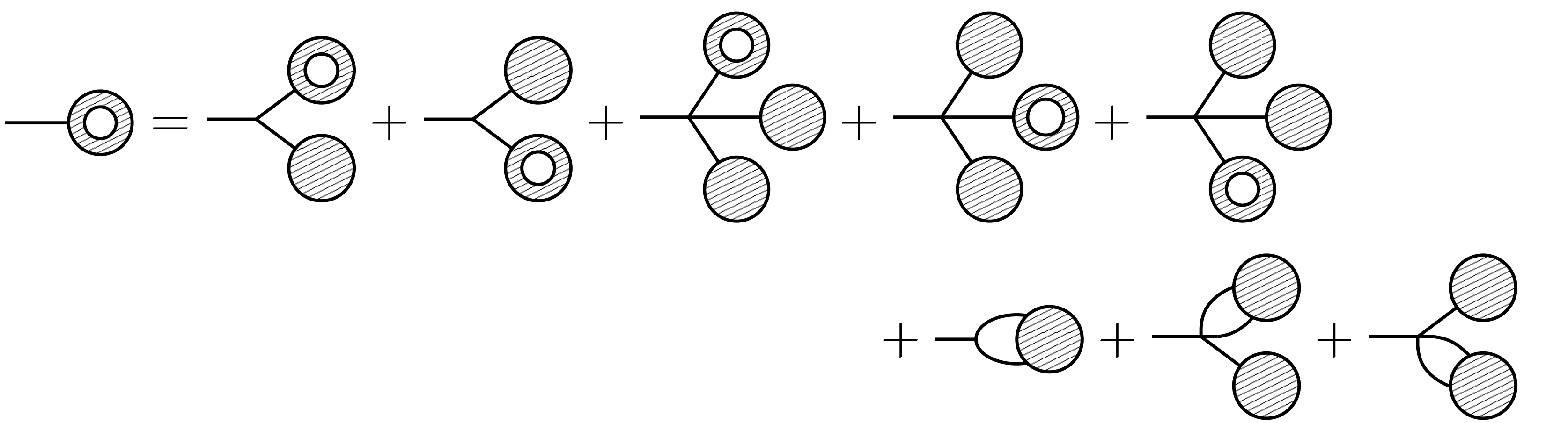
Primitive amplitudes

Color decomposition for one-loop gluon amplitudes Bern, Kosower '90

$$\mathcal{M}_n^{(1)} = \sum_{\pi \in S_n/Z_n} \mathrm{Tr}\left(T^{a_{\pi(1)}} \cdots T^{a_{\pi(n)}}\right) \mathcal{A}_n^{(1)}(\pi(1), \ldots, \pi(n))$$

$$+ \sum_{c=2}^{\lfloor n/2 \rfloor + 1} \sum_{\pi \in S_n/S_{n;c}} \mathrm{Tr}\left(T^{a_{\pi(1)}} \cdots T^{a_{\pi(c-1)}}\right) \mathrm{Tr}\left(T^{a_{\pi(c)}} \cdots T^{a_{\pi(n)}}\right) \times \mathcal{A}_n^{(c)}(\pi(1), \ldots, \pi(n))$$

Primitive amplitude $\mathcal{A}_n^{(1)}$ can be obtained from final o.s.c. in



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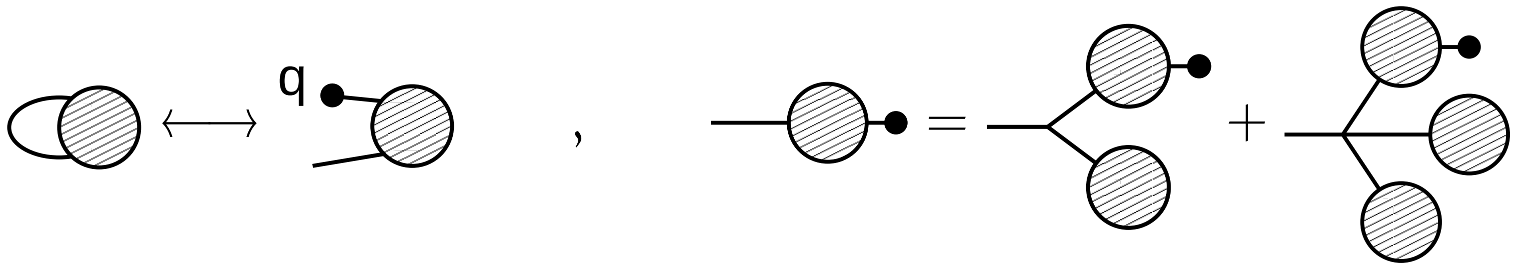
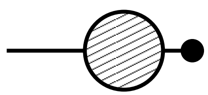




$$\mu \; [\text{diagram}] \; \lambda = \sum_{\mathcal{D}\subset\{i-1,i,\dots,j\}} \sum_{r=0}^{|\mathcal{D}|-1} \mathcal{G}^{\lambda\mu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D}) \, \frac{q^{\nu_1}q^{\nu_2}\cdots q^{\nu_r}}{\prod_{j\in\mathcal{D}}(q+p_{1,j})^2}$$

Separate the q-dependence from the 3-point vertex

$$V^{\mu}_{\nu\rho}(q+p_1,p_2) = V^{\mu}_{\nu\rho}(p_1,p_2) + X^{\mu}_{\sigma\nu\rho}q^{\sigma}$$

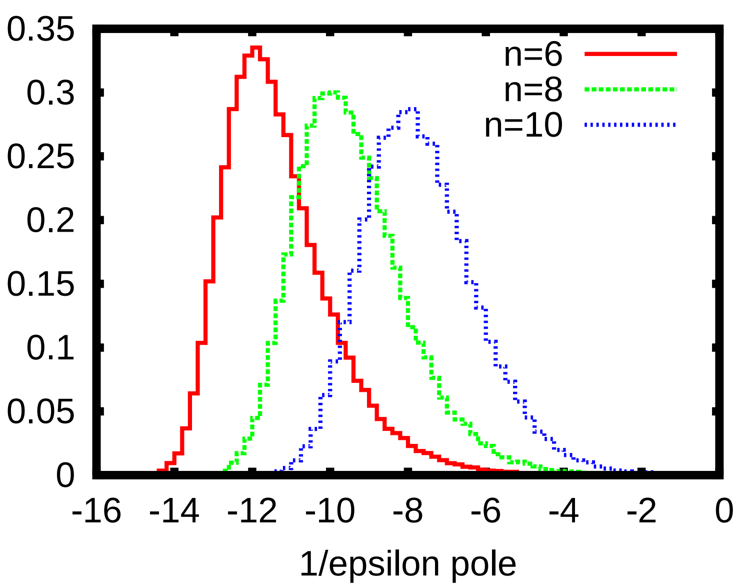
$$\mathcal{G}^{\lambda\mu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D},k,j) = \mathcal{G}^{\lambda\nu}_{\nu_1\nu_2\cdots\nu_r}(\mathcal{D},k)\left[V^{\mu}_{\nu\rho}A^{\rho}_{k+1,j} + \sum_{l=k+1}^{j-1} W^{\mu}_{\nu\rho\sigma}A^{\rho}_{k+1,l}A^{\sigma}_{l+1,j}\right] + \mathcal{G}^{\lambda\nu}_{\nu_1\nu_2\cdots\nu_{r-1}}(\mathcal{D},k)\,X^{\mu}_{\nu_r\nu\rho}A^{\rho}_{k+1,j}$$

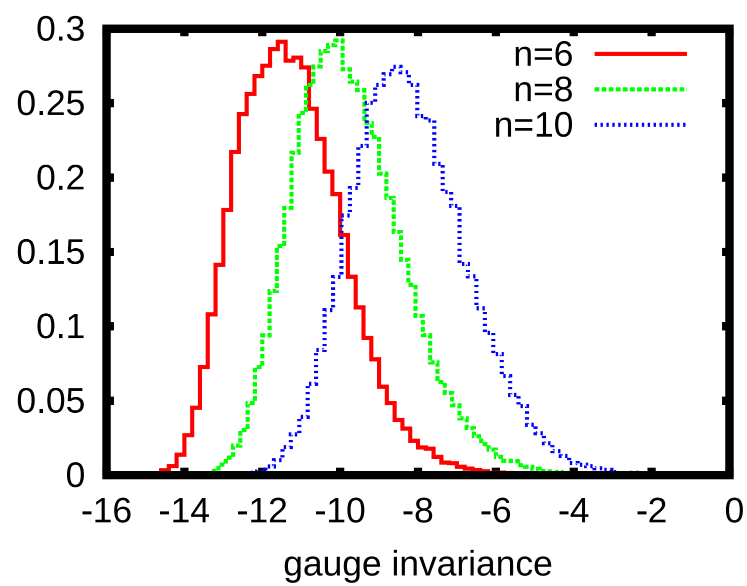
Tensor reduction

- Scalar 1-,2-,3-,4-point functions with `OneLOop` van Hameren, Papadopoulos, Pittau '09;
- 2-point tensor integrals also with `OneLOop`;
- 3-point tensor integrals with Passarino-Veltman reduction '78;
- 4-point tensor integrals with "Alternative PV-like reduction" Denner, Dittmaier '05 or integrand-level reduction del Aguila, Pittau '04;
- scalar $(n > 4)$-point integrals with OPP Ossola, Papadopoulos, Pittau '06;
- tensor $(n > 4)$-point integrals with integrand-level reduction del Aguila, Pittau '04.

Results

- Numbers from Giele, Zanderighi '08 were reproduced until 10 gluons up to at least 4 decimals.
- numerical stability at double-precision level:





$$ {}^{10}\log\left|\frac{\mathcal{A}^{(1/\epsilon)}_{n,\text{analytical}} - \mathcal{A}^{(1/\epsilon)}_{n}}{\mathcal{A}^{(1/\epsilon)}_{n,\text{analytical}}}\right| \qquad {}^{10}\log\left(\left|\frac{\text{Re}\,\mathcal{A}^{(1)}_{n}(\varepsilon_i \leftarrow p_i)}{\text{Re}\,\mathcal{A}^{(1)}_{n}}\right| + \left|\frac{\text{Im}\,\mathcal{A}^{(1)}_{n}(\varepsilon_i \leftarrow p_i)}{\text{Im}\,\mathcal{A}^{(1)}_{n}}\right|\right) $$

Results

- speed (on a 2.80GHz Intel Xeon processor)

n	4	5	6	7	8	9	10
$t_{\text{loop}}\,(\text{ms})$	2.762	10.15	34.37	109.8	335.1	965.2	2744
$t_{\text{loop}}\,(\mu s)/(n^4 2^n)$	0.6744	0.5077	0.4144	0.3573	0.3196	0.2873	0.2680
$t_{\text{loop}}/t_{\text{tree}}/10^3$	0.2990	0.6102	1.180	2.244	4.104	7.919	15.17

Conclusions

- Color-ordered multi-gluon one-loop amplitudes can be calculated efficiently using tensor integrals up to 10 gluons.
- The methods seem to be applicable to general amplitudes.