HELAC - PHEGAS : automatic helicity amplitude calculation and parton level generation

Costas G. Papadopoulos

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Web page

http://www.cern.ch/helac-phegas

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  hep-ph/0012004 and Tokyo 2001,(CPP2001) Computational particle physics, p. 20-25

  T. Gleisberg, *et al.* Eur. Phys. J. C 34 (2004) 173

Old **Feynman graphs $\longrightarrow$ computational cost $\sim n!$**

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P.Draggiotis, R.H.Kleiss and C.G.Papadopoulos, Phys. Lett. B439 (1998) 157

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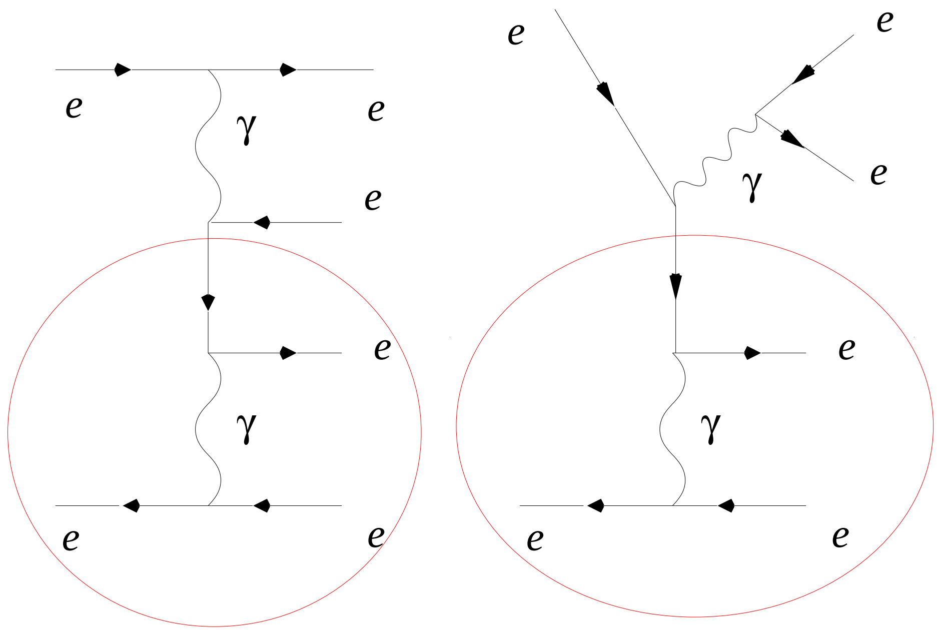
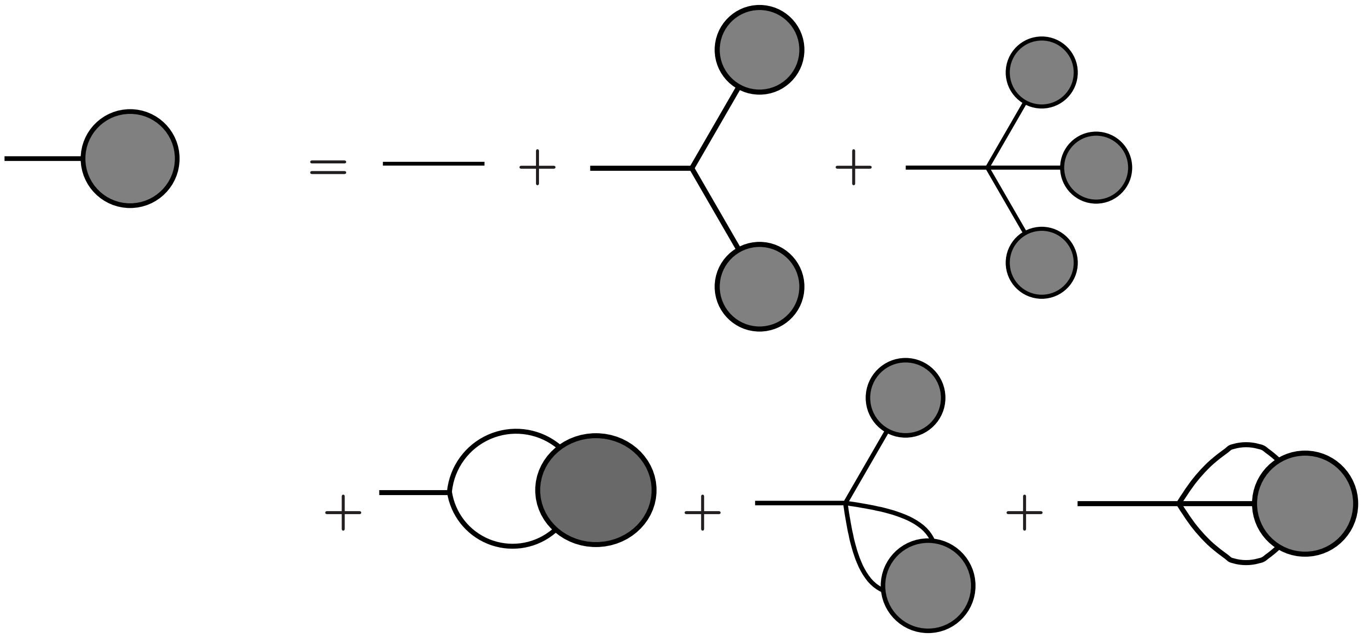
- ***Example:*** $e^- e^+ \longrightarrow e^- e^+ e^- e^+$ **in QED:**

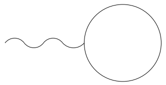
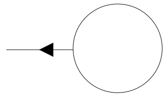
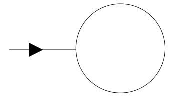


The Dyson-Schwinger recursion

- **Imagine a theory with 3- and 4- point vertices and just one field. Then it is straightforward to write an equation that gives the amplitude for 1 $\longrightarrow$ *n***



⇒ **Systematic approach:**

$b_\mu(P) =$  $\psi(P) =$  $\bar{\psi}(P) =$ 

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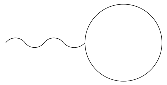
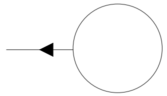
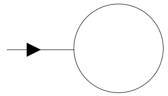
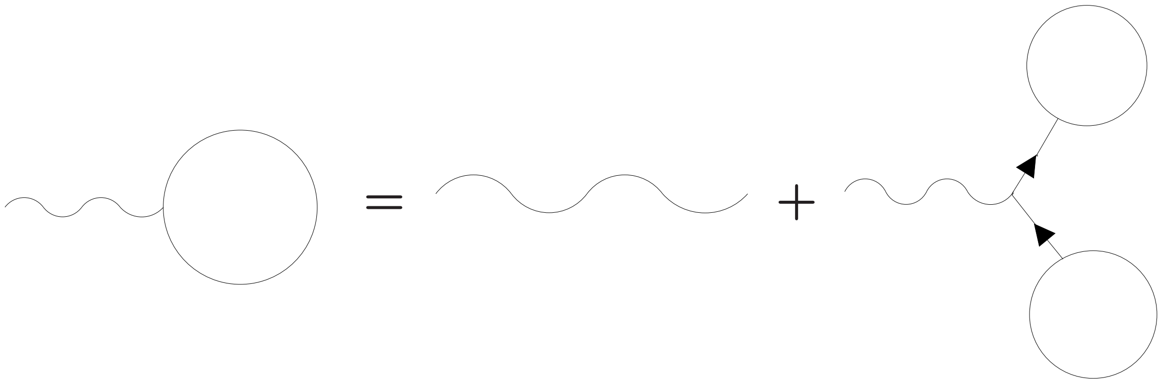
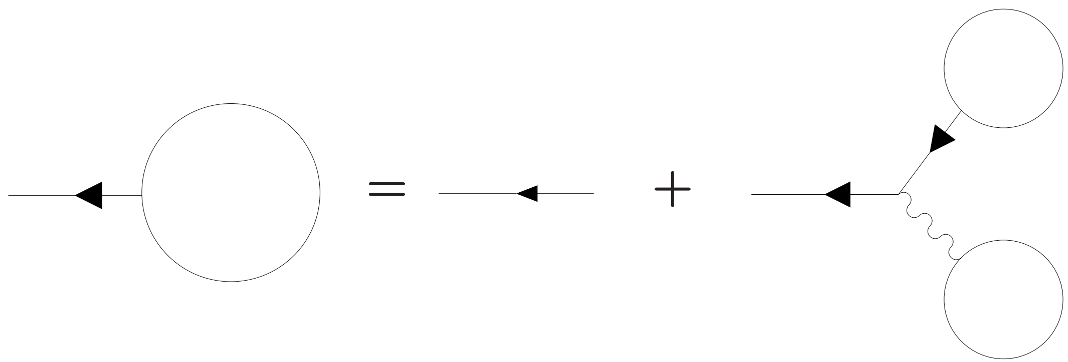
$b_\mu(P) =$  $\psi(P) =$  $\bar\psi(P) =$ 



$$b^\mu(P) = \sum_{i=1}^{n} \delta_{P=p_i} b^\mu(p_i) + \sum_{P=P_1+P_2} (ig)\Pi^\mu_\nu \; \bar\psi(P_2)\gamma^\nu \psi(P_1)\epsilon(P_1, P_2)$$

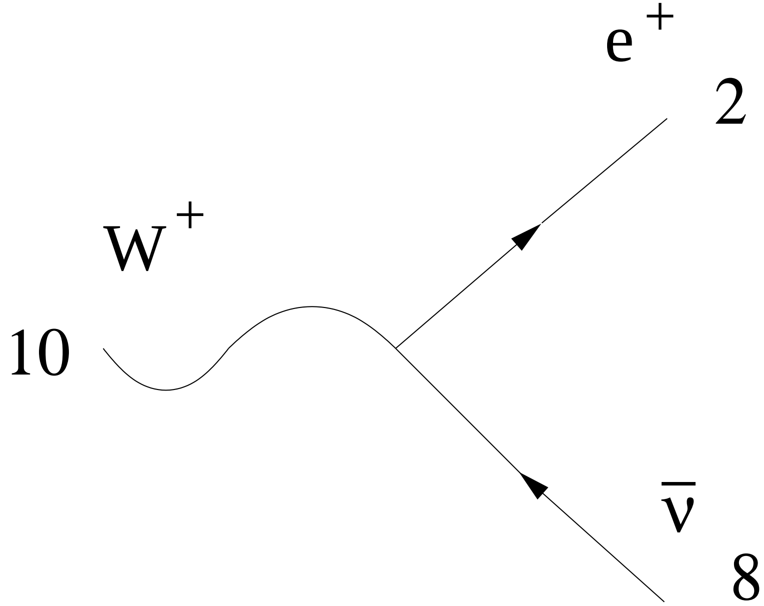


$$\psi(P) = \sum_{i=1}^{n} \delta_{P=p_i}\psi(p_i) + \sum_{P=P_1+P_2} (ig)\not{\psi}(P_2)\frac{(\not{P}_1 + m)}{P_1^2 - m^2}\psi(P_1)\epsilon(P_1, P_2)$$

The solution

$$e^-(1)\ \ e^+(2) \longrightarrow e^-(4)\ \ \bar{\nu}_e(8)\ \ u(16)\ \ \bar{d}(32)$$

1	10	33	2	-2	8	1
1	12	33	4	-2	8	1
1	48	34	16	-3	32	4
2	26	-4	10	33	16	-3
			...			
2	62	-2	10	33	52	-1
2	62	-2	12	33	50	-1
2	62	-2	58	31	4	-2
2	62	-2	58	32	4	-2
2	62	-2	60	31	2	-2
2	62	-2	60	32	2	-2


e^+
2
W^+
10
$\bar{\nu}$
8

HELAC

- **Construction of the skeleton solution of the Dyson-Schwinger equations. At this stage only integer arithmetic is performed. This is part of the initialization phase.**
- **Dressing-up the skeleton with momenta, provided by `PHEGAS` and wave functions, propagators, n-point functions in general.**
- **Unitary and Feynman gauges implemented. Due to multi-precision arithmetic,tests of gauge invariance can be extended to arbitrary precision.**
- **All fermions masses can be non-zero.**
- **All Electroweak and QCD vertices are implemented, including Higgs and would-be Goldstone bosons.**

Colour Configuration - EWK$\oplus$QCD

Colour Configuration - EWK⊕QCD

- **Ordinary approach $SU(N)$-type**

$$\mathcal{A}^{a_1 \ldots a_n} = \sum Tr(T^{a_{\sigma_1}} \ldots T^{a_{\sigma_n}}) \quad A(\sigma_1 \ldots \sigma_n)$$

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$$\mathcal{A}^{a_1 \dots a_n} = \sum Tr(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) \quad A(\sigma_1 \dots \sigma_n)$$

$$\mathcal{C}_{ij} = \sum Tr(T^{a_{\sigma_1}} \dots T^{a_{\sigma_n}}) Tr(T^{a_{\sigma'_1}} \dots T^{a_{\sigma'_n}})$$

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Quarks and gluons treated differently

Colour Configuration - EWK⊕QCD

- **New approach $U(N)$-type**

$$M\left(a_1,\ldots,a_{n_g},i_{1+n_g},\ldots,i_{n_q+n_g},j_{1+n_g},\ldots,j_{n_q+n_g}\right)$$

$$t^{a_1}_{i_1 j_1}\ldots t^{a_{n_g}}_{i_{n_g} j_{n_g}} \rightarrow M^{i_1\ldots}_{j_1\ldots}$$

$$M^{i_1\ldots}_{j_1\ldots}=\sum \delta_{i_1,\sigma_I(j_1)}\delta_{i_2,\sigma_I(j_2)}\ldots\delta_{i_n,\sigma_I(j_n)}\mathcal{A}_{\sigma_I}$$

⋆ **quarks** $1\ldots n$

⋆ **antiquarks** $\sigma_i(1\ldots n)$ **and**

⋆ **gluons** $= q\bar{q}$

Colour Configuration - EWK$\oplus$QCD

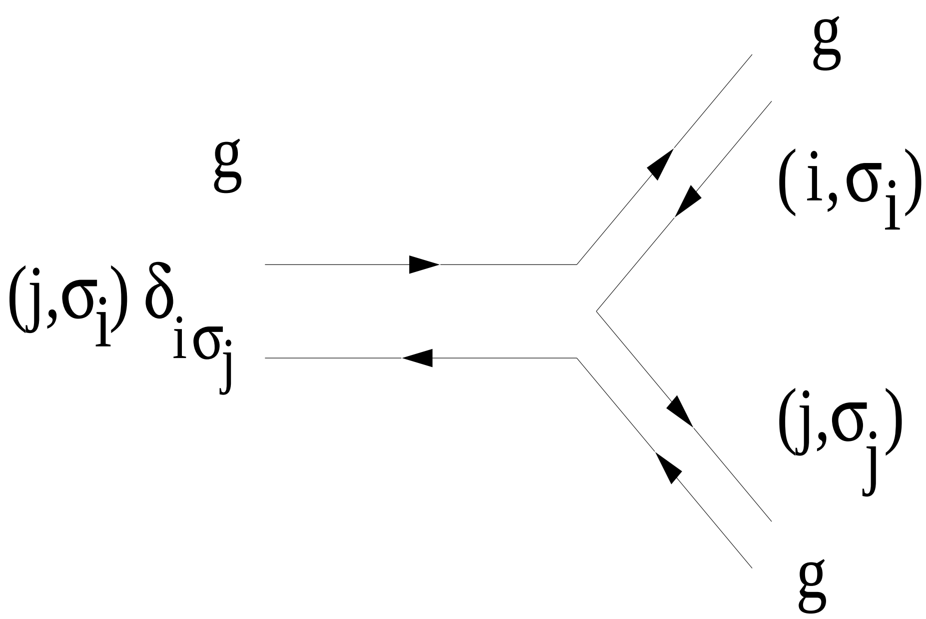
$$\sum |M^{i_1\ldots}_{j_1\ldots}|^2 = \sum \mathcal{C}_{IJ}\mathcal{A}^*_I\mathcal{A}_J$$

$$\mathcal{C}_{IJ} = \sum D_I D_J = N_c^{\alpha} \;, \quad \alpha = \langle \sigma_1, \sigma_2 \rangle$$

$$D_I = \delta_{i_1,\sigma_I(j_1)}\delta_{i_2,\sigma_I(j_2)}\ldots\delta_{i_n,\sigma_I(j_n)}$$

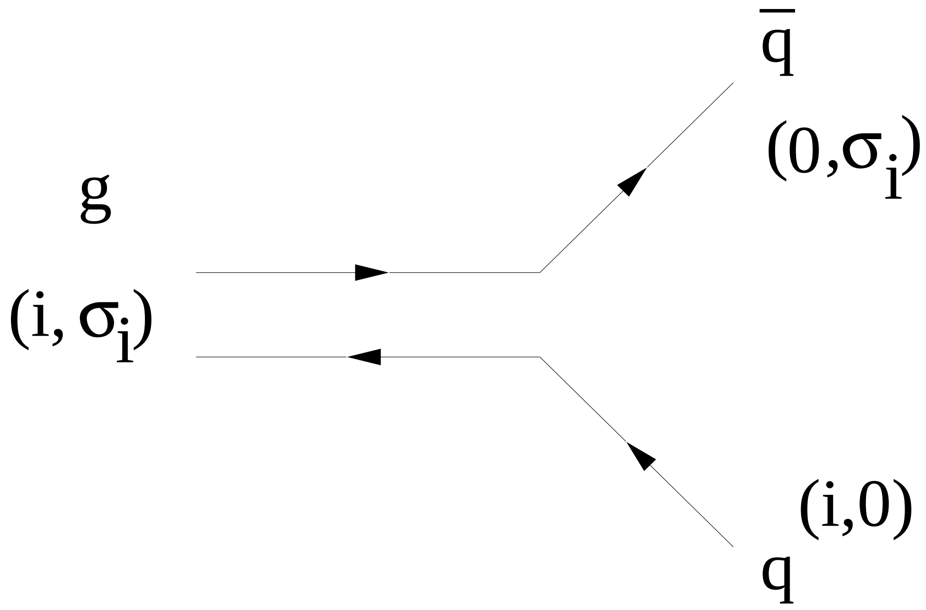
♠ **exact color treatment ⇒ low color charge**

Problem: number of colour connection configurations: $\sim$ $n!$ where n is the number of gluons or $q\bar{q}$ pairs. ⇒ Monte-Carlo over continuous colour-space.



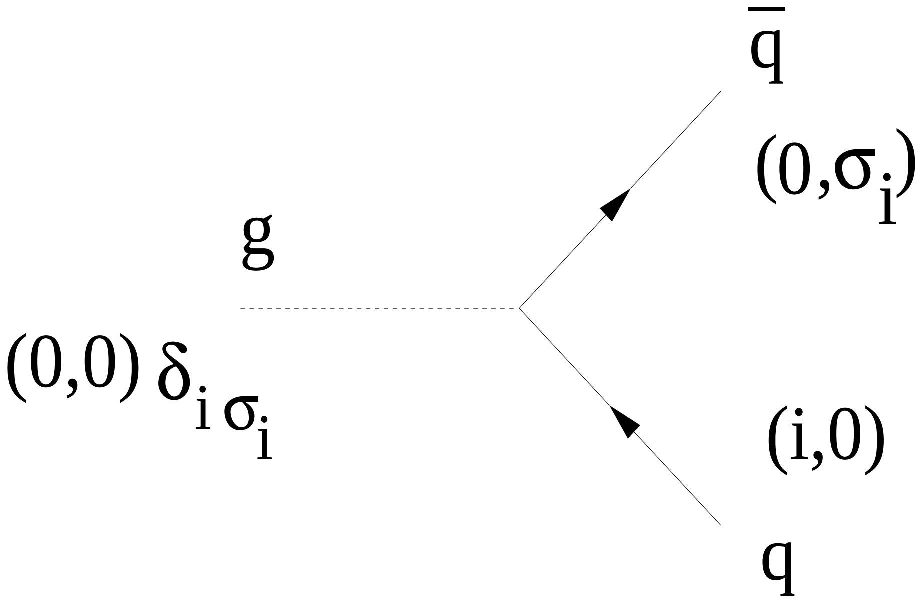
$$\sum f^{abc} t^a_{AB} t^b_{CD} t^c_{EF} = -\frac{i}{4}(\delta_{AD}\delta_{CF}\delta_{EB} - \delta_{AF}\delta_{CB}\delta_{ED})$$

$$\delta_{1\sigma_2}\delta_{2\sigma_3}\delta_{3\sigma_1}$$



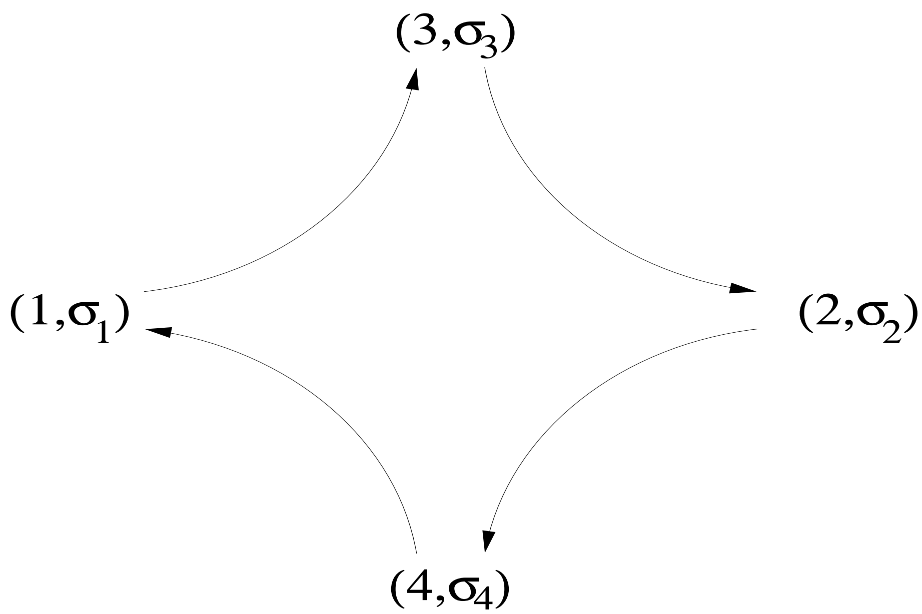
$$\sum t^{a}_{AB} t^{b}_{CD} = \frac{1}{2}(\delta_{AD}\delta_{CB} - \frac{1}{N_c}\delta_{AB}\delta_{AC})$$

$$\frac{1}{\sqrt{2}}$$



$$\sum t^a_{AB} t^b_{CD} = \frac{1}{2}(\delta_{AD}\delta_{CB} - \frac{1}{N_c}\delta_{AB}\delta_{AC})$$

$$\frac{1}{\sqrt{2N_c}}$$



$$\delta_{1\sigma_3}\delta_{3\sigma_2}\delta_{2\sigma_4}\delta_{4\sigma_1}$$

$$2g_{12}g_{34} - g_{13}g_{24} - g_{14}g_{23}$$

lxplus238.cern.ch - lxplus238 - SSH Secure Shell

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```
 the colour of particles ONE 2 1 0 4 0 3
 the colour of particles TWO 1 2 3 0 4 0
for the      8  colour conf. there are       0  subamplitudes
 the colour of particles ONE 2 3 0 1 0 4
 the colour of particles TWO 1 2 3 0 4 0
for the      9      7  colour conf. there are      30  subamplitudes
   1    2    6 -11    7    1    1    2   35    2    4 -11    3    0    0    0    1    1    1
   2    2    6 -11    7    0    1    2   35    2    4 -11    3    0    0    0    2    1    1
   3    1   48   31    8    1    1   16  -12    5   32   12    6    0    0    0    0    1    0
   4    1   48   32    9    1    1   16  -12    5   32   12    6    0    0    0    0    1    0
   5    1   48   35   10    1    1   16  -12    5   32   12    6    0    0    0    0    1    2
   6    2   52 -11   11    1    3   48   31    8    4 -11    3    0    0    0    1    1    0
   7    2   52 -11   11    0    3   48   31    8    4 -11    3    0    0    0    2    1    0
   8    2   52 -11   11    2    3   48   32    9    4 -11    3    0    0    0    1    1    0
   9    2   52 -11   11    0    3   48   32    9    4 -11    3    0    0    0    2    1    0
  10    2   52 -11   11    3    3   48   35   10    4 -11    3    0    0    0    1    1    2
  11    2   52 -11   11    0    3   48   35   10    4 -11    3    0    0    0    2    1    2
  12    3   56   11   12    1    3   48   31    8    8   11    4    0    0    0    1    1    0
  13    3   56   11   12    0    3   48   31    8    8   11    4    0    0    0    2    1    0
  14    3   56   11   12    2    3   48   32    9    8   11    4    0    0    0    1    1    0
  15    3   56   11   12    0    3   48   32    9    8   11    4    0    0    0    2    1    0
  16    3   56   11   12    3    3   48   35   10    8   11    4    0    0    0    1    1    2
  17    3   56   11   12    0    3   48   35   10    8   11    4    0    0    0    2    1    2
  18    2   54 -11   13    1    4    2   35    2   52 -11   11    0    0    0    1    1    1
  19    2   54 -11   13    0    4    2   35    2   52 -11   11    0    0    0    2    1    1
  20    2   54 -11   13    2    4   48   31    8    6 -11    7    0    0    0    1    1    0
  21    2   54 -11   13    0    4   48   31    8    6 -11    7    0    0    0    2    1    0
  22    2   54 -11   13    3    4   48   32    9    6 -11    7    0    0    0    1    1    0
  23    2   54 -11   13    0    4   48   32    9    6 -11    7    0    0    0    2    1    0
  24    2   54 -11   13    4    4   48   35   10    6 -11    7    0    0    0    1    1    2
  25    2   54 -11   13    0    4   48   35   10    6 -11    7    0    0    0    2    1    2
  26    1   60   35   14    1    2    4  -11    3   56   11   12    0    0    0    0    1    1
  27    1   60   35   14    2    2   52  -11   11    8   11    4    0    0    0    0    1    1
  28    4   62   35   15    1    3    2   35    2   60   35   14    0    0    0    0    1    1
  29    1   62   35   15    2    3    6  -11    7   56   11   12    0    0    0    0    1    1
  30    1   62   35   15    3    3   54  -11   13    8   11    4    0    0    0    0    1    1
 the number of Feynman graphs =  10
 the number of Feynman graphs =  10
 (feynman.f)the number of Feynman graphs is: 10
 the colour of particles ONE 2 3 0 4 0 1
 the colour of particles TWO 1 2 3 0 4 0
                                                                                    518,3         20%
```

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 ----------------------------
sigma=   0.274242D-02   0.170635D-01     23998     94000     94000
 ----------------------------
sigma=   0.274562D-02   0.169650D-01     24257     95000     95000
 ----------------------------
sigma=   0.274661D-02   0.168580D-01     24514     96000     96000
 ----------------------------
sigma=   0.274584D-02   0.167396D-01     24763     97000     97000
 ----------------------------
sigma=   0.274919D-02   0.166607D-01     25012     98000     98000
 ----------------------------
sigma=   0.275387D-02   0.165664D-01     25267     99000     99000
 ----------------------------
sigma=   0.276020D-02   0.165019D-01     25518    100000    100000
 ----------------------------
 out of  100000    100001  points have been used
 and   25518  points resulted to =/= 0 weight
 whereas   74483  points to 0 weight
  estimator x:     0.276017D-02
  estimator y:     0.207463D-08
  estimator z:     0.177629D-19
  average estimate :    0.276017D-02
                +\-     0.455481D-04
  variance estimate:    0.207463D-08
                +\-     0.133278D-09
 lwri: points have used 0.000000000000000E+00
 2212 2212 7000.000000000000 7000.000000000000 3 1
 % error: 1.650188706631237
 <w>/w_max,w_max 3.402995943569451E-03 0.8111008833504001
  0.811101E+00    0.472230E+00    0.213615E+00    0.413747E-01    0.113141E-01    0.626923E-02    0.000000E+00
  0.000000E+00    0.000000E+00    0.000000E+00
 <me>/memax,memax 2.824090411444549E-04 4.886440423606611E-03
 iwarning( 4 ) =  46
 number of w=1 events 0
 number of w>1 events 0
 maximum weight used for un:vs 0.000000000000000E+00 0.000000000000000E+00
0NO.199 LOG : INSIDE,UNDER,OVER       0      0      0
  NO ENTRIES INSIDE HISTOGRAM
 TIME= 2009 5 5 120 20 51 35 13
"output" 1992L, 118814C                                                                1992,2        Bot
```

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PHEGAS

- Phase space

$$d\Phi_n = (2\pi)^{4-3n} \prod_{i=1}^{n} \frac{d^3 p_i}{2E_i} \delta\left(\sum E_i - w\right) \delta^3\left(\sum \vec{p}_i\right)$$

- RAMBO, VEGAS-based nice but completely inefficient!

$$d\sigma_n = \text{FLUX} \times |\mathcal{M}_{2\to n}|^2 d\Phi_n$$

need appropriate mappings of peaking structures, plus optimization!

- Efficiency $\Rightarrow$ to a large number of generators, each one for a specific class of processes.

Multichannel approach

$$\mathcal{I} = \int f(\vec{x})d\mu(\vec{x}) = \int \frac{f(\vec{x})}{p(\vec{x})}p(\vec{x})d\mu(\vec{x})$$

$$p(\vec{x}) = \sum_{i=1}^{M_{ch}} \alpha_i \, p_i(\vec{x}) \qquad \sum_{i=1}^{M_{ch}} \alpha_i = 1$$

$$\mathcal{I} \to \left\langle \frac{f(\vec{x})}{p(\vec{x})} \right\rangle \qquad \mathcal{E}^2 N \to \left\langle \left(\frac{f(\vec{x})}{p(\vec{x})}\right)^2 - \mathcal{I}^2 \right\rangle$$

⋆ **Optimize $\alpha_i \Rightarrow$ Minimize $\mathcal{E}$** ⋆

R.Kleiss and R.Pittau, Comput. Phys. Commun. 83, 141 (1994).

New **Dyson-Schwinger equations: subamplitude is a combination of several peaking structures!**

problem unsolved?

QCD antennas

P.D.Draggiotis, A.van Hameren and R.Kleiss, hep-ph/0004047.

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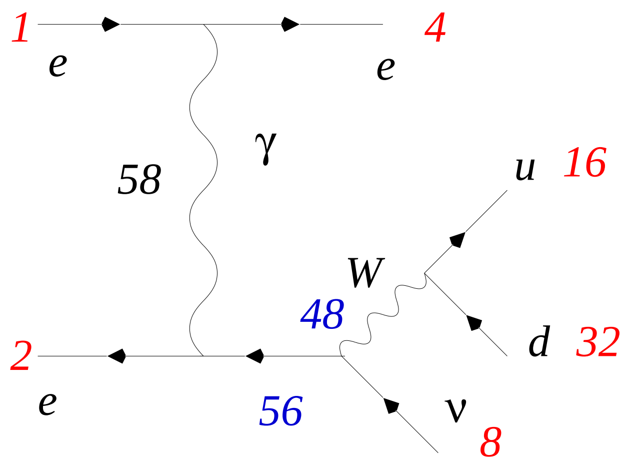
Old **Feynman graphs: exhibit single peaking structure!**

problem solved

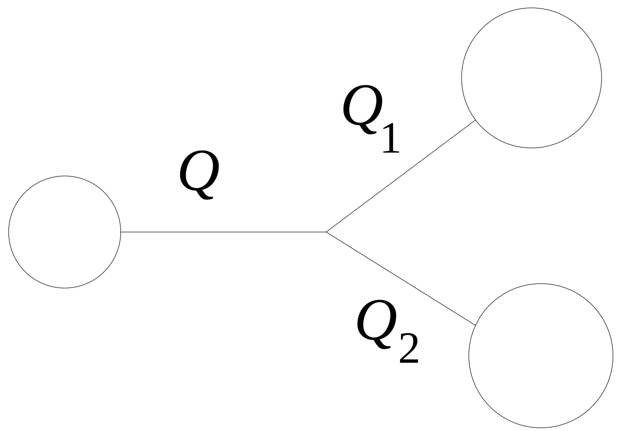
Back to Feynman graphs:



The corresponding intrinsic representation looks like

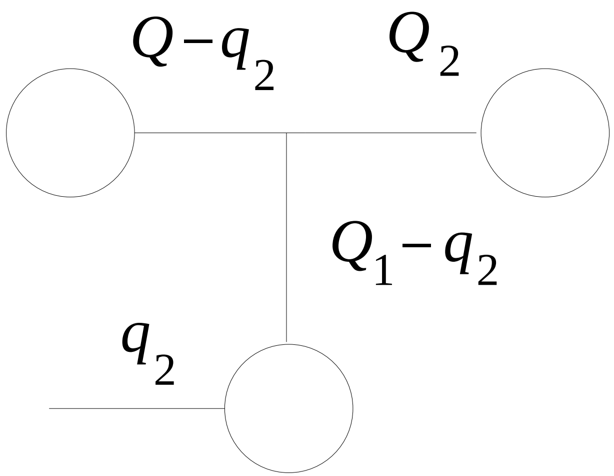
62	**-2**	**4**	**-2**	**58**	**31**
58	**31**	**2**	**-2**	**56**	**2**
56	**2**	**48**	**33**	**8**	**1**
48	**33**	**16**	**-3**	**32**	**4**

Time-like momenta $q^2 \geq 0$



$$
\begin{aligned}
d\Phi_n &= \ldots \frac{dQ_1^2}{2\pi}\,\frac{dQ_2^2}{2\pi}\, d\Phi_2(Q \to Q_1, Q_2) \ldots \\
&= \ldots \frac{dQ_1^2}{2\pi}\,\frac{dQ_2^2}{2\pi}\, d\cos\theta\, d\phi\, \frac{\lambda^{1/2}(Q^2, Q_1^2, Q_2^2)}{32\pi^2\, Q^2} \ldots
\end{aligned}
$$

Space-like momenta



$$
\begin{aligned}
d\Phi_n &= \ldots \frac{dQ_1^2}{2\pi}\,\frac{dQ_2^2}{2\pi}\; d\Phi_2(Q \to Q_1, Q_2) \ldots \\
&= \ldots \frac{dQ_1^2}{2\pi}\,\frac{dQ_2^2}{2\pi}\; dt\; d\phi\; \frac{1}{32\pi^2\; Q\; |\vec{q}_2|} \ldots
\end{aligned}
$$

$$
t = (Q_1 - q_2)^2 = m_2^2 + Q_1^2 - \frac{E_2}{Q}(Q^2 + Q_1^2 - Q_2^2) + \frac{\lambda^{1/2}}{Q}|\vec{q}_2| \cos\theta
$$

Current Status

- **Single process mode: all SM processes. Only limitation memory and CPU cost ! to be judged by the user. Experience with as many as 10 particles in the final state.**
- **Summation over processes mode: all SM processes with fl_{ini} and fl_{fin} flavors for 'jets'. Only limitation memory and CPU cost ! to be judged by the user. Parallelism !**
- **Complete generation for pp and $p\bar{p}$ collisions, including all sub-processes. We do not exclude any processes!**
- **Interfacing with Pythia, including CKKW-like reweighting and use of `UPVETO` à la MLM.**
- **Extra version with HG^n and $H\gamma^n$ couplings**

OPP Reduction - Intro

G. Ossola., C. G. Papadopoulos and R. Pittau, Nucl. Phys. B **763**, 147 (2007) – arXiv:hep-ph/0609007

and JHEP **0707** (2007) 085 – arXiv:0704.1271 [hep-ph]

R. K. Ellis, W. T. Giele and Z. Kunszt, JHEP **0803**, 003 (2008)

Any m-point one-loop amplitude can be written, before integration, as

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

A bar denotes objects living in $n = 4 + \epsilon$ dimensions

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2$$

$$\bar{q}^2 = q^2 + \tilde{q}^2$$

$$\bar{D}_i = D_i + \tilde{q}^2$$

External momenta p_i are 4-dimensional objects

The old "master" formula

$$
\begin{aligned}
\int A \;=\; & \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \\
+ & \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2) \\
+ & \sum_{i_0<i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1) \\
+ & \sum_{i_0}^{m-1} a(i_0) A_0(i_0) \\
+ & \text{ rational terms}
\end{aligned}
$$

THE OLD "MASTER" FORMULA

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

$$N(q) \to q^{\mu_1} \ldots q^{\mu_m} \to g^{\mu_1 \mu_2} p_i^{\mu_3} \ldots$$

G. Passarino and M. J. G. Veltman, "One Loop Corrections For E+ E- Annihilation Into Mu+ Mu- In The Weinberg Model," Nucl. Phys. B **160** (1979) 151.

THE OLD "MASTER" FORMULA

$$T^N_{\mu_1 \ldots \mu_P}(p_1, \ldots, p_{N-1}, m_0, \ldots, m_{N-1}) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \frac{q_{\mu_1} \cdots q_{\mu_P}}{D_0 D_1 \cdots D_{N-1}}$$

$$T^N_{\mu_1 \ldots \mu_P}(p_1, \ldots, p_{N-1}, m_0, \ldots, m_{N-1}) = \sum_{i_1, \ldots, i_P = 0}^{N-1} T^N_{i_1 \ldots i_P} p_{i_1 \mu_1} \cdots p_{i_P \mu_P}.$$

$$\begin{aligned}
D_\mu &= \sum_{i=1}^{3} p_{i\mu} D_i, \\
D_{\mu\nu} &= g_{\mu\nu} D_{00} + \sum_{i,j=1}^{3} p_{i\mu} p_{j\nu} D_{ij}, \\
D_{\mu\nu\rho} &= \sum_{i=1}^{3} (g_{\mu\nu} p_{i\rho} + g_{\nu\rho} p_{i\mu} + g_{\mu\rho} p_{i\nu}) D_{00i} + \sum_{i,j,k=1}^{3} p_{i\mu} p_{j\nu} p_{k\rho} D_{ijk}, \\
D_{\mu\nu\rho\sigma} &= (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) D_{0000} \\
&\quad + \sum_{i,j=1}^{3} (g_{\mu\nu} p_{i\rho} p_{j\sigma} + g_{\nu\rho} p_{i\mu} p_{j\sigma} + g_{\mu\rho} p_{i\nu} p_{j\sigma} \\
&\qquad + g_{\mu\sigma} p_{i\nu} p_{j\rho} + g_{\nu\sigma} p_{i\mu} p_{j\rho} + g_{\rho\sigma} p_{i\mu} p_{j\nu}) D_{00ij} \\
&\quad + \sum_{i,j,k,l=1}^{3} p_{i\mu} p_{j\nu} p_{k\rho} p_{l\sigma} D_{ijkl}.
\end{aligned}$$

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W. L. van Neerven and J. A. M. Vermaseren, "Large Loop Integrals," Phys. Lett. B **137**, 241 (1984)

The derivation of the reduction formula starts as in ref. [1] with the Schouten identity which is a relation between five Levi-Civita tensors:

$$\epsilon^{p_1 p_2 p_3 p_4} Q_\mu = \epsilon^{\mu p_2 p_3 p_4} Q \cdot p_1 + \epsilon^{p_1 \mu p_3 p_4} Q \cdot p_2 + \epsilon^{p_1 p_2 \mu p_4} Q \cdot p_3 + \epsilon^{p_1 p_2 p_3 \mu} Q \cdot p_4 \,. \quad (6)$$

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which yields the final formula for the scalar one-loop five-point function:

$$E_{01234}(w^2 - 4\Delta_4 m_0^2) = D_{1234}\,[2\Delta_4 - w\cdot(v_1 + v_2 + v_3 + v_4)]$$

$$+ D_{0234}\, v_1\cdot w + D_{0134}\, v_2\cdot w + D_{0124}\, v_3\cdot w + D_{0123}\, v_4\cdot w \,. \qquad (19)$$

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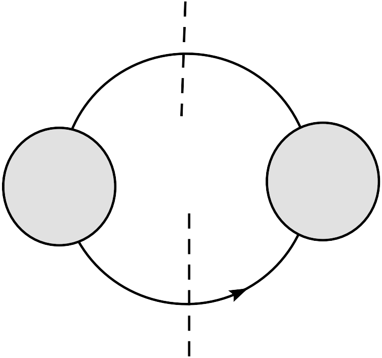
This method is completely different from the one used in ref. [3].

UNITARITY



Started in 90's, mainly QCD, amplitude level (analytical results)
Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower,
[arXiv:hep-ph/9403226].
Gluing tree amplitudes plus colinear limits → extract coefficients

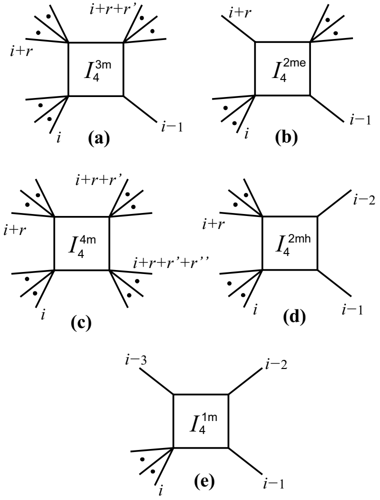
Unitarity

$$\mathcal{C} * \int A = \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \mathcal{C} * D_0(i_0 i_1 i_2 i_3)$$
$$+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \mathcal{C} * C_0(i_0 i_1 i_2)$$
$$+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \mathcal{C} * B_0(i_0 i_1)$$

Unitarity

	Integral	Unique Function
a	$I_4^{0\mathrm{m}}(s,t)$	$\ln(-s)\ln(-t)$
b	$I_3^{1\mathrm{m}}(s)$	$\ln(-s)^2$
c	$I_3^{1\mathrm{m}}(t)$	$\ln(-t)^2$
d	$I_2(s)$	$\ln(-s)$
e	$I_2(t)$	$\ln(-t)$

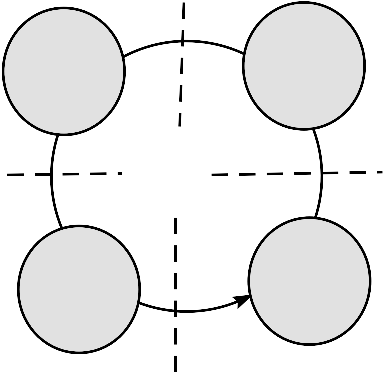
Table 1: The set of integral functions that may appear in a cut-constructible massless four-point amplitude, together with the independent logarithms.


i+r+r'
i+r
I_4^{3m}
i
i−1
(a)
i+r
I_4^{2me}
i
i−1
(b)
i+r+r'
i+r
I_4^{4m}
i+r+r'+r''
i
(c)
i−2
i+r
I_4^{2mh}
i
i−1
(d)
i−3
i−2
I_4^{1m}
i
i−1
(e)

	Integral	Unique Function
a	$I^{3\mathrm{m}}_{4:r,r';i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r+r'}^{[n-r-r'-1]})$
b	$I^{2\mathrm{m}\,e}_{4:r;i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r+1}^{[n-r-2]})$
c	$I^{4\mathrm{m}}_{4:r,r',r'';i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r+r'}^{[r'']})$
d	$I^{2\mathrm{m}\,h}_{4:r;i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r}^{[n-r-1]})$
e	$I^{1\mathrm{m}}_{4;i}$	$\ln(-t_i^{[r]})\ln(-t_i^{[r+1]})$
f	$I^{3\mathrm{m}}_{3:r,r';i}$	$\ln(-t_i^{[r]})\ln(-t_{i+r}^{[r']})$

Table 2: Following the ordering shown and taking large $t_i^{[r]}$ makes the proof of uniqueness of the cuts straightforward.

R. Britto, F. Cachazo and B. Feng, [arXiv:hep-th/0412103].
Quadruple cut with complex momenta $\rightarrow d(i_0 i_1 i_2 i_3)$



OPP "MASTER" FORMULA - I

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) \;=\; & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
+ & \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
+ & \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
+ & \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

OPP "MASTER" FORMULA - II

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) \qquad\qquad \right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c(i_0 i_1 i_2) \qquad\qquad] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) \qquad\qquad \right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1} [a(i_0) \qquad\qquad] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

- The quantities $d(i_0 i_1 i_2 i_3)$ are the coefficients of 4-point functions with denominators labeled by i_0, i_1, i_2, and i_3.
- $c(i_0 i_1 i_2)$, $b(i_0 i_1)$, $a(i_0)$ are the coefficients of all possible 3-point, 2-point and 1-point functions, respectively.

OPP "MASTER" FORMULA - II

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1)\right] \prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0)\right] \prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

The quantities $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ are the "spurious" terms

- They still depend on q (integration momentum)
- They should vanish upon integration

What is the explicit expression of the spurious term?

Spurious Terms - I

Following F. del Aguila and R. Pittau, arXiv:hep-ph/0404120

- Express any q in $N(q)$ as

$$q^\mu = -p_0^\mu + \sum_{i=1}^4 G_i\, \ell_i^\mu \;,\; {\ell_i}^2 = 0$$

$$k_1 = \ell_1 + \alpha_1 \ell_2\,, \quad k_2 = \ell_2 + \alpha_2 \ell_1\,,\; k_i = p_i - p_0$$
$$ {\ell_3}^\mu = < \ell_1|\gamma^\mu|\ell_2]\,,\; {\ell_4}^\mu = < \ell_2|\gamma^\mu|\ell_1]$$

- The coefficients G_i either reconstruct denominators D_i

$\rightarrow$ They give rise to d, c, b, a coefficients

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- The coefficients G_i either reconstruct denominators D_i or vanish upon integration

$\rightarrow$ They give rise to d, c, b, a coefficients
$\rightarrow$ They form the spurious $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ coefficients

Spurious Terms - II

- $\tilde{d}(q)$ term (only 1)

$$\tilde{d}(q) = \tilde{d}\, T(q)\,,$$

where $\tilde{d}$ is a constant (does not depend on q)

$$T(q) \equiv Tr[(\not{q} + \not{p}_0)\not{\ell}_1\not{\ell}_2\not{k}_3\gamma_5]$$

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- $\tilde{c}(q)$ terms (they are 6)

$$\tilde{c}(q) = \sum_{j=1}^{j_{max}} \left\{ \tilde{c}_{1j} [(q + p_0) \cdot \ell_3]^j + \tilde{c}_{2j} [(q + p_0) \cdot \ell_4]^j \right\}$$

In the renormalizable gauge, $j_{max} = 3$

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In the renormalizable gauge, $j_{max} = 3$

- $\tilde{b}(q)$ and $\tilde{a}(q)$ give rise to 8 and 4 terms, respectively

A simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 D_4}$$

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$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}$$

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- Melrose, Nuovo Cim. 40 (1965) 181
- G. Källén, J.Toll, J. Math. Phys. 6, 299 (1965)

A next to simple example

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$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

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$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

A next to simple example

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

A next to simple example

$$\begin{vmatrix} T_0^5 & -T_0^4(0) & -T_0^4(1) & -T_0^4(2) & -T_0^4(3) & -T_0^4(4) \\ 1 & Y_{00} & Y_{01} & Y_{02} & Y_{03} & Y_{04} \\ 1 & Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ 1 & Y_{20} & Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ 1 & Y_{30} & Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ 1 & Y_{40} & Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{vmatrix} = 0,$$

General strategy

Now we know the form of the spurious terms:

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1}\left[d(i_0 i_1 i_2 i_3)+\tilde{d}(q;i_0 i_1 i_2 i_3)\right]\prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1}\left[c(i_0 i_1 i_2)+\tilde{c}(q;i_0 i_1 i_2)\right]\prod_{i\neq i_0,i_1,i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1}\left[b(i_0 i_1)+\tilde{b}(q;i_0 i_1)\right]\prod_{i\neq i_0,i_1}^{m-1} D_i + \sum_{i_0}^{m-1}\left[a(i_0)+\tilde{a}(q;i_0)\right]\prod_{i\neq i_0}^{m-1} D_i
\end{aligned}
$$

Our calculation is now reduced to an algebraic problem

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\end{aligned}
$$

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating N(q) for a set of values of the integration momentum q

There is a very good set of such points: Use values of q for which a set of denominators D_i vanish → The system becomes "triangular": solve first for 4-point functions, then 3-point functions and so on

Example

$$N(q) = d + \tilde{d}(q) + \sum_{i=0}^{3} [c(i) + \tilde{c}(q;i)] D_i + \sum_{i_0<i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] D_{i_0} D_{i_1}$$

$$+ \sum_{i_0=0}^{3} [a(i_0) + \tilde{a}(q;i_0)] D_{i \neq i_0} D_{j \neq i_0} D_{k \neq i_0}$$

We look for a q of the form $q^\mu = -p_0^\mu + x_i \ell_i^\mu$ such that

$$D_0 = D_1 = D_2 = D_3 = 0$$

$\longrightarrow$ we get a system of equations in x_i that has two solutions $q_0^\pm$

Example

$$N(q) = d + \tilde{d}(q)$$

Our "master formula" for $q = q_0^{\pm}$ is:

$$N(q_0^{\pm}) = [d + \tilde{d}\ T(q_0^{\pm})]$$

$\longrightarrow$ solve to extract the coefficients d and $\tilde{d}$

Example

$$N(q) - d - \tilde{d}(q) = \sum_{i=0}^{3} [c(i) + \tilde{c}(q;i)] D_i + \sum_{i_0<i_1}^{3} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] D_{i_0} D_{i_1}$$

$$+ \sum_{i_0=0}^{3} [a(i_0) + \tilde{a}(q; i_0)] D_{i\neq i_0} D_{j\neq i_0} D_{k\neq i_0}$$

Then we can move to the extraction of c coefficients using

$$N'(q) = N(q) - d - \tilde{d}\, T(q)$$

and setting to zero three denominators (ex: $D_1 = 0$, $D_2 = 0$, $D_3 = 0$)

Example

$$N(q) - d - \tilde{d}(q) \quad = \quad [c(0) + \tilde{c}(q;0)]\, D_0$$

We have infinite values of q for which

$$D_1 = D_2 = D_3 = 0 \quad \text{and} \quad D_0 \neq 0$$

$\rightarrow$ Here we need 7 of them to determine $c(0)$ and $\tilde{c}(q;0)$

Rational Terms - I

- Let's go back to the integrand

$$A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

- Insert the expression for $N(q) \rightarrow$ we know all the coefficients

$$N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d + \tilde{d}(q) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0<i_1<i_2}^{m-1} [c + \tilde{c}(q)] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i + \cdots$$

- Finally rewrite all denominators using

$$\frac{D_i}{\bar{D}_i} = \bar{Z}_i \,, \qquad \text{with} \quad \bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i} \right)$$

Rational Terms - I

$$
\begin{aligned}
A(\bar{q}) \quad = \quad & \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{Z}_i \\
+ \quad & \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{Z}_i \\
+ \quad & \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \prod_{i \neq i_0, i_1}^{m-1} \bar{Z}_i \\
+ \quad & \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \prod_{i \neq i_0}^{m-1} \bar{Z}_i
\end{aligned}
$$

The rational part is produced, after integrating over $d^n q$, by the $\tilde{q}^2$ dependence in $\bar{Z}_i$

$$
\bar{Z}_i \equiv \left(1 - \frac{\tilde{q}^2}{\bar{D}_i}\right)
$$

RATIONAL TERMS - I

The "Extra Integrals" are of the form

$$I_{s;\mu_1\cdots\mu_r}^{(n;2\ell)} \equiv \int d^n q\, \tilde{q}^{2\ell} \frac{q_{\mu_1}\cdots q_{\mu_r}}{\bar{D}(k_0)\cdots\bar{D}(k_s)},$$

where

$$\bar{D}(k_i) \equiv (\bar{q}+k_i)^2 - m_i^2\,,\, k_i = p_i - p_0$$

These integrals:

- **have dimensionality** $\mathcal{D} = 2(1+\ell-s)+r$
- **contribute only when** $\mathcal{D} \geq 0$**, otherwise are of** $\mathcal{O}(\epsilon)$

Rational Terms - II

Expand in D-dimensions ?

$$\bar{D}_i = D_i + \tilde{q}^2$$

Rational Terms - II

Expand in D-dimensions ?

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2)\right] \prod_{i\neq i_0,i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2)\right] \prod_{i\neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

Rational Terms - II

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$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2)\right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

$$m_i^2 \to m_i^2 - \tilde{q}^2$$

Polynomial dependence on $\tilde{q}^2$

$$b(ij;\tilde{q}^2) \;=\; b(ij) + \tilde{q}^2 b^{(2)}(ij)\,, \quad c(ijk;\tilde{q}^2) \;=\; c(ijk) + \tilde{q}^2 c^{(2)}(ijk)\,.$$

Rational Terms - II

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$$\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} \;=\; -\frac{i\pi^2}{2}\left[m_i^2 + m_j^2 - \frac{(p_i-p_j)^2}{3}\right] + \mathcal{O}(\epsilon)\,,$$

$$\int d^n\bar{q}\,\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} \;=\; -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)\,, \quad \int d^n\bar{q}\,\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} \;=\; -\frac{i\pi^2}{6} + \mathcal{O}(\epsilon)\,.$$

Rational Terms - II

Furthermore, by defining

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) \equiv \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3;\tilde{q}^2) + \tilde{d}(q;i_0 i_1 i_2 i_3;\tilde{q}^2) \right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i \,,$$

the following expansion holds

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) = \sum_{j=2}^{m} \tilde{q}^{(2j-4)} d^{(2j-4)}(q)\,,$$

where the last coefficient is independent on q

$$d^{(2m-4)}(q) = d^{(2m-4)}\,.$$

Rational Terms - II

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

$$
\begin{aligned}
\mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\
&- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) .
\end{aligned}
$$

G. Ossola, C. G. Papadopoulos and R. Pittau,arXiv:0802.1876 [hep-ph]

Rational Terms - R_2

A different source of Rational Terms, called R_2, can also be generated from the ϵ-dimensional part of $N(q)$

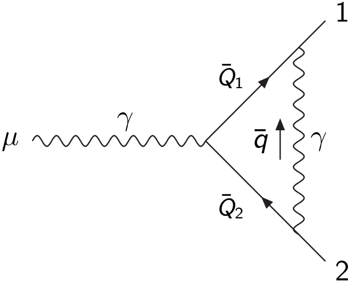
$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, \epsilon; q)$$

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q} \frac{\tilde{N}(\tilde{q}^2, \epsilon; q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\mathcal{R}_2$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

New vertices/particles or GKM-approach

Rational Terms - R_2



$$\bar{Q}_1 = \bar{q} + p_1 = Q_1 + \tilde{q}$$
$$\bar{Q}_2 = \bar{q} + p_2 = Q_2 + \tilde{q}$$

$$\bar{D}_0 = \bar{q}^2$$

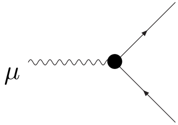
$$\bar{D}_1 = (\bar{q} + p_1)^2$$

$$\bar{D}_2 = (\bar{q} + p_2)^2$$

$$\begin{aligned} \bar{N}(\bar{q}) &\equiv e^3 \left\{ \bar{\gamma}_{\bar{\beta}} \, (\bar{\not{Q}}_1 + m_e) \, \gamma_\mu \, (\bar{\not{Q}}_2 + m_e) \, \bar{\gamma}^{\bar{\beta}} \right\} \\ &= e^3 \, \{ \gamma_\beta (\not{Q}_1 + m_e) \gamma_\mu (\not{Q}_2 + m_e) \gamma^\beta \\ &- \epsilon \, (\not{Q}_1 - m_e) \gamma_\mu (\not{Q}_2 - m_e) + \epsilon \tilde{q}^2 \, \gamma_\mu - \tilde{q}^2 \, \gamma_\beta \gamma_\mu \gamma^\beta \} \; , \end{aligned}$$

$$\int d^n\bar{q}\frac{\tilde{q}^2}{\bar{D}_0\bar{D}_1\bar{D}_2} = -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon)\,,$$
$$\int d^n\bar{q}\frac{q_\mu q_\nu}{\bar{D}_0\bar{D}_1\bar{D}_2} = -\frac{i\pi^2}{2\epsilon}g_{\mu\nu} + \mathcal{O}(1)\,,$$

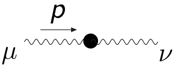
$$\mathrm{R}_2 = -\frac{ie^3}{8\pi^2}\gamma_\mu + \mathcal{O}(\epsilon)\,,$$



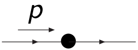
$$= -\frac{ie^3}{8\pi^2}\gamma_\mu$$

Rational Terms - R_2

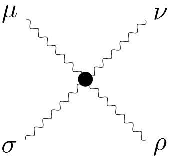
Rational counterterms



$$= -\frac{ie^2}{8\pi^2} g_{\mu\nu} \left(2m_e^2 - p^2/3\right)$$



$$= \frac{ie^2}{16\pi^2}(-\not{p} + 2m_e)$$



$$= \frac{ie^4}{12\pi^2}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right)$$

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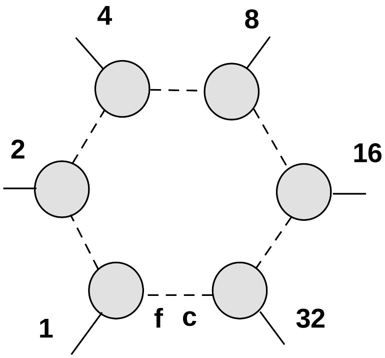
Evaluate scalar integrals

- massive integrals $\longrightarrow$ FF [G. J. van Oldenborgh]
- massless+massive integrals $\longrightarrow$ OneLOop [A. van Hameren]


4
8
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1
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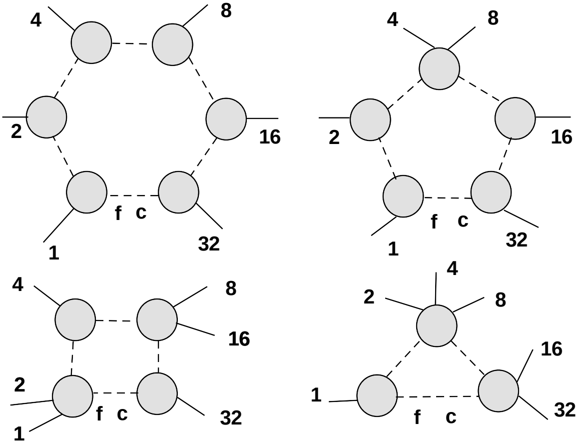
4
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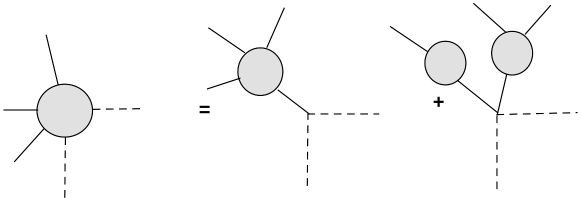
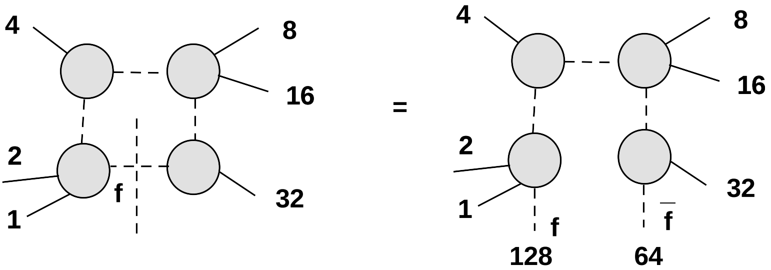
f c
32

HELAC 1-LOOP




4
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4
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1
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f
f̄
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64

HELAC Color treatment

$$\mathcal{M}^{a_1,i_2,\ldots,i_k}_{j_2,\ldots,j_k} t^{a_1}_{i_1 j_1} \rightarrow \mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}$$

$$\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} = \sum_{\sigma} \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} A_{\sigma}$$

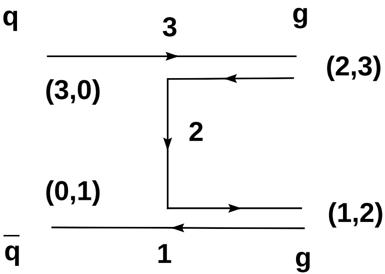
$$\sum_{\{i\},\{j\}} |\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}|^2$$

$$\sum_{\sigma,\sigma'} A^*_{\sigma} \mathcal{C}_{\sigma,\sigma'} A_{\sigma'}$$

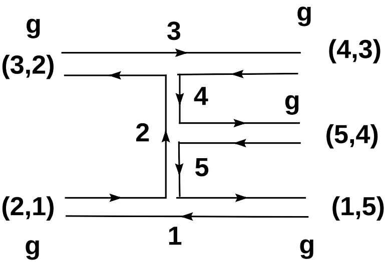
$$\mathcal{C}_{\sigma,\sigma\prime} \equiv \sum_{\{i\},\{j\}} \delta_{i_{\sigma_1},j_1} \delta_{i_{\sigma_2},j_2} \ldots \delta_{i_{\sigma_k},j_k} \delta_{i_{\sigma'_1},j_1} \delta_{i_{\sigma'_2},j_2} \ldots \delta_{i_{\sigma'_k},j_k}$$

$$(x_1, y_1) \ldots (x_n, y_n)$$

where y_i take the values $\{1, 2, \ldots, n_l\}$ if i is a gluon or an outgoing quark (incoming anti-quark) otherwise $y_i = 0$, whereas x_i take the values $\{\sigma_1, \sigma_2, \ldots, \sigma_{n_l}\}$ if i is a gluon or an incoming quark (outgoing anti-quark) otherwise $x_i = 0$. So for instance for a $q\bar{q} \to gg$ process, $n_l = 3$ and a possible color connection is given by $(3,0)(0,1)(1,2)(2,3)$



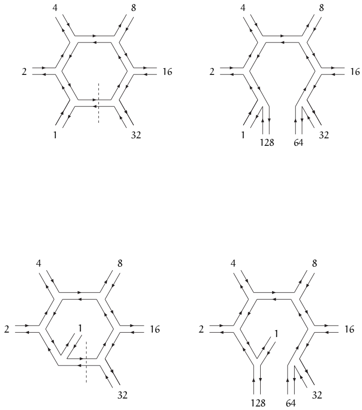
whereas for $gg \rightarrow ggg$, $n_l = 5$ and a possible color connection is given by $(2,1)(3,2)(4,3)(5,4)(1,5)$

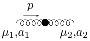


$$\mathcal{C}_{\sigma,\sigma'} = N_c^{m(\sigma,\sigma')}$$

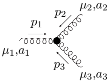
where $m(\sigma,\sigma')$ count the number of common cycles of the two permutations.

HELAC Color treatment - 1 loop

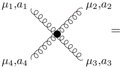




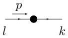
$$= \frac{ig^2 N_{col}}{48\pi^2}\,\delta_{a_1a_2}\left[\frac{p^2}{2}g_{\mu_1\mu_2} + \lambda_{HV}\left(g_{\mu_1\mu_2}p^2 - p_{\mu_1}p_{\mu_2}\right) + \frac{N_f}{N_{col}}\,(p^2 - 6\,m_q^2)\,g_{\mu_1\mu_2}\right]$$



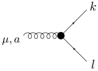
$$= -\frac{g^3 N_{col}}{48\pi^2}\left(\frac{7}{4} + \lambda_{HV} + 2\frac{N_f}{N_{col}}\right) f^{a_1a_2a_3}\,V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$$



$$= -\frac{ig^4 N_{col}}{96\pi^2}\sum_{P(234)}\Bigg\{\Big[\frac{\delta_{a_1a_2}\delta_{a_3a_4} + \delta_{a_1a_3}\delta_{a_4a_2} + \delta_{a_1a_4}\delta_{a_2a_3}}{N_{col}} + 4\,Tr(t^{a_1}t^{a_3}t^{a_2}t^{a_4} + t^{a_1}t^{a_4}t^{a_2}t^{a_3})\,(3+\lambda_{HV}) - Tr(\{t^{a_1}t^{a_2}\}\{t^{a_3}t^{a_4}\})\,(5 + 2\lambda_{HV})\Big]\,g_{\mu_1\mu_2}g_{\mu_3\mu_4} + 12\frac{N_f}{N_{col}}Tr(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\left(\frac{5}{3}g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_2\mu_3}g_{\mu_1\mu_4}\right)\Bigg\}$$

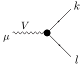


$$= \frac{ig^2}{16\pi^2}\frac{N_{col}^2 - 1}{2N_{col}}\delta_{kl}(-\not{p} + 2m_q)\,\lambda_{HV}$$

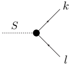


$$= \frac{ig^3}{16\pi^2}\frac{N_{col}^2 - 1}{2N_{col}}\,t^a_{kl}\gamma_\mu\,(1 + \lambda_{HV})$$

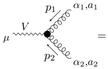
HELAC R2 TERMS



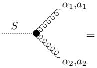
$$= -\frac{g^2}{16\pi^2}\frac{N_{col}^2-1}{2N_{col}}\,\delta_{kl}\gamma_\mu\,(v+a\gamma_5)\,(1+\lambda_{HV})$$



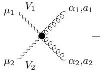
$$= -\frac{g^2}{8\pi^2}\frac{N_{col}^2-1}{2N_{col}}\,\delta_{kl}\,(c+d\gamma_5)\,(1+\lambda_{HV})$$



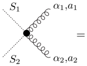
$$= a\,\frac{ig^2}{12\pi^2}\,\delta_{a_1a_2}\epsilon_{\mu\alpha_1\alpha_2\beta}\,(p_1-p_2)^\beta$$



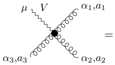
$$= c\,\frac{g^2}{8\pi^2}\,\delta_{a_1a_2}g_{\alpha_1\alpha_2}\,m_q$$



$$= -\frac{ig^2}{24\pi^2}\,\delta_{a_1a_2}(v_1v_2+a_1a_2)\,(g_{\mu_1\mu_2}g_{\alpha_1\alpha_2}+g_{\mu_1\alpha_1}g_{\mu_2\alpha_2}+g_{\mu_1\alpha_2}g_{\mu_2\alpha_1})$$



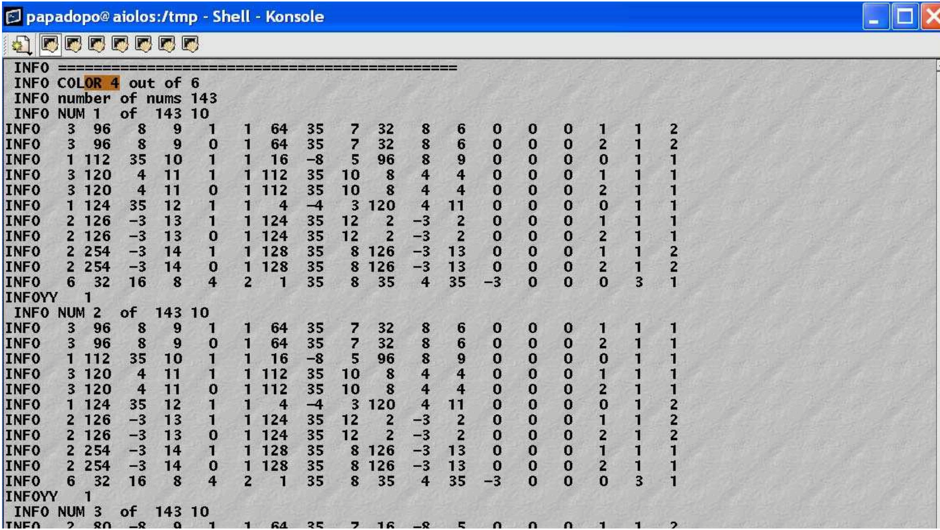
$$= \frac{ig^2}{8\pi^2}\,\delta_{a_1a_2}\,(c_1c_2-d_1d_2)\,g_{\alpha_1\alpha_2}$$



$$= -\frac{g^3}{24\pi^2}\,\{v\,Tr(t^{a_1}\{t^{a_2}t^{a_3}\})(g_{\mu\alpha_1}g_{\alpha_2\alpha_3}+g_{\mu\alpha_2}g_{\alpha_1\alpha_3}+g_{\mu\alpha_3}g_{\alpha_1\alpha_2})$$

$$-i9a\,\left[Tr(t^{a_1}t^{a_2}t^{a_3})-Tr(t^{a_1}t^{a_3}t^{a_2})\right]\,\epsilon_{\mu\alpha_1\alpha_2\alpha_3}\}$$

```
 kmax,ncc,imax 6 24 6 10
 INFO ==========================================
 INFO COLOR 1 out of 6
INFO   1  12  35   7   1   1   4  -4   3   8   4   4   0   0   0   0   1   2
INFO   1  48  35   8   1   1  16  -8   5  32   8   6   0   0   0   0   1   2
INFO   2  14  -3   9   1   1  12  35   7   2  -3   2   0   0   0   1   1   2
INFO   2  14  -3   9   0   1  12  35   7   2  -3   2   0   0   0   2   1   2
INFO   2  28  -8  10   1   1  12  35   7  16  -8   5   0   0   0   1   1   2
INFO   2  28  -8  10   0   1  12  35   7  16  -8   5   0   0   0   2   1   2
INFO   3  44   8  11   1   1  12  35   7  32   8   6   0   0   0   1   1   2
INFO   3  44   8  11   0   1  12  35   7  32   8   6   0   0   0   2   1   2
INFO   2  50  -3  12   1   1  48  35   8   2  -3   2   0   0   0   1   1   2
INFO   2  50  -3  12   0   1  48  35   8   2  -3   2   0   0   0   2   1   2
INFO   2  52  -4  13   1   1  48  35   8   4  -4   3   0   0   0   1   1   2
INFO   2  52  -4  13   0   1  48  35   8   4  -4   3   0   0   0   2   1   2
INFO   3  56   4  14   1   1  48  35   8   8   4   4   0   0   0   1   1   2
INFO   3  56   4  14   0   1  48  35   8   8   4   4   0   0   0   2   1   2
INFO   1  60  35  15   1   4   4  -4   3  56   4  14   0   0   0   0   1   2
INFO   1  60  35  15   2   4  16  -8   5  44   8  11   0   0   0   0   1   2
INFO   1  60  35  15   3   4  28  -8  10  32   8   6   0   0   0   0   1   2
INFO   1  60  35  15   4   4  52  -4  13   8   4   4   0   0   0   0   1   2
INFO   2  62  -3  16   1   3  12  35   7  50  -3  12   0   0   0   1   1   2
INFO   2  62  -3  16   0   3  12  35   7  50  -3  12   0   0   0   2   1   2
INFO   2  62  -3  16   2   3  48  35   8  14  -3   9   0   0   0   1   1   2
INFO   2  62  -3  16   0   3  48  35   8  14  -3   9   0   0   0   2   1   2
INFO   2  62  -3  16   3   3  60  35  15   2  -3   2   0   0   0   1   1   2
INFO   2  62  -3  16   0   3  60  35  15   2  -3   2   0   0   0   2   1   2
 INFO ==========================================
 INFO COLOR 2 out of 6
INFO   1  12  35   7   1   1   4  -4   3   8   4   4   0   0   0   0   1   1
INFO   1  48  35   8   1   1  16  -8   5  32   8   6   0   0   0   0   1   1
```




```
 INFO ==============================================
 INFO COLOR 4 out of 6
 INFO number of nums 143
 INFO NUM 1  of  143 10
INFO   3  96   8   9   1   1  64  35   7  32   8   6   0   0   0   1   1   2
INFO   3  96   8   9   0   1  64  35   7  32   8   6   0   0   0   2   1   2
INFO   1 112  35  10   1   1  16  -8   5  96   8   9   0   0   0   0   1   1
INFO   3 120   4  11   1   1 112  35  10   8   4   4   0   0   0   1   1   1
INFO   3 120   4  11   0   1 112  35  10   8   4   4   0   0   0   2   1   1
INFO   1 124  35  12   1   1   4  -4   3 120   4  11   0   0   0   0   1   1
INFO   2 126  -3  13   1   1 124  35  12   2  -3   2   0   0   0   1   1   1
INFO   2 126  -3  13   0   1 124  35  12   2  -3   2   0   0   0   2   1   1
INFO   2 254  -3  14   1   1 128  35   8 126  -3  13   0   0   0   1   1   2
INFO   2 254  -3  14   0   1 128  35   8 126  -3  13   0   0   0   2   1   2
INFO   6  32  16   8   4   2   1  35   8  35   4  35  -3   0   0   0   3   1
INFOYY   1
 INFO NUM 2  of  143 10
INFO   3  96   8   9   1   1  64  35   7  32   8   6   0   0   0   1   1   1
INFO   3  96   8   9   0   1  64  35   7  32   8   6   0   0   0   2   1   1
INFO   1 112  35  10   1   1  16  -8   5  96   8   9   0   0   0   0   1   1
INFO   3 120   4  11   1   1 112  35  10   8   4   4   0   0   0   1   1   1
INFO   3 120   4  11   0   1 112  35  10   8   4   4   0   0   0   2   1   1
INFO   1 124  35  12   1   1   4  -4   3 120   4  11   0   0   0   0   1   2
INFO   2 126  -3  13   1   1 124  35  12   2  -3   2   0   0   0   1   1   2
INFO   2 126  -3  13   0   1 124  35  12   2  -3   2   0   0   0   2   1   2
INFO   2 254  -3  14   1   1 128  35   8 126  -3  13   0   0   0   1   1   1
INFO   2 254  -3  14   0   1 128  35   8 126  -3  13   0   0   0   2   1   1
INFO   6  32  16   8   4   2   1  35   8  35   4  35  -3   0   0   0   3   1
INFOYY   1
 INFO NUM 3  of  143 10
INFO   2  80  -8   9   1   1  64  35   7  16  -8   5   0   0   0   1   1   2
```

HELAC 1-LOOP

papadopo@aiolos:/tmp - Shell - Konsole

```
 INFO NUM 127   of   143  15
INFO    1  48  35   9   1   1  16  -8   5  32   8   6   0   0   0   0   1   1
INFO    3 112   3  10   1   1  48  35   9  64   3   7   0   0   0   1   1   1
INFO    3 112   3  10   0   1  48  35   9  64   3   7   0   0   0   2   1   1
INFO    1  12  35  11   1   1   4  -4   3   8   4   4   0   0   0   0   1   1
INFO    1 240  35  12   1   1 128  -3   8 112   3  10   0   0   0   0  -1   1
INFO    2 242  -3  13   1   1 240  35  12   2  -3   2   0   0   0   1   1   1
INFO    2 242  -3  13   0   1 240  35  12   2  -3   2   0   0   0   2   1   1
INFO    3 248   4  14   1   1 240  35  12   8   4   4   0   0   0   1   1   1
INFO    3 248   4  14   0   1 240  35  12   8   4   4   0   0   0   2   1   1
INFO    1 252  35  15   1   2   4  -4   3 248   4  14   0   0   0   0   1   1
INFO    4 252  35  15   2   2  12  35  11 240  35  12   0   0   0   0   1   1
INFO    2 254  -3  16   1   2  12  35  11 242  -3  13   0   0   0   1   1   1
INFO    2 254  -3  16   0   2  12  35  11 242  -3  13   0   0   0   2   1   1
INFO    2 254  -3  16   2   2 252  35  15   2  -3   2   0   0   0   1   1   1
INFO    2 254  -3  16   0   2 252  35  15   2  -3   2   0   0   0   2   1   1
INFO    2  48  15   3   3   0   0   0   0   0   0   0   0   0   0   0   2   5
INFOYY     5
 INFO NUM 128   of   143  11
INFO    1  12  35   7   1   1   4  -4   3   8   4   4   0   0   0   0   1   1
INFO    1  48  35   8   1   1  16  -8   5  32   8   6   0   0   0   0   1   1
INFO    2  28  -8   9   1   1  12  35   7  16  -8   5   0   0   0   1   1   1
INFO    2  28  -8   9   0   1  12  35   7  16  -8   5   0   0   0   2   1   1
INFO    3  56   4  10   1   1  48  35   8   8   4   4   0   0   0   1   1   1
INFO    3  56   4  10   0   1  48  35   8   8   4   4   0   0   0   2   1   1
INFO    1  60  35  11   1   3   4  -4   3  56   4  10   0   0   0   0   1   1
INFO    4  60  35  11   2   3  12  35   7  48  35   8   0   0   0   0   1   1
INFO    1  60  35  11   3   3  28  -8   9  32   8   6   0   0   0   0   1   1
INFO   25  62  -3  12   1   1  60  35  11   2  -3   2   0   0   0   1   1   1
INFO   25  62  -3  12   0   1  60  35  11   2  -3   2   0   0   0   2   1   1
INFO    0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   1
INFOYY     1
 INFO NUM 129   of   143  12
INFO   23  12  35   7   1   1   4  -4   3   8   4   4   0   0   0   0   1   1
```

HELAC RESULTS

$pp \to t\bar{t}b\bar{b}$			
$u\bar{u} \to t\bar{t}b\bar{b}$			
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-2.347908989000179E-07	-2.082520105681483E-07	3.909384299635230E-07
$I(\epsilon)$	-2.347908989000243E-07	-2.082520105665445E-07	
$gg \to t\bar{t}b\bar{b}$			
HELAC-1L	-1.435108168334016E-06	-2.085070773763073E-06	3.616343483497464E-06
$I(\epsilon)$	-1.435108168334035E-06	-2.085070773651439E-06	

	p_x	p_y	p_z	E
$u(g)$	0	0	250	250
$\bar{u}(g)$	0	0	-250	250
t	12.99421901255723	-9.591511769543683	75.05543670827210	190.1845561691092
$\bar{t}$	53.73271578143694	-0.2854146459513714	17.68101382654795	182.9642163285034
b	-41.57664370692741	3.895531135098977	-91.94931862397770	100.9874727883170
$\bar{b}$	-25.15029108706678	5.981395280396083	-0.7871319108423604	25.86375471407044

HELAC RESULTS

$pp \to VVb\bar{b}$ and $pp \to VV+$ 2 jets			
$u\bar{u} \to W^+W^-b\bar{b}$			
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-2.493916939359002E-07	-4.885901774740355E-07	1.592538533368835E-07
$I(\epsilon)$	-2.493916939359001E-07	-4.885901774752593E-07	
$gg \to W^+W^-b\bar{b}$			
HELAC-1L	-2.686310592221201E-07	-6.078682316434646E-07	-2.431624440346638E-07
$I(\epsilon)$	-2.686310592221206E-07	-6.078682340168020E-07	

	p_x	p_y	p_z	E
$u(g)$	0	0	250	250
$\bar{u}(g)$	0	0	-250	250
W^+	22.40377113462118	-16.53704884550758	129.4056091248114	154.8819879118765
W^-	92.64238702192333	-0.4920930146078141	30.48443210132545	126.4095336206695
b	-71.68369328357026	6.716416578342183	-158.5329205583824	174.1159068988160
$\bar{b}$	-43.36246487297426	10.31272528177322	-1.357120667754454	44.59257156863792

HELAC RESULTS

$pp \to V +$ 3 jets			
$u\bar{d} \to W^+ggg$			
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-1.995636628164684E-05	-5.935610843551600E-05	-5.323285370666314E-05
$I(\epsilon)$	-1.995636628164686E-05	-5.935610843566534E-05	
$u\bar{u} \to Zggg$			
HELAC-1L	-7.148261887172997E-06	-2.142170009323704E-05	-1.906378375774021E-05
$I(\epsilon)$	-7.148261887172976E-06	-2.142170009540120E-05	

	p_x	p_y	p_z	E
u	0	0	250	250
$\bar{d}$	0	0	-250	250
W^+	23.90724239064912	-17.64681636854432	138.0897548661186	162.5391101447744
g	98.85942812363483	-0.5251163702879512	32.53017998659339	104.0753327455388
g	-76.49423931754684	7.167141557113385	-169.1717405928078	185.8004692730082
g	-46.27243119673712	11.00479118171890	-1.448194259904179	47.58508783667868

HELAC RESULTS

$pp \to t\bar{t}+$ 2 jets			
$u\bar{u} \to t\bar{t}gg$			
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-6.127108113312741E-05	-1.874963444741646E-04	-3.305349683690902E-04
$I(\epsilon)$	-6.127108113312702E-05	-1.874963445081074E-04	
$gg \to t\bar{t}gg$			
HELAC-1L	-3.838786514961561E-04	-9.761168899507888E-04	-5.225385984750410E-04
$I(\epsilon)$	-3.838786514961539E-04	-9.761168898436521E-04	

	p_x	p_y	p_z	E
$u(g)$	0	0	250	250
$\bar{u}(g)$	0	0	-250	250
t	12.99421901255723	-9.591511769543683	75.05543670827210	190.1845561691092
$\bar{t}$	53.73271578143694	-0.2854146459513714	17.68101382654795	182.9642163285034
g	-41.57664370692741	3.895531135098977	-91.94931862397770	100.9874727883170
g	-25.15029108706678	5.981395280396083	-0.7871319108423604	25.86375471407044

HELAC RESULTS

$pp \to b\bar{b}b\bar{b}$			
$u\bar{u} \to b\bar{b}b\bar{b}$			
	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
HELAC-1L	-9.205269484951069E-08	-2.404679886692200E-07	-2.553568662778129E-07
$I(\epsilon)$	-9.205269484951025E-08	-2.404679886707971E-07	
$gg \to b\bar{b}b\bar{b}$			
HELAC-1L	-2.318436429821683E-05	-6.958360737366907E-05	-7.564212339279291E-05
$I(\epsilon)$	-2.318436429821662E-05	-6.958360737341511E-05	

	p_x	p_y	p_z	E
$u(g)$	0	0	250	250
$\bar{u}(g)$	0	0	-250	250
b	24.97040523056789	-18.43157602837212	144.2306511496888	147.5321146846735
$\bar{b}$	103.2557390255471	-0.5484684659584054	33.97680766420219	108.7035966213640
b	-79.89596300367462	7.485866671764871	-176.6948628845280	194.0630765341365
$\bar{b}$	-48.33018125244035	11.49417782256567	-1.512595929362970	49.70121215982584

$pp \longrightarrow t\bar{t}b\bar{b}$

$pp \to t\bar{t}b\bar{b}$

- Virtual corrections

$pp \longrightarrow t\bar{t}b\bar{b}$

- Virtual corrections
- $I(\epsilon)$ operator or integrated dipoles

Applications

$pp \to t\bar{t}b\bar{b}$

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- SF collinear counter-terms

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- Re-weighting

$pp \to t\bar{t}b\bar{b}$

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- Re-weighting
- MC over colors

$$P(x) = \frac{g(x)}{\int dx g(x)}$$

$$\int dx f(x) = \int dx g(x) \frac{f(x)}{g(x)} = \int dx P(x) \frac{f(x)}{g(x)} \int dx g(x)$$

$$\int dx P(x) \frac{f(x)}{g(x)} = \langle \frac{f(x)}{g(x)} \rangle$$

MC-COLOR

$$\sum |M^{i_1 \ldots i_n}_{j_1 \ldots j_n}|^2$$

$$M^{i_1 \ldots i_n}_{j_1 \ldots j_n} = \sum \delta_{\sigma(i_1),j_1} \ldots \delta_{\sigma(i_n),j_n} \mathcal{A}_\sigma$$

Choose a color configuration

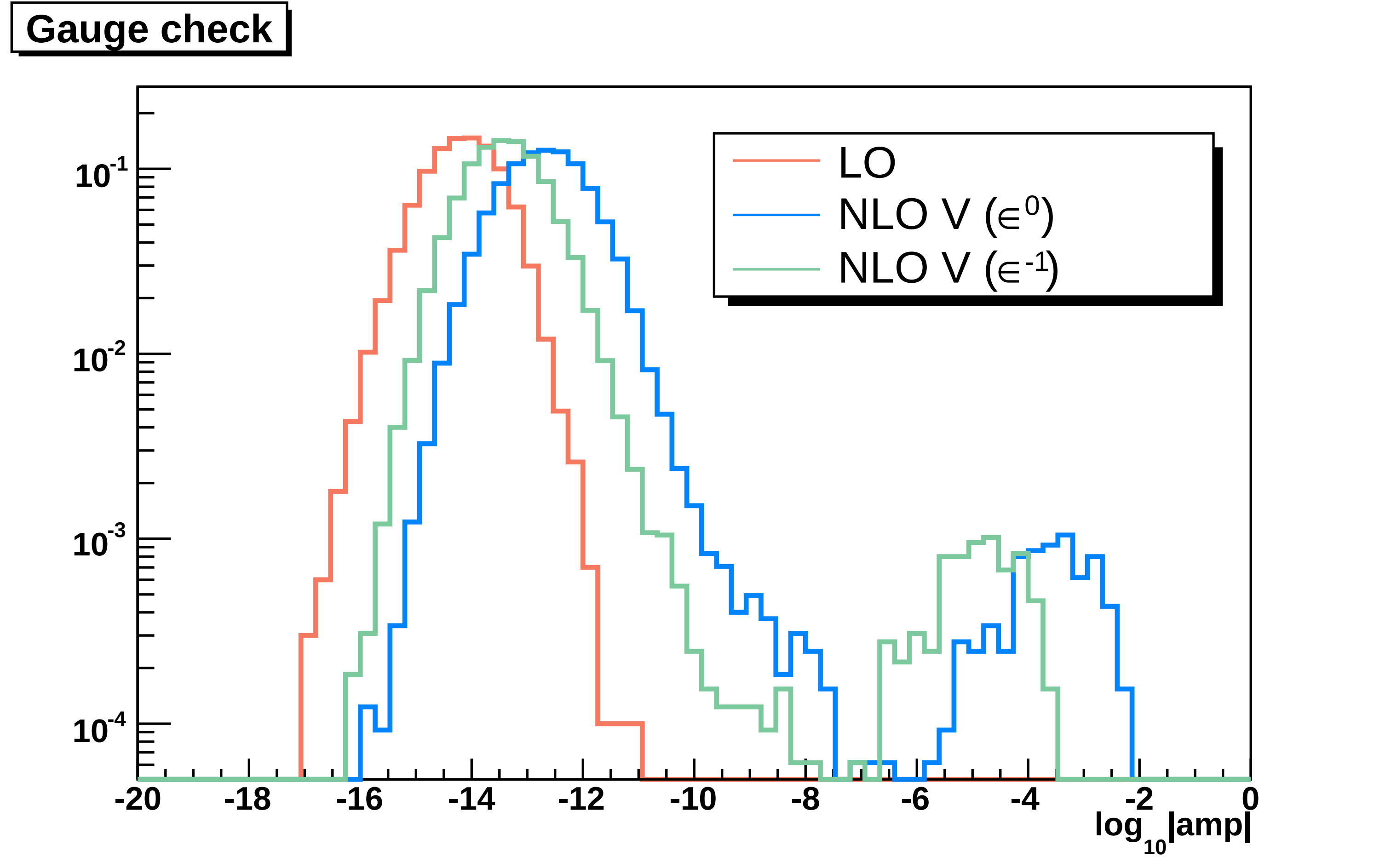
$$i_1 \ldots i_n \text{ and } \mathrm{j}_1 \ldots \mathrm{j}_\mathrm{n}$$

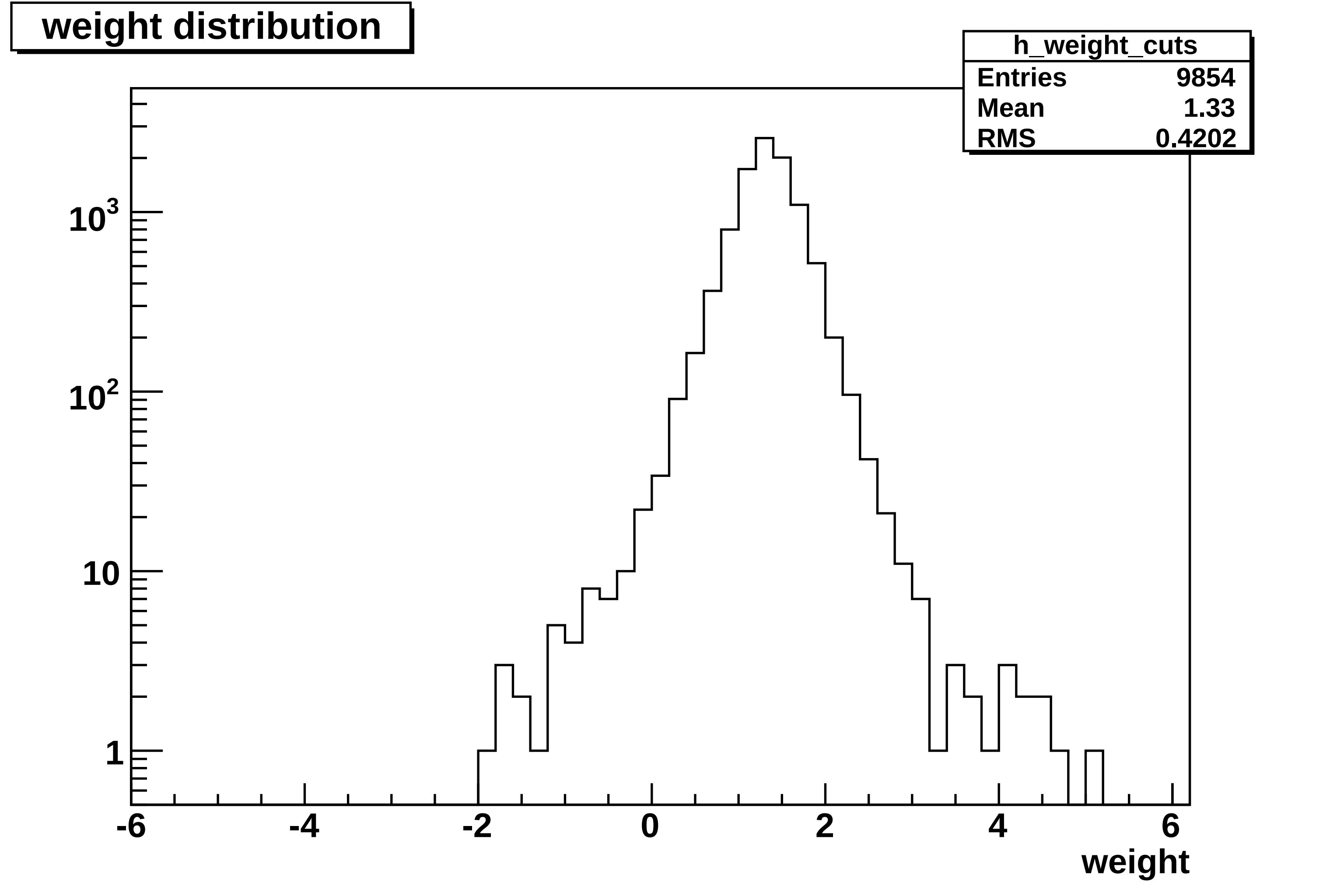
$$M^{i_1 \ldots i_n}_{j_1 \ldots j_n} = (\mathcal{A}_8 + \mathcal{A}_9 + \mathcal{A}_{23} + \mathcal{A}_{24})$$

$$M^{i_1 \ldots i_n}_{j_1 \ldots j_n} = T^{i_1 \ldots i_n}_{j_1 \ldots j_n} + L^{i_1 \ldots i_n}_{j_1 \ldots j_n}$$

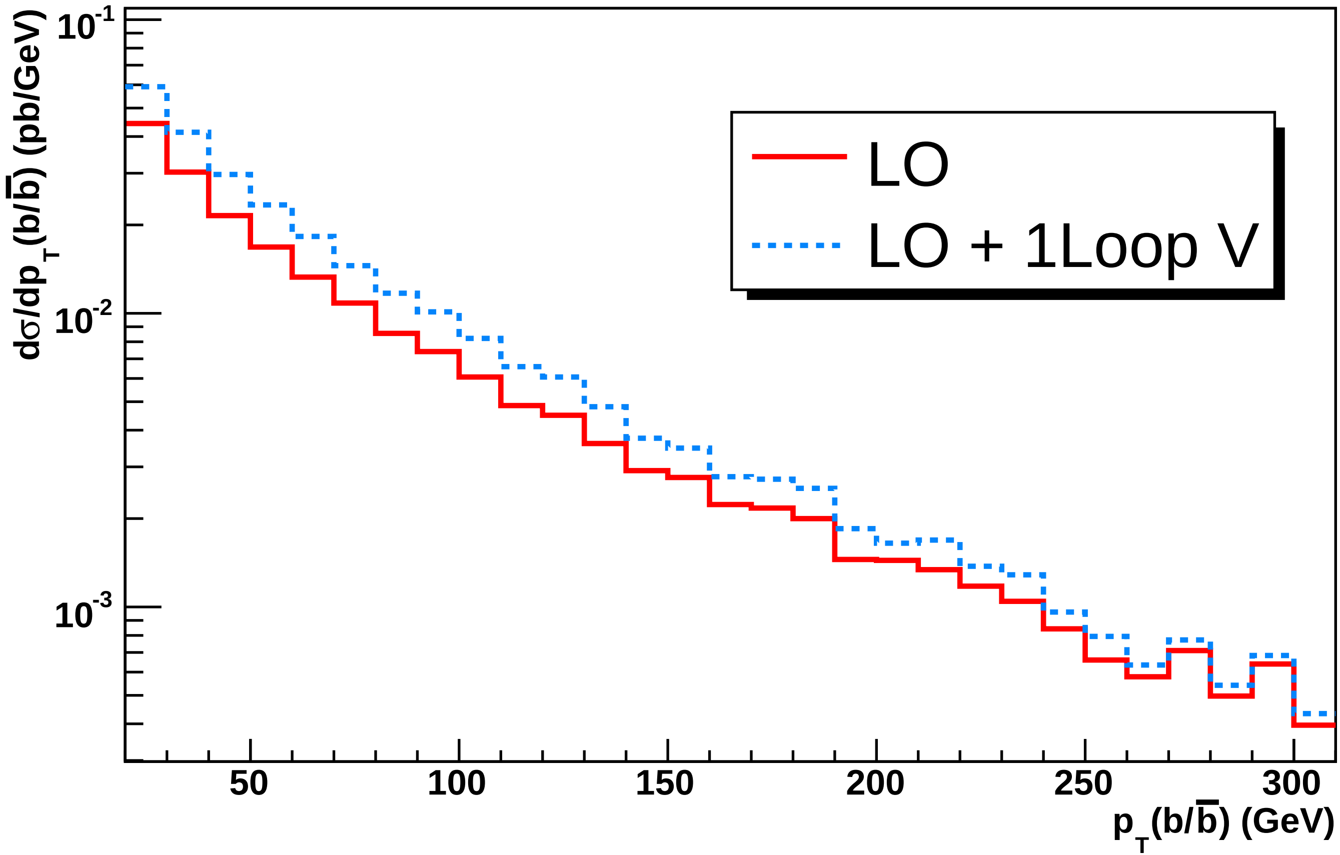
$$|M|^2 = |T|^2 (1 + \frac{T^* L + L^* T}{|T|^2})$$

	Integrated	Exact sum	Montecarlo sum
$gg \to b\bar{b}$	-0.748951	-0.744077	-0.743839
$gg \to t\bar{t}$	0.247375	0.251082	0.237591
$u\bar{u} \to b\bar{b}$	-0.463479	-0.465309	-0.458668
$u\bar{u} \to t\bar{t}$	0.0448277	0.0442139	0.0473160

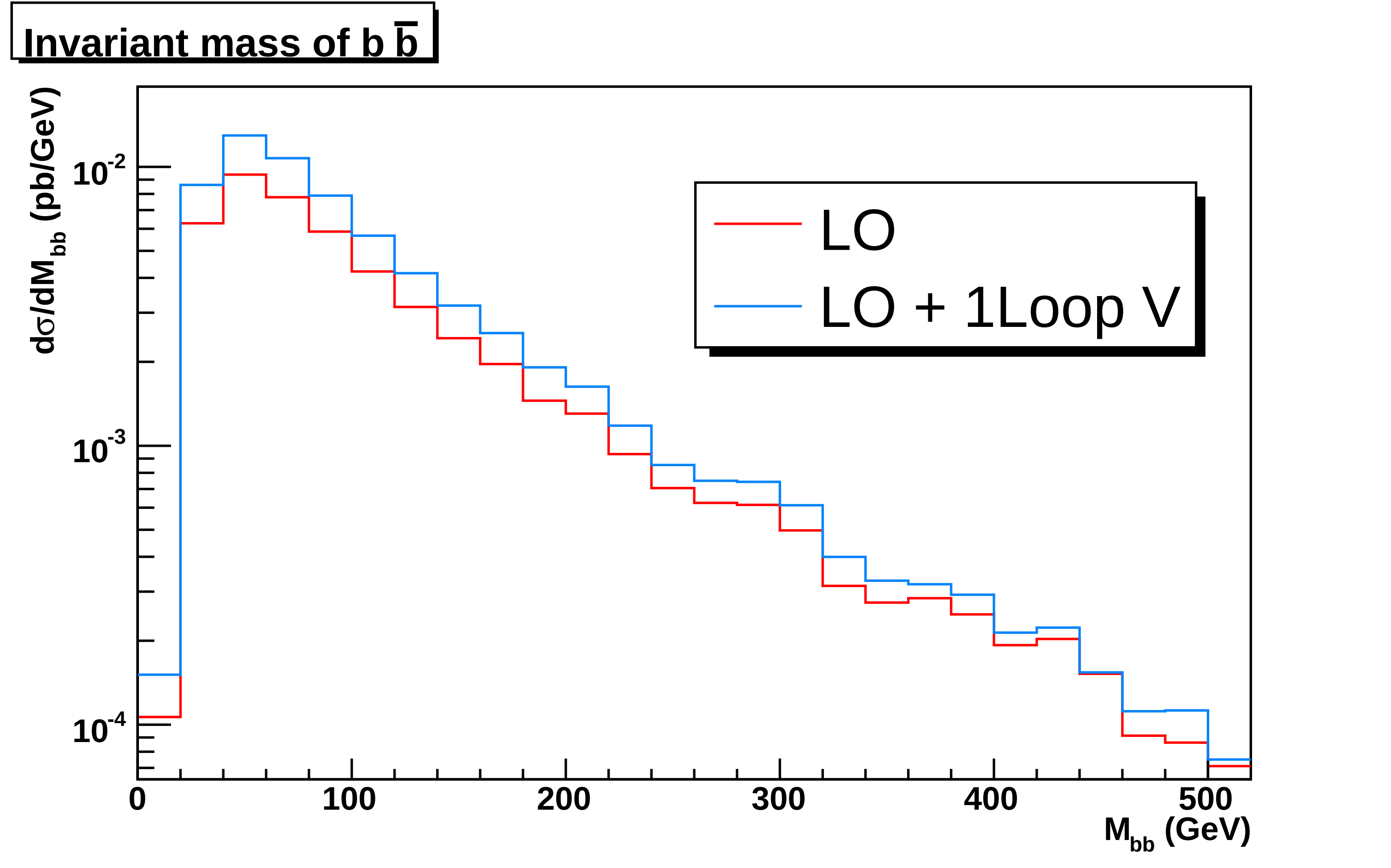

Gauge check
LO
NLO V (∈⁰)
NLO V (∈⁻¹)
10⁻¹
10⁻²
10⁻³
10⁻⁴
-20
-18
-16
-14
-12
-10
-8
-6
-4
-2
0
log10|amp|



weight distribution
h_weight_cuts
Entries 9854
Mean 1.33
RMS 0.4202
10^3
10^2
10
1
-6
-4
-2
0
2
4
6
weight



$d\sigma/dp_T(b/\bar{b})$ (pb/GeV)
10^{-1}
10^{-2}
10^{-3}
50
100
150
200
250
300
$p_T(b/\bar{b})$ (GeV)
LO
LO + 1Loop V



Invariant mass of b $\bar{b}$
$d\sigma/dM_{bb}$ (pb/GeV)
10^{-2}
10^{-3}
10^{-4}
LO
LO + 1Loop V
0
100
200
300
400
500
M_{bb} (GeV)

Outlook

OPP

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- changes the computational approach at one loop

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- Numerical but still algebraic: speed and precision not a problem

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- Automatize through Dyson-Schwinger equations Full one-loop amplitudes (color, rational)

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- Automatize the real contributions (dipoles)

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A generic NLO calculator *ante portas*

Tools 2009 ?

BlackHat

C.F. Berger, Z. Bern, L.J. Dixon, F. Febres Cordero, D. Forde, H. Ita, D.A. Kosower, D. Maitre, arXiv:0803.4180 [hep-ph]

Rocket

W. T. Giele and G. Zanderighi, arXiv:0805.2152 [hep-ph]

CutTools

G. Ossola, C. G. Papadopoulos and R. Pittau, [arXiv:0711.3596 [hep-ph]].

HELAC-1LOOP

A. van Hameren, C. G. Papadopoulos and R. Pittau [arXiv:0903.4665 [hep-ph]].