PHYSICS AT THE TERA SCALE
Helmholtz Alliance

NLO Subtraction Scheme

http://www.terascale.de/mc

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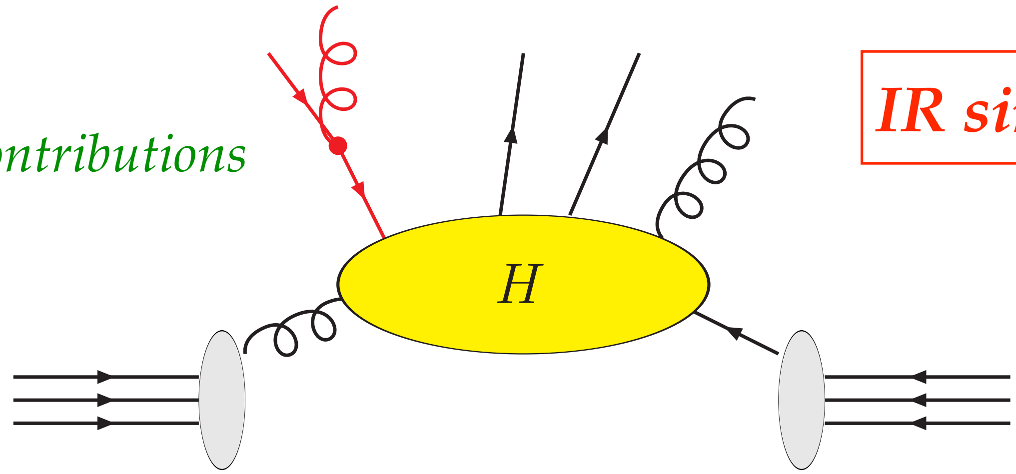
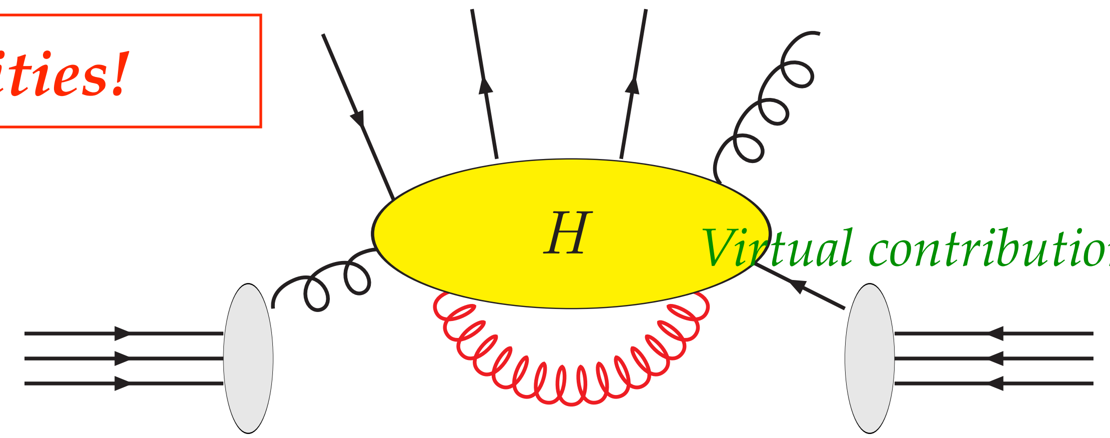
NLO Cross Section

Real contributions



IR singularities!

+



Virtual contributions

$$\sigma_{\mathrm{NLO}} = \int_N d\sigma^B + \int_{N+1} \left[d\sigma^R - d\sigma^A \right]_{\epsilon=0} + \int_N \left[d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0}$$

$$d\sigma^A \sim d\Gamma(\{p\}_{N+1}) \underbrace{V \otimes |\mathcal{M}(\{\tilde{p}\}_N)|^2}_{\text{Based on soft and collinear factorization}} F_J(\{\tilde{p}\}_N)$$

This is not a new problem and it has been already solved by Catani and Seymour in a very general way. *Why do we need a new NLO scheme?*

Motivation

Catani-Seymour dipole subtraction terms has the following structure:

$$d\sigma^{A} \sim d\Gamma(\{\hat{p}\}_{m+1}) \sum_{\substack{i,j \\ \text{pairs}}} \sum_{k\neq i,j} \mathcal{D}_{i,j,k}(\{\hat{p}\}_{m+1}) F_J(\{p\}_m^{(i,j,k)})$$

The number of the subtraction terms:

$$\sim \frac{N^3}{2}$$ where N is the number of the jets

We should minimize this number. In our scheme the number of the counter-terms is equal to *number of the possible collinear pairs*.

$$\sim \frac{N^2}{2}$$

Motivation

The physical cross section is

$$\sigma[F] = \sum_m \int [d\{p,f\}_m] \overbrace{f_{a/A}(\eta_{\rm a},\mu_F^2)\, f_{b/B}(\eta_{\rm b},\mu_F^2)}^{\text{parton distributions}} \frac{1}{2\eta_{\rm a}\eta_{\rm b} p_A\cdot p_B}$$
$$\times \big\langle \mathcal{M}(\{p,f\}_m)\big| \underbrace{F(\{p,f\}_m)}_{\text{observable}} \underbrace{\big|\mathcal{M}(\{p,f\}_m)\big\rangle}_{\text{matrix element}}$$

It is useful to write this as trace in the color and spin space

$$\sigma[F] = \sum_m \int [d\{p,f\}_m]\, \mathrm{Tr}\{\underbrace{\rho(\{p,f\}_m)}_{\text{density operator in color} \otimes \text{spin space}} F(\{p,f\}_m)\}$$

Motivation

The density operator is

$$\rho(\{p,f\}_m) = \big|\mathcal{M}(\{p,f\}_m)\big\rangle \frac{f_{a/A}(\eta_{\rm a},\mu_F^2) f_{b/B}(\eta_{\rm b},\mu_F^2)}{2\eta_{\rm a}\eta_{\rm b} p_A\cdot p_B} \big\langle \mathcal{M}(\{p,f\}_m)\big|$$

or expanding it on a color and spin basis

$$\rho(\{p,f\}_m) = \sum_{s,c}\sum_{s',c'} \big|\{s,c\}_m\big\rangle\, \rho(\{p,f,s',c',s,c\}_m)\, \big\langle\{s',c'\}_m\big|$$

If we define the cross section based on the density operator, then

- ✓ in the NLO calculation we can do *Monte Carlo spin and color sum everywhere*
- ✓ we can do the *matching to parton shower* in a straightforward way

Motivation

In the subtraction terms we want to capture the soft and collinear physics more precisely

- *correct color structure* even for the $g \to q + qbar$ collinear singularities
- better momentum mapping that can be generalized for NNLO calculation
- define the splitting function in a more systematic way
  (try to avoid structures those lead to spurious singularities in the NNLO calculations)

Actually, first we defined the parton shower algorithm and then basically without any modification we could do the NLO scheme. The NLO subtraction procedure is just *one step of the inverse parton shower* algorithm.

Splitting Operator

The physical states are given by the density matrix which is the direct product of the matrix element vector:

$$\rho(\{p,f\}_{m+1}) \sim \big|\mathcal{M}(\{p,f\}_{m+1})\big\rangle\big\langle\mathcal{M}(\{p,f\}_{m+1})\big|$$

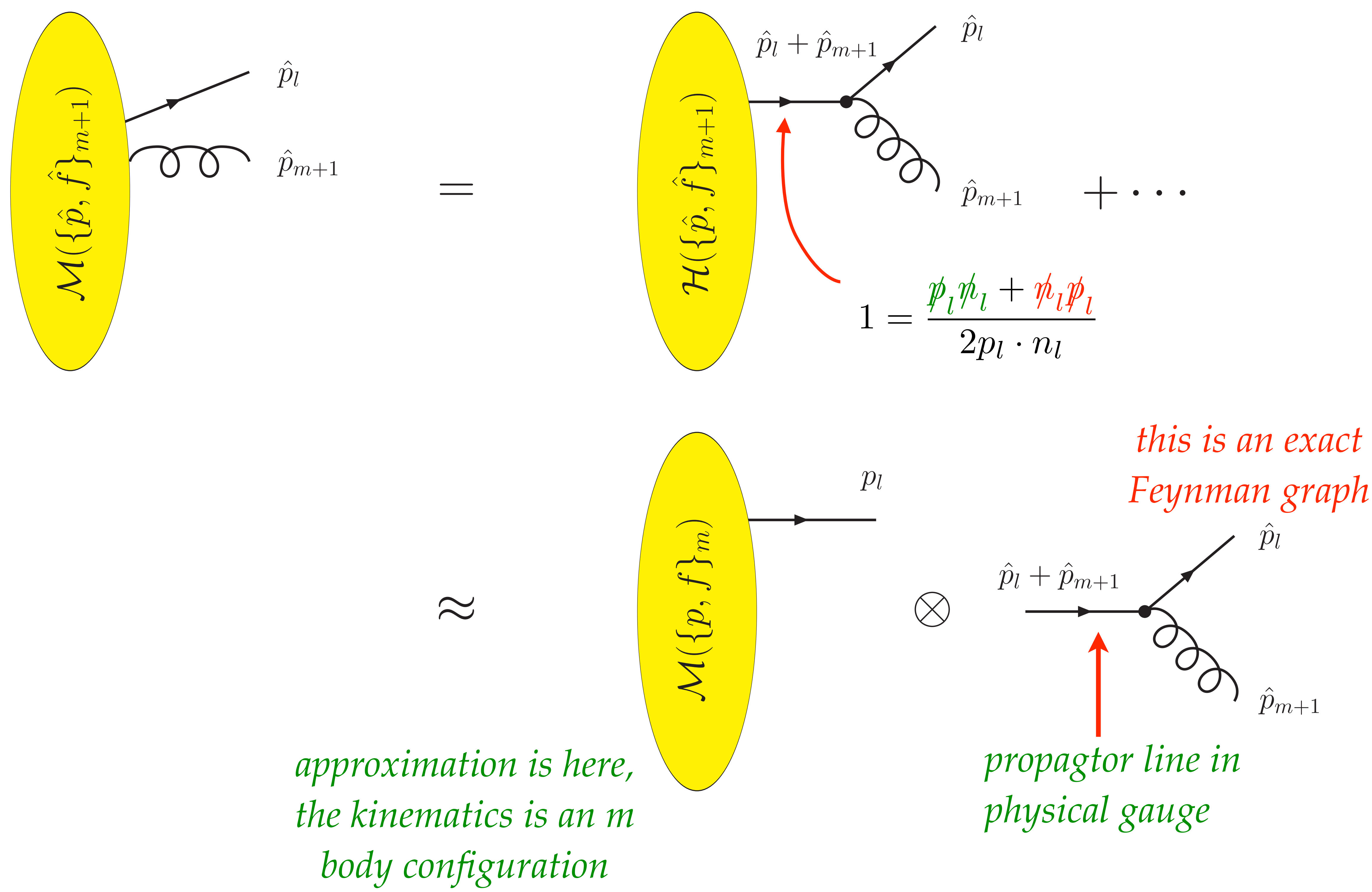
The splitting operator is based on the soft collinear factorization of the matrix element. Thus in the limit when parton l and $m+1$ become collinear we have

$$\big|M(\{\hat{p},\hat{f}\}_{m+1})\big\rangle \sim \underbrace{t_l^\dagger(f_l \to \hat{f}_l + \hat{f}_{m+1})}_{\textit{Color operator}}\ \underbrace{V_l^\dagger(\{\hat{p},\hat{f}\}_{m+1})}_{\textit{Spin operator}}\ \big|M(\{p,f\}_m)\big\rangle$$

And in the soft limit we have

$$\big|M(\{\hat{p},\hat{f}\}_{m+1})\big\rangle \sim \sum_l t_l^\dagger(f_l \to \hat{f}_l + \hat{f}_{m+1})\, V_l^\dagger(\{\hat{p},\hat{f}\}_{m+1})\,\big|M(\{p,f\}_m)\big\rangle$$

Splitting Operator



Splitting Operators

For the splitting operator we have

$$\left\langle \{\hat{s}\}_{m+1} \middle| V_{ij}^{\dagger}(\{\hat{p},\hat{f}\}_{m+1}) \middle| \{s\}_m \right\rangle = \left(\prod_{k \notin \{i,j\}} \delta_{\hat{s}_k, s_{\tilde{k}}} \right) v_{ij}(\{\hat{p},\hat{f}\}_{m+1}, \hat{s}_j, \hat{s}_i, s_{\tilde{\imath}})$$

and the collinear splitting function for the $q \to q + g$ in the final state is

$$v_{ij} = \sqrt{4\pi\alpha_{\mathrm{s}}}\,\varepsilon_\mu(\hat{p}_j, \hat{s}_j; \hat{Q})^* \frac{\overline{U}(\hat{p}_i, \hat{s}_i)\gamma^\mu [\hat{\not{p}}_i + \hat{\not{p}}_j + m(f_{\tilde{\imath}})] \not{n}_{\tilde{\imath}} U(p_{\tilde{\imath}}, s_{\tilde{\imath}})}{2p_{\tilde{\imath}} \cdot n_{\tilde{\imath}} \; [(\hat{p}_i + \hat{p}_j)^2 - m^2(f_{\tilde{\imath}})]}$$

For the soft singularities we can use the eikonal approximation, that is

$$v_{ij}^{\mathrm{soft}} = \sqrt{4\pi\alpha_{\mathrm{s}}}\,\delta_{\hat{s}_i, s_{\tilde{\imath}}} \frac{\varepsilon(\hat{p}_i, \hat{s}_j; \hat{Q})^* \cdot \hat{p}_i}{\hat{p}_i \cdot \hat{p}_j}$$

Momentum Mapping

What about the momentum mapping and phase space?

$$\hat{p}_l + \hat{p}_{m+1} = \lambda p_l + \frac{1-\lambda+y}{2a_l}\, Q$$

Emitter

Spectator

$$\lambda = \sqrt{\frac{(1+y)^2 - 4a_l(y+b_l)}{1-4a_l b_l}} \qquad a_l = \frac{Q^2}{2p_l\cdot Q}\,, \qquad b_l = \frac{m^2(f_l)}{2p_l\cdot Q}$$

For the spectators we have

$$\hat{p}_j^\mu = \Lambda^\mu{}_\nu(Q-\hat{p}_l-\hat{p}_{m+1}, Q-p_l)\; p_j^\nu \qquad j\notin\{l, m+1\}$$

and the Lorentz transformation is

$$\Lambda(\hat{K},K)^\mu{}_\nu = g^\mu_\nu - \frac{2(\hat{K}+K)^\mu(\hat{K}+K)_\nu}{(\hat{K}+K)^2} + \frac{2\hat{K}^\mu K_\nu}{K^2}$$

Phase Space Factorization

The phase space measure factorizes

$$\int [d\{\hat{p},\hat{f}\}_{m+1}]\; g(\{\hat{p},\hat{f}\}_{m+1}) = \int [d\{p,f\}_m] \sum_{\zeta_{\mathrm{f}}\in S_{\mathrm{f}}(f_i)} \int d\zeta_{\mathrm{p}}\; \theta(\zeta_{\mathrm{p}} \in S_{\mathrm{p}}(\{p\}_m,\zeta_{\mathrm{f}}))\; g(\{\hat{p},\hat{f}\}_{m+1})$$

For the final state splittings the integral measure for the unresolved phase space is basically a two body phase space integral

$$d\zeta_{\mathrm{p}} \equiv dy\; \theta(y_{\min} < y < y_{\max})\; \lambda^{d-3}\, \frac{p_{\tilde{\imath}}\cdot Q}{\pi} \times \frac{d^d\hat{p}_i}{(2\pi)^d}\, 2\pi\delta_+(\hat{p}_i^2 - m^2(\hat{f}_i))\; \frac{d^d\hat{p}_j}{(2\pi)^d}\, 2\pi\delta_+(\hat{p}_j^2 - m^2(\hat{f}_j)) \times (2\pi)^d\, \delta\!\left(\hat{p}_i + \hat{p}_j - \lambda p_{\tilde{\imath}} - \frac{1-\lambda+y}{2a_i}\, Q\right)$$

Subtraction terms

Now the subtraction procedure is

$$
\begin{aligned}
\sigma^{\mathrm{R-A}} &= \int_{m+1} \left[d\sigma^{\mathrm{R}} - d\sigma^{\mathrm{A}} \right] \\
&= \frac{1}{(m+1)!} \int \left[d\{\hat{p},\hat{f}\}_{m+1} \right] \frac{f_{\hat{a}/A}(\hat{\eta}_{\mathrm{a}},\mu_{\mathrm{F}}^2)\, f_{\hat{b}/B}(\hat{\eta}_{\mathrm{b}},\mu_{\mathrm{F}}^2)}{4n_{\mathrm{c}}(\hat{a})n_{\mathrm{c}}(\hat{b})\, 2\hat{\eta}_{\mathrm{a}}\hat{\eta}_{\mathrm{b}} p_{\mathrm{A}}\cdot p_{\mathrm{B}}} \\
&\quad \times \Bigg[\big\langle \mathcal{M}(\{\hat{p},\hat{f}\}_{m+1}) \big| \mathcal{M}(\{\hat{p},\hat{f}\}_{m+1}) \big\rangle\, F_{m+1}(\{\hat{p}\}_{m+1}) \\
&\qquad - \sum_{i\in\{\mathrm{a},\mathrm{b},1,\dots,m+1\}} \sum_{\substack{j\in\{1,\dots,m+1\} \\ j>i}} \mathcal{D}_{ij}(\{\hat{p},\hat{f}\}_{m+1})\, F_m(\{p\}_m^{ij}) \Bigg]
\end{aligned}
$$

where

$$
\mathcal{D}_{ij}(\{\hat{p},\hat{f}\}_{m+1}) = \big\langle M(\{p,f\}_m^{ij}) \big| \boldsymbol{P}_{ij}(\{\hat{p},\hat{f}\}_{m+1}) \big| M(\{p,f\}_m^{ij}) \big\rangle
$$

Subtraction terms

And the splitting kernel we have

$$\boldsymbol{P}_{ij}(\{\hat{p},\hat{f}\}_{m+1}) = C(\hat{f}_i,\hat{f}_j)\; V_{ij}(\{\hat{p},\hat{f}\}_{m+1})\, V^{\dagger}_{ij}(\{\hat{p},\hat{f}\}_{m+1})$$
$$+ \sum_{\tilde{k}\neq i,j} \boldsymbol{T}_{\tilde{i}}\cdot\boldsymbol{T}_{\tilde{k}}\,\Big\{ 2A_{ik}(\{\hat{p}\}_{m+1}) V^{\mathrm{soft}}_{kj}(\{\hat{p},\hat{f}\}_{m+1}) V^{\dagger,\mathrm{soft}}_{ij}(\{\hat{p},\hat{f}\}_{m+1})$$
$$+\,\theta(i\geq 1)\; 2A_{jk}(\{\hat{p}\}_{m+1}) V^{\mathrm{soft}}_{ki}(\{\hat{p},\hat{f}\}_{m+1}) V^{\dagger,\mathrm{soft}}_{ji}(\{\hat{p},\hat{f}\}_{m+1})\Big\}$$

Here the A_{lk} functions are *arbitrary partitioning functions*. They distribute the soft contributions along the collinear directions. The requirement for them is

$$A_{lk} + A_{kl} = 1$$

Trivial choice is of course

$$A_{lk} = \frac{1}{2}$$

Integral of the Subtraction terms

Now, we have integrate analytically the subtraction terms over the unresolved phase space

$$\int d\sigma^{\mathrm{A}} = \frac{1}{m!}\int [d\{p,f\}_m]\,\frac{f_{a/A}(\eta_{\mathrm{a}},\mu_{\mathrm{F}}^2)\,f_{b/B}(\eta_{\mathrm{b}},\mu_{\mathrm{F}}^2)}{n_{\mathrm{c}}(a)n_{\mathrm{c}}(b)\,n_{\mathrm{s}}(\hat{a})n_{\mathrm{s}}(\hat{b})\,2\eta_{\mathrm{a}}\eta_{\mathrm{b}}p_{\mathrm{A}}\cdot p_{\mathrm{B}}} \times \big\langle M(\{p,f\}_m)\big|\boldsymbol{I}(\{p,f\}_m)\big|M(\{p,f\}_m)\big\rangle \ .$$

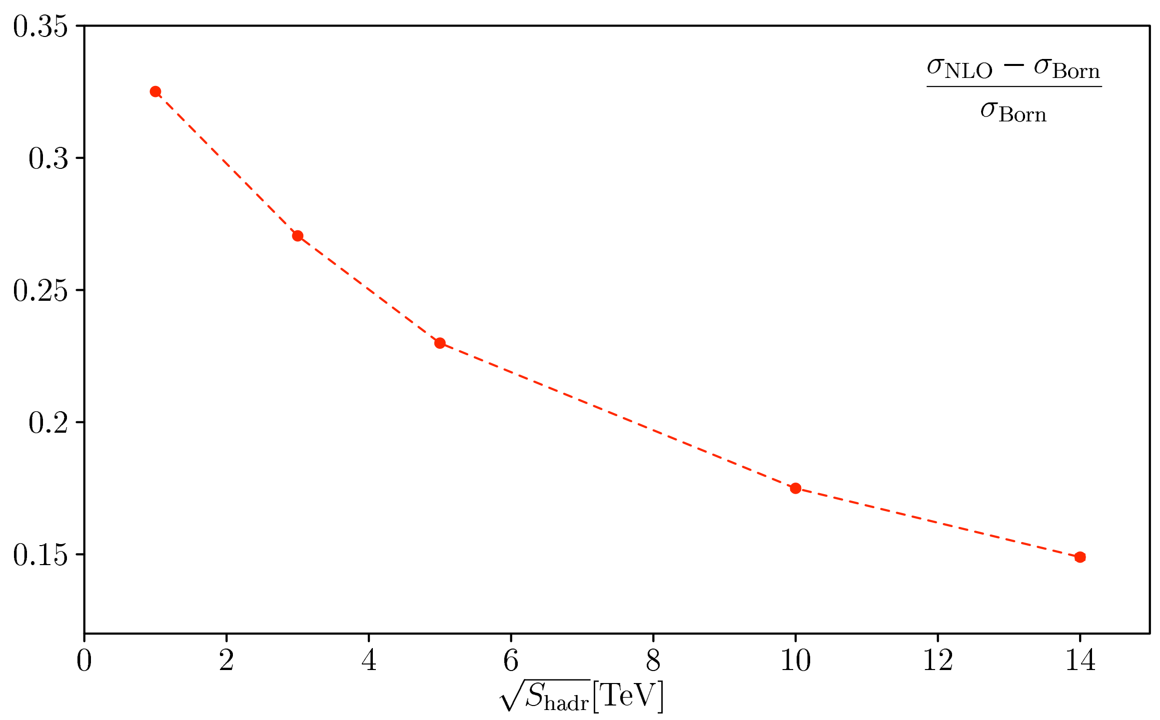
The singular operator $\boldsymbol{I}$ is

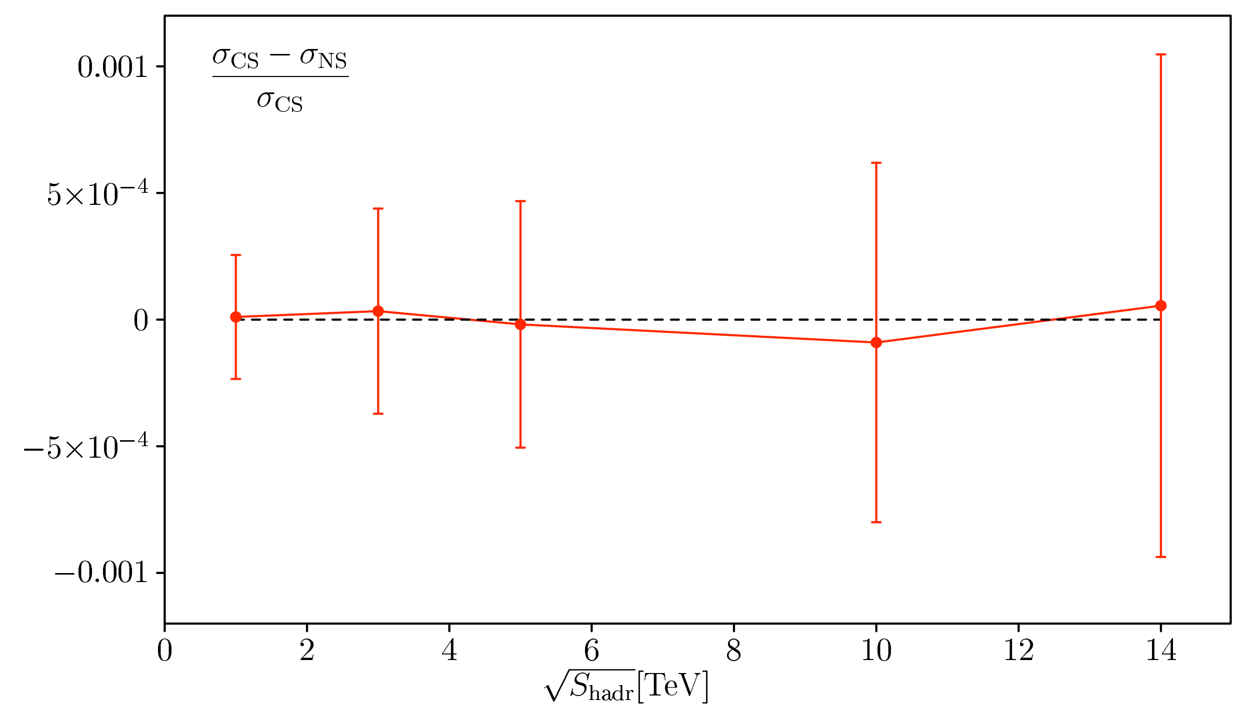
$$\begin{aligned}\boldsymbol{I}(\{p,f\}_m) = {} & \sum_{l=\mathrm{a},\mathrm{b},1,\dots,m}\ \sum_{\zeta_{\mathrm{f}}\in\Phi_l(f_l)} \int d\zeta_{\mathrm{p}}\,\theta(\zeta_{\mathrm{p}}\in\Gamma_l(\{p\}_m,\zeta_{\mathrm{f}})) \\ & \times \frac{n_{\mathrm{c}}(a)n_{\mathrm{c}}(b)}{n_{\mathrm{c}}(\hat{a})n_{\mathrm{c}}(\hat{b})}\,\frac{n_{\mathrm{s}}(a)n_{\mathrm{s}}(b)}{n_{\mathrm{s}}(\hat{a})n_{\mathrm{s}}(\hat{b})}\,\frac{\eta_{\mathrm{a}}\eta_{\mathrm{b}}}{\hat{\eta}_{\mathrm{a}}\hat{\eta}_{\mathrm{b}}}\,\frac{f_{\hat{a}/A}(\hat{\eta}_{\mathrm{a}},\mu_{\mathrm{F}}^2)\,f_{\hat{b}/B}(\hat{\eta}_{\mathrm{b}},\mu_{\mathrm{F}}^2)}{f_{a/A}(\eta_{\mathrm{a}},\mu_{\mathrm{F}}^2)\,f_{b/B}(\eta_{\mathrm{b}},\mu_{\mathrm{F}}^2)} \\ & \times \Big\{\theta(\hat{f}_{m+1}\neq \mathrm{g})\,T_{\mathrm{R}}\,\overline{w}_{ll}(\{\hat{p},\hat{f}\}_{m+1}) \\ & \qquad - \theta(\hat{f}_{m+1} = \mathrm{g})\sum_{k\neq l}\boldsymbol{T}_l\cdot\boldsymbol{T}_k \\ & \qquad\quad \times \big[\overline{w}_{ll}(\{\hat{p}\}_{m+1}) - 2A_{lk}(\{\hat{p}\}_{m+1})\overline{w}_{lk}(\{\hat{p}\}_{m+1})\big]\Big\}\end{aligned}$$

This integral is rather complicated but it can be done and leads to well know singularity structure which is cancelled after we combine with the virtual contributions.

Simple examples

Same simple cross sections have been calculated with this scheme, like e+e- → 2jet and the Drell-Yan process.





Tania and Cheng-Han are working on the e+e- → 3jet process and we have some progress also on the general implementation.

Summary

- ✓ General phase space generator implemented with multi channel important sampling in all singular region.
- ✓ It is also generates the color structure randomly, so MC color sum is available.
- ✓ General structure, mapping, splitting functions
- ✗ Missing the virtual part + finite remaining part
- ✗ Interface to the matrix element generators