Tensor reduction of one-loop pentagons and hexagons

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(DESY, ZEUTHEN)

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Based on [**arXiv:hep-ph/0807.2984, 0812.2134, 0901.4455]**

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Motivation and goals

- Recent years have seen the emergence of first results for $2 \rightarrow 4$ scattering processes

- One of the challenges posed is the need to compute one-loop tensor integrals with up to 6 legs

- To provide compact analytic formulas for the complete reduction of tensor pentagons and hexagons to scalar master integrals, free of leading inverse Gram determinants

  We start from results of:

  J.Fleischer, F.Jegerlehner, and O.V. Tarasov, Nucl. Phys. **B566** (2000) 423-440

- Implementation to Fortran and Mathematica code

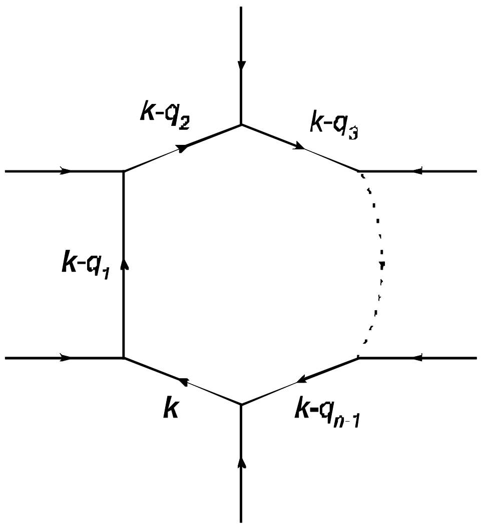
Notations

We consider one-loop, (N)-point tensor integrals of rank R in d-dimensional space-time,

$$J^{(N)}_{\mu_1\ldots\mu_R}(d;\nu_1\ldots\nu_N)=\int\frac{d^dk}{i\pi^{d/2}}\frac{k_{\mu_1}\ldots k_{\mu_R}}{D_1^{\nu_1}\ldots D_N^{\nu_N}}$$

with propagator denominators:

$$D_j=(k-q_j)^2-m_j^2+i\varepsilon$$



We decompose these tensor integrals into a basis of symmetric tensors constructed from g and the momenta q_j

$$J^{(N)}_{\mu_1\ldots\mu_R}(d;\nu_1\ldots\nu_N) = \sum_{\lambda,\kappa_1,\ldots\kappa_N} coeff \times \{[g]^{\lambda}[q_1]^{\kappa_1}\ldots[q_N]^{\kappa_N}\}_{\mu_1\ldots\mu_R}$$

$$\times J^{(N)}(d+2(R-\lambda);\nu_1+\kappa_1,\ldots,\nu_N+\kappa_N) \quad : scalar$$

A.I. Davydychev, Phys. Lett. B **263** (1991) 107

- The next step is the usage of recurrence relations to reduce the scalar coefficients $J^{(N)}$ appearing in the decomposition to a set of master integrals
- Combining integration by parts identities, with relations connecting integrals in different space-time dimensions, one obtains the following basic recurrence relations:

$$()_N \nu_j j^+ J^{(N)}(d+2) = \left[-\binom{j}{0}_N + \sum_{k=1}^{n} \binom{j}{k} k^- \right] J^{(N)}(d),$$

$$(d - \sum_{i=1}^{n} \nu_i + 1)\, ()_N\, J^{(N)}(d+2) = \left[\binom{0}{0}_N - \sum_{k=1}^{n} \binom{0}{k}_N k^- \right] J^{(N)}(d)$$

$$\binom{0}{0}_N \nu_j j^+ J^{(N)}(d) = \sum_{k=1}^{n} \binom{0j}{0k}_N \times \left[d - \sum_{i=1}^{n} \nu_i (k^- i^+ + 1) \right] J^{(N)}(d)$$

- Where the operator $j^{\pm}$ acts by shifting the index ν_j by ± 1

O.V. Tarasov, Phys. Rev. D **54** (1996) 6479

Notations continue …

Where:

$$
()_N = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & Y_{11} & Y_{12} & \cdots & Y_{1N} \\ 1 & Y_{12} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & Y_{1N} & Y_{2N} & \cdots & Y_{NN} \end{vmatrix} = -2^{N-1} \times \begin{vmatrix} q_1 \cdot q_1 & q_1 \cdot q_2 & \cdots & q_1 \cdot q_{N-1} \\ q_2 \cdot q_1 & q_2 \cdot q_2 & \cdots & q_2 \cdot q_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N-1} \cdot q_1 & q_{N-1} \cdot q_2 & \cdots & q_{N-1} \cdot q_{N-1} \end{vmatrix}
$$

An $(N+1)\times(N+1)$ matrix known as the modified Cayley determinant
(D.B. Melrose, Nuovo Cim. **40** (1965) 181)

with coefficients:

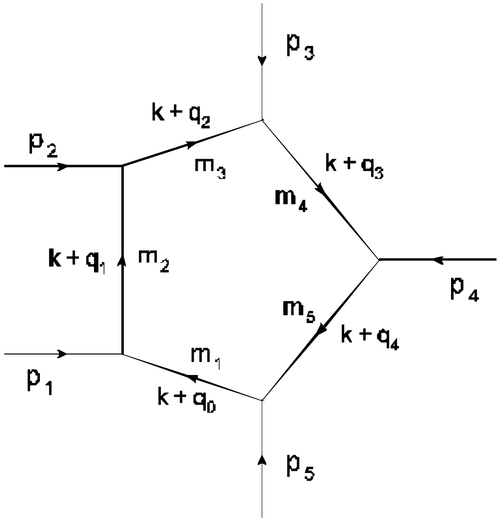
$$
Y_{ij} = -(q_i - q_j)^2 + m_i^2 + m_j^2, \quad (i, j = 1 \ldots N)
$$

Notations continue …

$$\begin{pmatrix} i \\ j \end{pmatrix}_N = \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & Y_{11} & \cdots & Y_{1j} & \cdots & Y_{1n} \\ 1 & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & Y_{i1} & \cdots & Y_{ij} & \cdots & Y_{in} \\ 1 & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & Y_{n1} & \cdots & Y_{nj} & \cdots & Y_{nn} \end{vmatrix}$$

Diagrams

- Pentagons



We will restrict to a third rank tensor $(I_5^{\mu\nu\lambda})$ with indices:

$$\nu_1 = \nu_2 = \nu_3 = \nu_4 = \nu_5 = 1$$

$$J^{(N)}_{\mu_1 \ldots \mu_R}(d;\nu_1 \ldots \nu_N) = \int \frac{d^d k}{i\pi^{d/2}} \frac{k_{\mu_1} \ldots k_{\mu_R}}{D_1^{\nu_1} \ldots D_N^{\nu_N}}$$

(Assuming loop momentum k has been shifted so $q_N = 0$)

Applying Davydychev's equation gives integrals in d+4 and d+6 dimensions and with increased indices.

They are reduced back to the generic dimension $d = 4 - 2\varepsilon$ by the first 2 recurrence relations:

$$()_N \nu_j \mathbf{j}^+ J^{(N)}(d+2) = \left[-\begin{pmatrix} j \\ 0 \end{pmatrix}_N + \sum_{k=1}^{n} \begin{pmatrix} j \\ k \end{pmatrix}_N \mathbf{k}^- \right] J^{(N)}(d),$$

$$\left(d - \sum_{i=1}^{n} \nu_i + 1\right) ()_N J^{(N)}(d+2) = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}_N - \sum_{k=1}^{n} \begin{pmatrix} 0 \\ k \end{pmatrix}_N \mathbf{k}^- \right] J^{(N)}(d)$$

It involves division by a Gram determinant $()_N$ at each step

- The leading Gram determinant $()_5$ can be avoided if one is only interested in contractions of the tensor integral with 4-dimensional objects.

- It is achieved by using the following decomposition of the metric tensor:

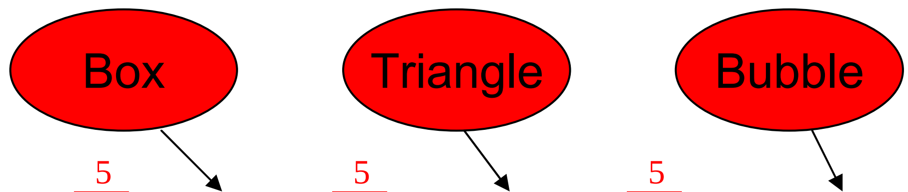
$$g^{\mu\nu} = 2\sum_{i,j=1}^{N-1} \frac{\begin{pmatrix} i \\ j \end{pmatrix}}{()_N} q_i^{\mu} q_j^{\nu}$$

- We actually rearrange things until we see the combination above and then we replace

After further simplifications we obtain:

$$I_5^{\mu\nu\lambda} = \sum_{i,j,k=1}^{4} q_i^{\mu} q_j^{\nu} q_k^{\lambda} \mathrm{E}_{ijk} + \sum_{k=1}^{4} g^{[\mu\nu} q_k^{\lambda]} \mathrm{E}_{00k}$$

With scalar coefficients defined by:



$$E_{ijk} = \sum_{s=1}^{5} S_{ijk}^{4,s} I_4^{s} + \sum_{s,t=1}^{5} S_{ijk}^{3,st} I_3^{st} + \sum_{s,t,u=1}^{5} S_{ijk}^{2,stu} I_2^{stu},$$

Where s,t,u are the internal lines that are cut from the initial pentagon to produce the relative, box (s), triangle (s,t), or bubble (s,t,u).

$I_4^{s}, I_3^{st}, I_2^{stu}$ are scalar master integrals

The remaining coefficients are defined as:

$$S^{4,s}_{ijk} = \frac{1}{3\binom{0}{0}_5 \binom{s}{s}_5^2} \times$$
$$\left\{ -\binom{0s}{0k}_5 \left[\binom{0s}{is}_5 \binom{0s}{js}_5 + \binom{is}{js}_5 \binom{0s}{0s}_5 \right] + \binom{0s}{0s}_5 \left[\binom{0i}{sk}_5 \binom{0s}{js}_5 + \binom{0j}{sk}_5 \binom{0s}{is}_5 \right] \right\}$$
$$+ (i \leftrightarrow k) + (j \leftrightarrow k)$$

$$S^{3,st}_{ijk} = \frac{1}{3\binom{0}{0}_5 \binom{s}{s}_5^2} \left\{ \binom{0s}{0k}_5 \left[\binom{ts}{is}_5 \binom{0s}{js}_5 + \binom{is}{js}_5 \binom{ts}{0s}_5 + \frac{\binom{s}{s}_5 \binom{0st}{ist}_5}{\binom{st}{st}_5} \binom{ts}{js}_5 \right] \right.$$
$$- \left[\binom{0i}{sk}_5 \binom{0s}{js}_5 + \binom{0j}{sk}_5 \binom{0s}{is}_5 \right] \binom{ts}{0s}_5$$
$$- \left[\binom{0i}{sk}_5 \binom{ts}{js}_5 + \binom{0j}{sk}_5 \binom{ts}{is}_5 \right]$$
$$\left. \times \frac{\binom{s}{s}_5 \binom{0st}{0st}_5}{2\binom{st}{st}_5} \right\} + (i \leftrightarrow k) + (j \leftrightarrow k),$$

$$S^{2,stu}_{ijk} = -\frac{1}{3\binom{0}{0}_5 \binom{s}{s}_5 \binom{st}{st}_5} \times$$
$$\left\{ \binom{0s}{0k}_5 \binom{ts}{js}_5 \binom{ust}{ist}_5 - \frac{1}{2} \left[\binom{0j}{sk}_5 \binom{ust}{ist}_5 + \binom{0i}{sk}_5 \binom{ust}{jst}_5 \right] \binom{ts}{0s}_5 \right\}$$
$$+ (i \leftrightarrow k) + (j \leftrightarrow k),$$

$$E_{00j} = \frac{1}{6\binom{0}{0}_5} \left\{ -\sum_{s=1}^{5} \frac{1}{\binom{s}{s}_5^2} \right.$$
$$\times \left[3\binom{s}{0}_5 \binom{0s}{js}_5 - \binom{s}{j}_5 \binom{0s}{0s}_5 \right] \binom{0s}{0s}_5 I_4^s$$
$$+ \sum_{s,t=1}^{5} \frac{1}{\binom{s}{s}_5^2}$$
$$\times \left[3\binom{s}{0}_5 \binom{0s}{js}_5 - \binom{s}{j}_5 \frac{\binom{ts}{0s}_5^2}{\binom{st}{st}_5} \right] \binom{ts}{0s}_5 I_3^{st}$$
$$\left. - \sum_{s,t,u=1}^{5} \binom{s}{j}_5 \frac{\binom{ust}{0st}_5}{\binom{s}{s}_5 \binom{st}{st}_5} \binom{ts}{0s}_5 I_2^{stu} \right\}.$$

- This decomposition is similar to the one found in the:

A.Denner and S. Dittmaier, Nucl. Phys. B **658** (2003) 175

where the coefficients E_{ijk} and E_{00j} are expressed in tensor 4-point functions

- Detailed discussion on second rank pentagon can be found in

J.Fleischer, J.Gluza, K.Kajda and T.Riemann, Acta Phys. Polon. B **38** (2007) 3529

T.Diakonidis, J.Fleischer, J.Gluza, K.Kajda and T.Riemann, B.Tausk hep-ph/**0812.2134**

Hexagons

If the external momenta of a hexagon are 4-dimensional

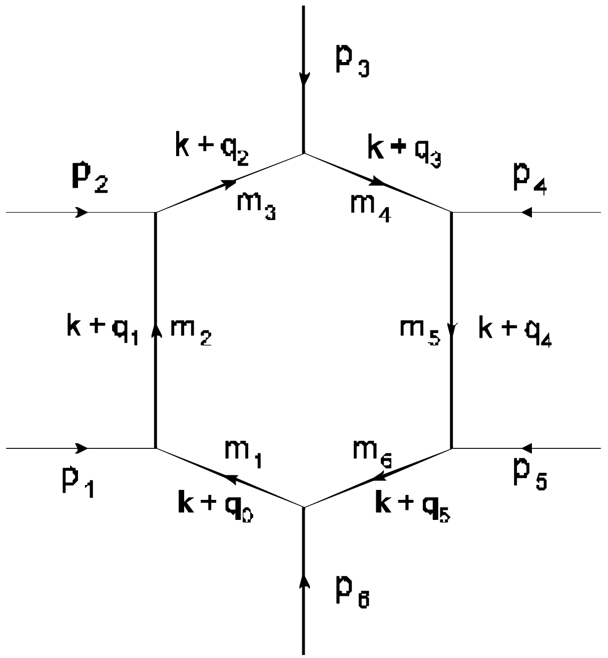
Due to $()_6 = 0$:

$$1 = \sum_{j=1}^{6} \frac{\binom{0}{j}_6}{\binom{0}{0}_6} D_j$$

Any hexagon integral can be reduced to pentagons (e.g.):

$$scalar : I_6 = \sum_{r=1}^{6} \frac{\binom{0}{r}_6}{\binom{0}{0}_6} I_5^r$$

D.B. Melrose, Nuovo Cim. **40** (1965) 181



It was also noticed that a reduction directly to tensor pentagons of rank R-1 is also possible:

$$I_6^{\mu_1\cdots\mu_R} = \sum_{r=1}^{6} u_r^{\mu_1} \mathrm{I}_5^{\mu_2\cdots\mu_R, r}$$

Where

$$u_r^{\mu} \equiv -\frac{1}{\binom{0}{0}_6} \sum_{i=1}^{5} \binom{0i}{0r}_6 q_i^{\mu}$$

J. Fleischer, F. Jegerlehner, and O.V. Tarasov, Nucl. Phys. **B566** (2000) 423

See also:

T. Binoth, J.P. Guillet, G. Heinrich, E. Pilon and C. Schubert, JHEP **0510** (2005) 015

A more general proof can be found in:

A. Denner and S. Dittmaier, Nucl. Phys. B **734** (2006) 62

Substituting our reduction formulas for tensor pentagons, we can express tensor hexagons in terms of scalar master integrals

Numerical results (Fortran)

- For five and six point tensor integrals, we have a Fortran implementation package (Th. Diakonidis & B. Tausk)

The present implementation includes:

- Six point functions up to rank four (Hexagon.F)
- Five point functions up to rank three (Pentagon.F)

It is able to output the full result for:

- Six or five point tensor integral
- A specific coefficient for a given rank

- The code so far uses:

Looptools 2.2 (by Thomas Hahn)
(calculates only the finite part)
QCDLoop (R.K. Ellis and G. Zanderighi)
(Finite part and $1/\varepsilon$ and $1/\varepsilon^2$ terms)

To calculate the scalar master integrals after the reduction

(The first is restricted to massive cases but the second can be implemented for massless cases too)

- It can be adapted to any Fortran package for 1,2,3,4 point functions

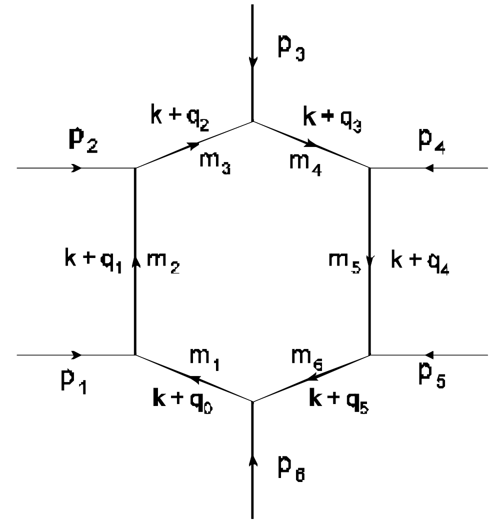
- A lot of cross checks have been done so far (shown after) and we also cross checked the results with an independent code by Peter Uwer

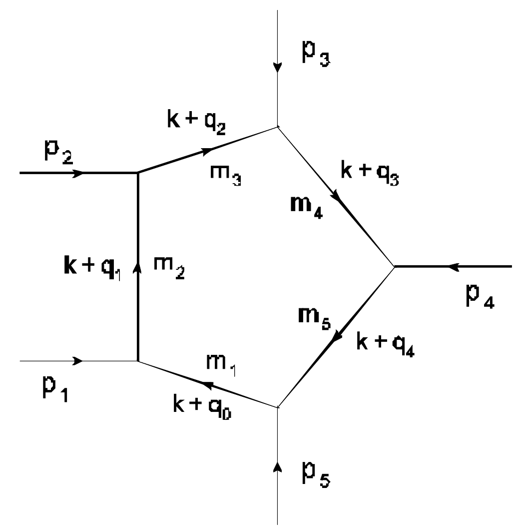
More about the programs

- They provide coefficients of Lorentz-covariant tensors, and work in a basis of $g^{\mu\nu}$ and internal momenta q_i

$$q_0 = 0, \quad q_n = \sum_{i=1}^{n} p_i$$

- In terms of these coefficients, the tensor decomposition of pentagons E and hexagons F reads:





$$
\begin{aligned}
E^{\mu} &= \sum_{i=1}^{4} q_i^{\mu} E_i, \\
E^{\mu\nu} &= \sum_{i,j=1}^{4} q_i^{\mu} q_i^{\nu} E_{ij} + g^{\mu\nu} E_{00}, \\
E^{\mu\nu\lambda} &= \sum_{i,j,k=1}^{4} q_i^{\mu} q_i^{\nu} q_k^{\lambda} E_{ijk} + \sum_{i=1}^{4} g^{[\mu\nu} q_i^{\lambda]} E_{00i}, \\
F^{\mu} &= \sum_{i=1}^{5} q_i^{\mu} F_i, \\
F^{\mu\nu} &= \sum_{i,j=1}^{5} q_i^{\mu} q_i^{\nu} F_{ij}, \\
F^{\mu\nu\lambda} &= \sum_{i,j,k=1}^{5} q_i^{\mu} q_i^{\nu} q_k^{\lambda} F_{ijk} + \sum_{i=1}^{5} g^{\mu\nu} q_i^{\lambda} F_{00i}, \\
F^{\mu\nu\lambda\rho} &= \sum_{i,j,k,l=1}^{5} q_i^{\mu} q_i^{\nu} q_k^{\lambda} q_l^{\rho} F_{ijkl} \\
&+ \sum_{i,j=1}^{5} q_i^{\mu} q_j^{[\nu} g^{\lambda\rho]} F_{00ij}
\end{aligned}
$$

Some sample results

For the randomly chosen phase space point given by:

$$
\begin{aligned}
p_1 &= (0.21774554E+03, \quad 0, \quad 0, \quad 0.21774554E+03) \\
p_2 &= (0.21774554E+03, \quad 0, \quad 0, \quad -0.21774554E+03) \\
p_3 &= (-0.20369415E+03, \quad -0.47579512E+02, \quad 0.42126823E+02, \quad 0.84097181E+02) \\
p_4 &= (-0.20907237E+03, \quad 0.55215961E+02, \quad -0.46692034E+02, \quad -0.90010087E+02) \\
p_5 &= (-0.68463308E+01, \quad 0.53063195E+01, \quad 0.29698267E+01, \quad -0.31456871E+01) \\
p_6 &= (-0.15878244E+02, \quad -0.12942769E+02, \quad 0.15953850E+01, \quad 0.90585932E+01) \\
& m_1 = 110.0, \; m_2 = 120.0, \; m_3 = 130.0, \; m_4 = 140.0, \; m_5 = 150.0, \; m_6 = 160.0
\end{aligned}
$$

Results for scalar, vector and 2^{nd} rank six point functions:

	RESULTS	
	REAL	IM
	F_0	
	-0.223393E-18	-0.396728E-19
μ	F^μ	
0	0.192487E-17	0.972635E-17
1	-0.363320E-17	-0.11940E-17
2	0.365514E-17	0.106928E-17
3	0.239793E-16	0.341928E-17
μ ν	$F^{\mu\nu}$	
0 0	0.599459E-14	-0.114601E-14
0 1	0.323869E-15	0.423754E-15
0 2	-0.294252E-15	-0.375481E-15
0 3	-0.255450E-14	-0.195640E-14
1 1	-0.164562E-14	-0.993796E-16
1 2	0.920944E-16	0.706487E-17
1 3	0.347694E-15	-0.127190E-16
2 2	-0.163339E-14	-0.994148E-16
2 3	-0.341773E-15	0.818678E-17
3 3	-0.413909E-14	0.670676E-15

3^{rd} rank 6 point functions

μ	ν	λ	$F^{\mu\nu\lambda}$
0	0	0	– 0.227754 E–11 – i 0.267244 E–12
0	0	1	0.140271 E–13 – i 0.119448 E–12
0	0	2	– 0.201270 E–13 + i 0.101968 E–12
0	0	3	0.102976 E–12 + i 0.624467 E–12
0	1	1	0.183904 E–12 + i 0.142429 E–12
0	1	2	– 0.131028 E–13 – i 0.610343 E–14
0	1	3	– 0.543316 E–13 – i 0.158809 E–13
0	2	2	0.181352 E–12 + i 0.141686 E–12
0	2	3	0.506408 E–13 + i 0.163568 E–13
0	3	3	0.600542 E–12 + i 0.130733 E–12
1	1	1	– 0.563539 E–13 + i 0.178403 E–13
1	1	2	0.210641 E–13 – i 0.584990 E–14
1	1	3	0.120482 E–12 – i 0.574688 E–13
1	2	2	– 0.201182 E–13 + i 0.620591 E–14
1	2	3	– 0.686164 E–14 + i 0.205457 E–14
1	3	3	– 0.447329 E–13 + i 0.193180 E–13
2	2	2	0.582201 E–13 – i 0.163889 E–13
2	2	3	0.119659 E–12 – i 0.570084 E–13
2	3	3	0.457464 E–13 – i 0.181141 E–13
3	3	3	0.557081 E–12 – i 0.374359 E–12

4^{th} rank 6-point

μ	ν	λ	ρ	REAL $F^{\mu\nu\lambda\rho}$	IM $F^{\mu\nu\lambda\rho}$
0	0	0	0	0.666615D-09	0.247562D-09
0	0	0	1	-0.200049D-10	0.294036D-10
0	0	0	2	0.200975D-10	-0.237333D-10
0	0	0	3	0.645477D-10	-0.162236D-09
0	0	1	1	-0.116956D-10	-0.516760D-10
0	0	1	2	0.160357D-11	0.222284D-11
0	0	1	3	0.792692D-11	0.729502D-11
0	0	2	2	-0.111838D-10	-0.513133D-10
0	0	2	3	-0.681086D-11	-0.708933D-11
0	0	3	3	-0.804454D-10	-0.801909D-10
0	1	1	1	0.100498D-10	-0.151735D-13
0	1	1	2	-0.348984D-11	-0.195436D-12
0	1	1	3	-0.211111D-10	0.295212D-11
0	1	2	2	0.357455D-11	0.662809D-14
0	1	2	3	0.121595D-11	-0.807388D-13
0	1	3	3	0.825803D-11	-0.142086D-11
0	2	2	2	-0.958961D-11	-0.585948D-12

μ	ν	λ	ρ	REAL $F^{\mu\nu\lambda\rho}$	IM $F^{\mu\nu\lambda\rho}$
0	2	2	3	-0.209232D-10	0.289031D-11
0	2	3	3	-0.802359D-11	0.994701D-12
0	3	3	3	-0.102576D-09	0.378476D-10
1	1	1	1	-0.246426D-10	0.276326D-10
1	1	1	2	0.915670D-12	-0.660629D-12
1	1	1	3	0.303529D-11	-0.287480D-11
1	1	2	2	-0.822697D-11	0.919635D-11
1	1	2	3	-0.116294D-11	0.100024D-11
1	1	3	3	-0.146918D-10	0.183799D-10
1	2	2	2	0.908296D-12	-0.654735D-12
1	2	2	3	0.109510D-11	-0.100875D-11
1	2	3	3	0.717342D-12	-0.557293D-12
1	3	3	3	0.450661D-11	-0.485065D-11
2	2	2	2	-0.245154D-10	0.274313D-10
2	2	2	3	-0.318500D-11	0.279750D-11
2	2	3	3	-0.146317D-10	0.182912D-10
2	3	3	3	-0.477335D-11	0.477368D-11
3	3	3	3	-0.730168D-10	0.112865D-09

More results (massless case)

For the phase space point given by:

p1 =(1, 0, 0, 0)
p2 =(-0.19178191,-0.12741180,-0.08262477,-0.11713105)
p3 =(-0.33662712, 0.06648281, 0.31893785, 0.08471424)
p4 =(-0.21604814, 0.20363139,-0.04415762,-0.05710657)
p5 =-(p1+p2+p3+p4)

M1=0, M2=0, M3=0, M4=0, M5=0

Golem95: T.Binoth, J.-Ph.Guillet, G. Heinrich, E.Pilon, T.Reiter [arXiv:hep-ph/0810.0992]

Comparisons with golem95

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
E_0	(202.168496, 3211.04072)	(1022.10601 , 972.027061)	(309.405823 ,0)
E_3	(-264.996441,303.068452)	(96.4696846,149.228472)	(47.5008979,0)
E_{44}	(1780.58042 , 2914.50734)	(927.71650 , 568.572069)	(180.982111 ,0)
E_{00}	(9.56327810 , 0)	(0 , 0)	(0,0)
E_{555}	(1035.29689 , 1422.01085)	(452.640112 , 254.226520)	(80.9228146 , 0)
E_{001}	(0.84742102 ,0)	(0,0)	(0,0)

Complete agreement to all the numbers shown

(QCDLoop was used for the scalar master integrals)

Mixed case (Hexagon)

p_1	0.21774554 E+01	0.0	0.0	0.21774554 E+01
p_2	0.21774554 E+01	0.0	0.0	– 0.21774554 E+01
p_3	– 0.20369415 E+01	– 0.47579512 E+00	0.42126823 E+00	0.84097181 E+00
p_4	– 0.20907237 E+01	0.55215961 E+00	– 0.46692034 E+00	– 0.90010087 E+00
p_5	– 0.68463308 E–01	0.53063195 E–01	0.29698267 E–01	– 0.31456871 E–01
p_6	– 0.15878244 E+00	– 0.12942769 E+00	0.15953850 E–01	0.90585932 E–01
$m_1 = 0.0,\ m_2 = 0.0,\ m_3 = 0.0,\ m_4 = 1.7430,\ m_5 = 0.0,\ m_6 = 0.0$				

It corresponds to the reaction : $gg \to t\bar{t} + q\bar{q}$

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
F_0	0.2403558675 E+04 – i 0.2058213187 E+03	0.7315208677 E+02 – i 0.4276718518 E+02	– 0.7543148872 E+01 + i 0.0
F^2	0.1112747404 E+03 – i 0.6809282900 E+01	0.4419243474 E+01 – i 0.1201033663 E+01	– 0.1044856909 E+00 + i 0.0
F^{13}	– 0.1014018623 E+02 + i 0.1797332619 E+01	– 0.5914958485 E–01 + i 0.3275539398 E+00	0.7678550480 E–01 + i 0.0
F^{123}	– 0.5007216712 E+00 + i 0.4194342396 E–01	– 0.1642316924 E–01 + i 0.7789453935 E–02	0.1225024390 E–02 + i 0.0
F^{3210}	0.1263455978 E+00 – i 0.6509987460 E–02	0.4610567958 E–02 – i 0.1506637282 E–02	– 0.1945123881 E–03 + i 0.0

Mixed case (Pentagon)

p_1	0.21774554 E+01	0.0	0.0	0.21774554 E+01
p_2	0.21774554 E+01	0.0	0.0	– 0.21774554 E+01
p_3	– 0.20369415 E+01	– 0.47579512 E+00	0.42126823 E+00	0.84097181 E+00
p_4	– 0.20907237 E+01	0.55215961 E+00	– 0.46692034 E+00	– 0.90010087 E+00
p_5	– 0.68463308 E–01	0.53063195 E–01	0.29698267 E–01	– 0.31456871 E–01
p_6	– 0.15878244 E+00	– 0.12942769 E+00	0.15953850 E–01	0.90585932 E–01
$m_1 = 0.0,\ m_2 = 0.0,\ m_3 = 0.0,\ m_4 = 1.7430,\ m_5 = 0.0,\ m_6 = 0.0$				

Produced by adding together the external momenta : $p_1' = p_1 + p_2$

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
E_0	– 0.289852933 E+04 + i 0.228935552 E+03	– 0.945038648 E+02 + i 0.454178453 E+02	0.7112330546 E+01 + i 0.0
E^3	0.168344624 E+03 – i 0.181758172 E+02	0.4242553725 E+01 – i 0.338838829 E+01	– 0.6442770877 E+00 + i 0.0
E^{23}	– 0.79409571852 E+01 + i 0.5445326927 E+00	– 0.3008645503 E+00 + i 0.9457613783 E–01	0.1027869989 E–01 + i 0.0
E^{012}	0.2472148936 E+01 – i 0.127011969 E+00	0.9699262574 E-01 – i 0.2560545796 E–01	– 0.2331885086 E–02 + i 0.0
E^{2130}	0.2733228280 E+02 – i 0.519106421 E+02	– 0.909476582 E+01 + i 0.1744459753 E–02	0.2112313083 E–03 + i 0.0

Numerical results (Mathematica)

Mathematica package hexagon.m (by K. Kajda)

The present implementation includes:

- Six point functions up to rank four
- Five point functions up to rank three

It is able to output the full result for:

- Six or five point tensor integral
- A specific coefficient for a given rank
- A list of all coefficients of a given rank

Functions used in the package

Six point functions		Five point functions	
RedF0	scalar 6pt integral	RedE0	scalar 5pt integral
RedF1	vector 6pt integral	RedE1	vector 5pt integral
RedF2	rank two 6pt tensor integral	RedE2	rank two 5pt tensor integral
RedF3	rank three 6pt tensor integral	RedE3	rank three 5pt tensor integral
RedF4	rank four 6pt tensor integral		
RedFcoef	coefficient of given 6pt	RedEcoef	coefficient of given 5pt
RedFget	all coefficients of given 6pt	RedEget	all coefficients of given 5pt

The basic functions have the following arguments, here $s_{ij} = (p_i + p_j)^2$, $s_{ijk} = (p_i + p_j + p_k)^2$:

RedF0[$p_1^2, \ldots, p_6^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}, m_1^2, \ldots, m_6^2$]

RedE0[$p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$]

Numerical cross checks

1. Comparison with AMBRE & MB.m $p_1^\mu p_2^\nu p_3^\lambda E_{\mu\nu\lambda}$

Point:

$p_1^2 = p_2^2 = p_3^2 = p_5^2 = 1,\ p_4^2 = 0,\ m_1^2 = m_3^2 = 0,\ m_2^2 = m_4^2 = m_5^2 = 1,$

$s_{12} = -3,\ s_{23} = -6,\ s_{34} = -5,\ s_{45} = -7,\ s_{15} = -2$

In: RedE3[$p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$]/.{D4->D0,C3->C0,B2->B0}

Out: 0.218741

2. Comparison with Sector Decomposition : F_0

Point:

$p_1^2 = p_2^2 = p_3^2 = p_4^2 = p_5^2 = p_6^2 = -1,\ m_1^2 = m_2^2 = m_3^2 = m_4^2 = m_5^2 = m_6^2 = 1,$

$s_{12} = s_{23} = s_{34} = s_{45} = s_{56} = s_{16} = s_{123} = s_{234} = -1,\ s_{345} = -5/2$

In: RedF0[$p_1^2, \ldots, p_6^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}, m_1^2, \ldots, m_6^2$]/.{D4->D0}

Out: 0.013526

1. J. Gluza, K. Kajda and T. Riemann, Comput. Phys. Comm. **177** (2007) 879
   M. Czakon, Comput. Phys. Commun. **175** (2006) 559

2. C. Bogner and S. Weinzierl, Comput. Phys. Commun. **178** (2008) 596
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Numerical cross checks

3. Comparison with LoopTools : E_0, E_1, E_2, E_3, E_4, E_{34}, E_{123}, E_{002}

Point:

$p_1^2 = p_2^2 = 0$, $p_3^2 = p_5^2 = 49/256$, $p_4^2 = 9/100$, $m_1^2 = m_2^2 = m_3^2 = 49/256$, $m_4^2 = m_5^2 = 81/1600$,
$s_{12} = 4$, $s_{23} = -1/5$, $s_{34} = 1/5$, $s_{45} = 3/10$, $s_{15} = -1/2$

In: RedE0[$p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$]/.D4->D0

Out: 41.3403 - 45.9721*I

In: RedEget[rank1 , $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$]/.D4->D0

Out: ee1 =-2.38605 + 5.27599*I, ee2 =-5.80875 + 0.597891*I,
ee3 =-14.4931 + 20.8149*I, ee4 =-11.3362 + 18.1593*I

In: RedEcoef[ee34 , $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$]/.{D4->D0,C3->C0}

Out: 7.1964 + 3.10115*I

In: RedEcoef[ee123 , $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$]/.{D4->D0,C3->C0,B2->B0}

Out:-0.149527 - 0.31059*I

In: RedEcoef[ee002 , $p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$]/.{D4->D0,C3->C0,B2->B0}

Out: 0.154517 - 0.387727*I

3. T. Hahn and M. Perez-Victoria, Comput. Phys. Commun. **118** (1999) 153
   T. Hahn and M. Rauch, Nucl. Phys. Proc. Suppl. **157** (2006) 236

Conclusions

- An analytical reduction of one-loop tensor integrals with 5 or 6 legs down to scalar master integrals has been described
- Result for the tensor pentagon rank 3 is shown explicitly
- Reduction formulas have been implemented in a Mathematica and a Fortran program
- Mathematica program publicly available at:

http://www-zeuthen.desy.de/theory/research/CAS.html

BACKUP SLIDES

- The determinants shown above are signed minors of the modified Cayley determinant, constructed by deleting m rows and m columns from $()_N$ and multiplying with a sign factor.

- Denoted by:

$$\begin{pmatrix} j_1 & j_2 & \cdots & j_m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix}_N \equiv (-1)^{\sum_l (j_l + k_l)}$$

$$\mathrm{sgn}_{\{j\}}\,\mathrm{sgn}_{\{k\}} \begin{vmatrix} \textit{rows } j_1 \ldots j_m \textit{ deleted} \\ \textit{columns } k_1 \ldots k_m \textit{ deleted} \end{vmatrix}$$

- Where $\mathrm{sgn}_{\{j\}}$ and $\mathrm{sgn}_{\{k\}}$ are the signs of permutations that sort the deleted rows $j_1 \ldots j_m$ and columns $k_1 \ldots k_m$ into ascending order