MANCHESTER 1824

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Statistics for Particle Physics: Intervals

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Summary

Concepts

- Confidence and Probability
- Chi squared
- p-values
- Likelihood
- Bayesian Probability

Techniques

- $\Delta X^2=1$, $\Delta \ln L=-½$
- 1D and 2+D
- Integrating and/or profiling

Simple example

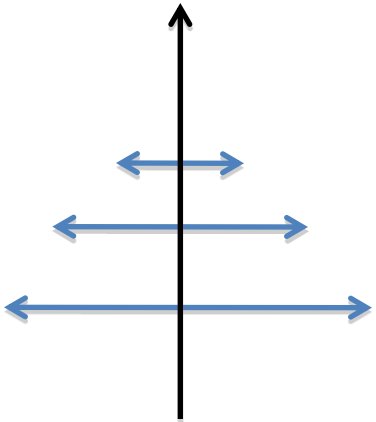
Measurement: value and Gaussian Error

171.2 ± 2.1 means:

169.9 to 173.3 @ 68%

167.8 to 175.4 @ 95%

165.7 to 177.5 @ 99.7% etc



Thus provides a whole set of intervals and associated probability/confidence values

Aside (1): Why Gaussian?

Central Limit Theorem:

The cumulative effect of many different uncertainties gives a Gaussian distribution – whatever the form of the constituent distributions.

Moral: don't worry about nonGaussian distributions. They will probably be combined with others.

Aside(2) Probability and confidence

"169.9 to 173.3 @ 68%"

What does this mean? Number either is in this interval or it isn't. Probability is either 0 or 1. This is not like population statistics.

Reminder: basic definition of probability as limit of frequency $P(A)= \text{Limit } N(A)/N$

Interpretation. 'The statement "Q lies in the range 169.9 to 173.3" has a 68% probability of being true.' Statement made with 68% confidence

Illustration

Simple straight line fit $y=a\,x$

Estimate $a=\Sigma\, x_i y_i \,/\, \Sigma\, x_i^2$

Error on a given by $\sigma/\sqrt{(\Sigma x_i^2)}$

(combination of errors)

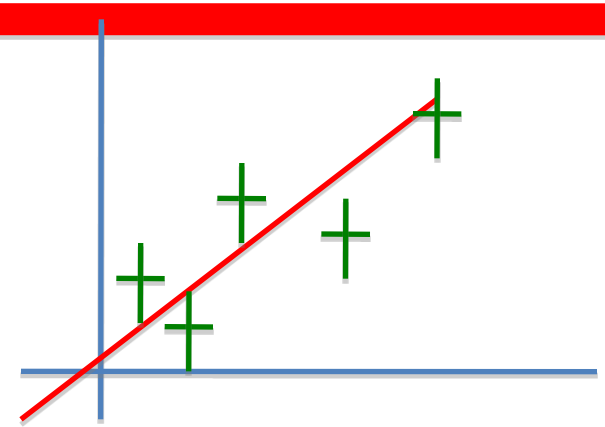
Also look at $\chi^2=\Sigma\,(y_i-ax_i)^2/\sigma^2$

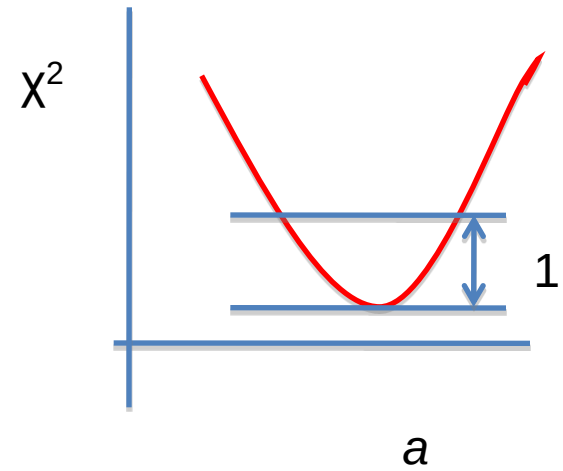
Size contains information on quality of fit

Parabolic function of a

2nd derivative gives error on a

Can be read off from points where χ^2= increases by 1





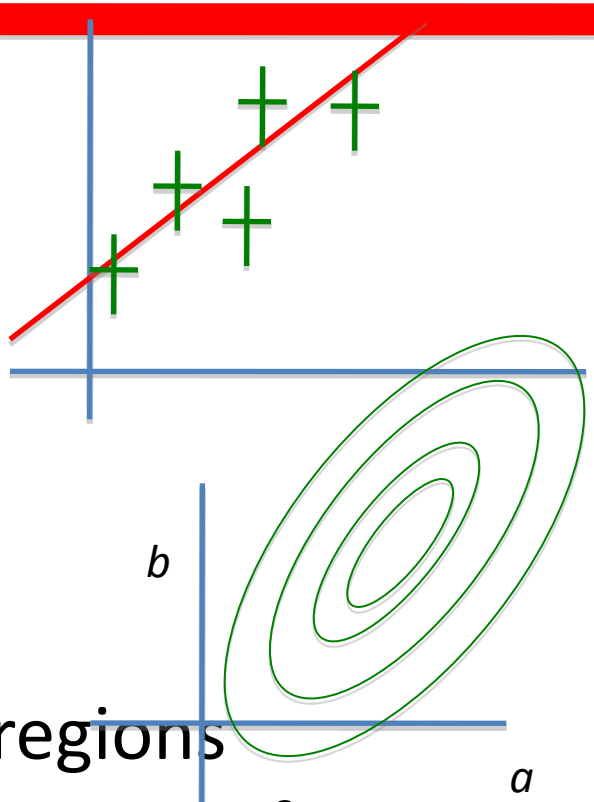
Illustration

Simple straight line fit $y=a\,x+b$

Estimate a, b

Errors on a,b and correlation given by combination of errors

Also look at $\chi^2=\Sigma\,(y-ax-b)^2/\sigma^2$



Parabolic function of a and b

χ^2 contours map out confidence regions

Values 2.30 for 68%, 5.99 for 95%, 11.83 for 99.73%

χ^2

X^2 Distribution is convolution of N Gaussians

Expected $\chi^2 \approx N$ If $\chi^2 >> N$ the model is implausible.

Quantify this using standard function $F(\chi^2 ;N)$

Fitting a parameter just reduces N by 1

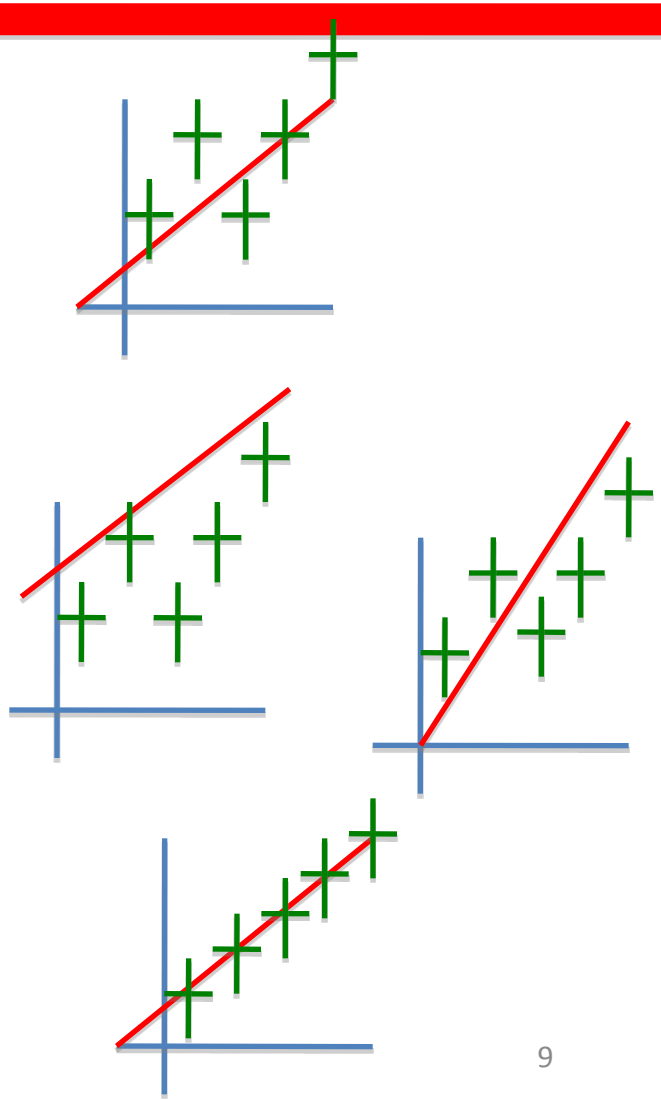
Chi squared probability and p values

$p(\chi^2;N)$=Integral of F from χ^2 to ∞

An example of a *p-value* :the probability that the true model would give a result this bad, or worse.

Correct p-values are distributed uniformly between 0 and 1

Notice the choice to be made as to what is 'bad'



Likelihood

$L(a;\underline{x})=\Pi P(x_i;a)$

$\ln L(a;x)=\Sigma \ln P(x_i;a)$

Regarded as function of *a* for given data *x*.

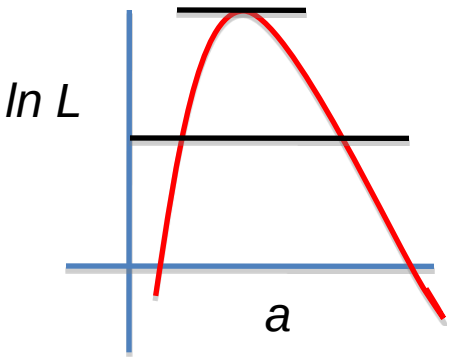
For set of Gaussian measurements, clearly

$\ln L = -\frac{1}{2} \chi^2$

So -2 ln L behaves like a χ^2 distribution

Generalisation (Wilks' Theorem) this is true in other cases

Find 1-σ confidence interval by $\Delta \ln L = -\frac{1}{2}$



OK for parabolic likelihood function

Extension to nonparabolic functions is not rigorous but everybody does it

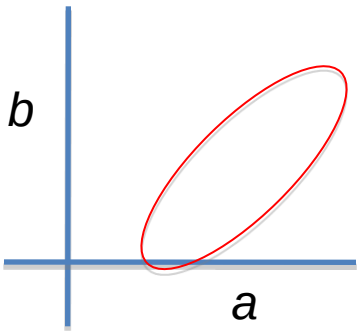
Extend to several variables

Map out region in parameter space where likelihood is above appropriate value

Appears in many presentations of results]

Sometimes both/all parameters are important

Sometimes not…

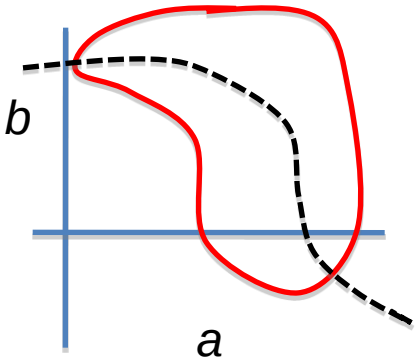


"Nuisance Parameters", or systematic errors

Basic rule is to say what you're doing.

Can use profile likelihood technique to include effect.

Or integrate. Dubious but probably OK. Bayesian



Bayes theorem

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Example: Particle ID

Bayesian Probability

$$P(\text{Theory}|\text{Data}) = \frac{P(\text{Data}|\text{Theory})\,P(\text{Theory})}{P(\text{Data})}$$

Example: bets on tossing a coin

P(Theory): Prior P(Theory|Data): Posterior

Apparatus all very nice but prior is **subjective**.

Bayes and distributions

Extend method. For parameter *a* have prior probability distribution *P(a)* and then posterior probability distribution *P(a|x)*

Intervals can be read off directly.

In simple cases, Bayesian and frequentist approach gives the same results and there is no real reason to use a Bayesian analysis.

Nuisance parameters

L(a,b;x) and *b* is of no interest (e.g. experimental resolution).

May have additional knowledge e.g. from another channel

$$L'(a;x)=\int L(a,b;x)\, P(b)\, db$$

Seems natural – but be careful

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