Walking behavior, weakly first order phase transitions and complex CFTs

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Work with Victor Gorbenko and Slava Rychkov, 1805.xxxxx

Walking behavior

Four dimensional $SU(N_c)$ gauge theory.

Add N_f massless fermions in the fundamental representation of the gauge group.

Global symmetry group: $SU(N_f)xSU(N_f)xU(1)$

WHAT HAPPENS IN THE IR?

1-loop Yang-Mills beta function

$$\beta_{\text{1-loop}}(g) = -A\left(\frac{11}{2}N_c - N_f\right)g^3 + \ldots \qquad A > 0$$

[Politzer '73, Gross, Wilczek '74]

$N_f < \frac{11}{2}N_C$: asymptotic freedom, the theory is free in the UV.

$N_f > \frac{11}{2}N_C$: loss of asymptotic freedom, the theory in the IR is free.

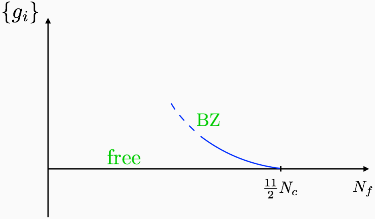
2-loops Yang-Mills beta function

$$\beta_{\text{2-loops}}(g) = -A\left(\frac{11}{2}N_c - N_f\right)g^3 + Bg^5 + \dots$$

[Caswell '74, Jones '74]

$B > 0$ (in neighborhood of $\frac{11}{2}N_c$)

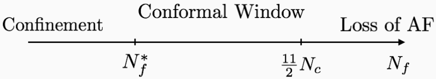
Non trivial fixed point at $g_*^2 = \frac{A}{B}\left(\frac{11}{2}N_c - N_f\right)$: **Banks-Zaks fixed point.** [Banks, Zaks '82]



Conformal window of QCD

We can trust the computation as long as the fixed point is weakly interacting, $g_*^2 = \frac{A}{B}\left(\frac{11}{2}N_c - N_f\right) << 1 \leftrightarrow N_f \lesssim \frac{11}{2}N_c$

For smaller N_f, resort to lattice studies.



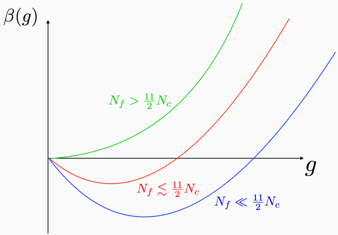
Value of N_f* only accessible through lattice studies, conflicting results: [review Nogradi, Patella '16]

$N_c = 2: \ N_f^* \sim 6$

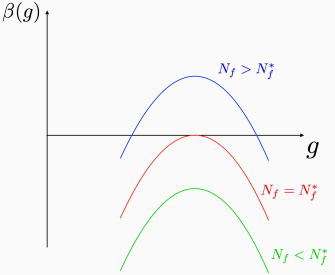
$N_c = 3: \ N_f^* \sim 8, 10$

How does the BZ fixed point disappear?

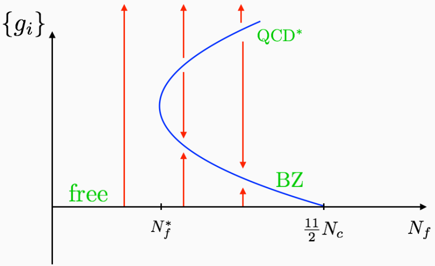
On upper end of conformal window, it merges with the free theory



On lower end, it annihilates with another fixed point [Kaplan, Lee, Son, Stephanov '09]



How does the BZ fixed point disappears?



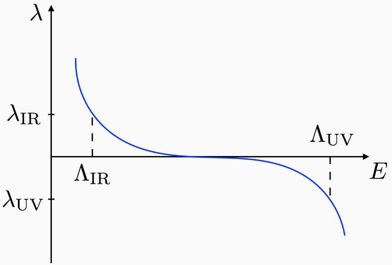
Generic form of the beta function close to N_f^*:

$$\beta(\lambda) = ax - b\lambda^2 + \dots \qquad x \sim N_f - N_f^* \qquad a, b > 0$$

For $N_f > N_f^*$, two fixed points. What happens for $N_f \lesssim N_f^*$?

Walking behavior

Walking behavior



$N_f \lesssim N_f^*$: x small and negative.

A beta function of the kind $\beta(\lambda) = -a|x| - b\lambda^2$ gives an exponentially large hierarchy

$$\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}} = \exp\left(\int_{\lambda_{\rm IR}}^{\lambda_{\rm UV}} \frac{d\lambda}{\beta(\lambda)}\right) \approx \exp\frac{\pi}{\sqrt{ab|x|}} \qquad |x| \ll 1\,.$$

$\lambda_{\rm IR} \simeq O(1)$ and $\lambda_{\rm UV} \simeq -O(1)$.

Walking behavior

Walking behavior relevant for BSM: walking technicolor [Holdom '85; ...; Luty, Okui '04]. Theory approximately invariant between Λ_{UV} and Λ_{IR}, approximated by some unitary CFT. Higgs is part of this theory, requirement $[H^\dagger H] \sim 4$ (hierarchy) and $[H] \sim 1$ (suppression of FCNC).

Walking technicolor motivated the first modern bootstrap paper [Rattazzi, Rychkov, Tonni, Vichi '08]: study this unitary CFT, rule out most theory space.

Weakly first order phase transitions

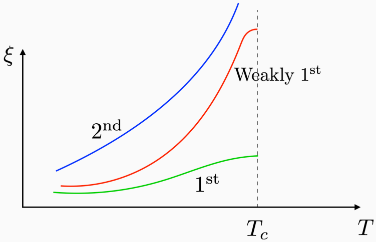
Continuous and discontinuous phase transitions

Similar behavior happens in statistical physics. The correlation length ξ at the critical temperature, compared to the lattice spacing a, is

2^{nd} order: $\xi/a = \infty$

1^{st} order: $\xi/a \sim O(1)$

Weakly 1^{st} order: $\xi/a \gg 1$

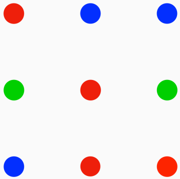


Weakly 1^{st} order phase transitions show the same large separation of scales between UV and IR as walking in QCD.

The Potts model: spin formulation

An example of weakly first order phase transition is given by the **Potts model** [Potts '51]. Spin formulation:

$$H = -\sum_{\langle i,j \rangle} \delta_{s_i, s_j} \qquad s_i = 1, \dots, Q$$



Symmetry group: S_Q, shuffling the states around.
$Q = 2$ is the Ising model.

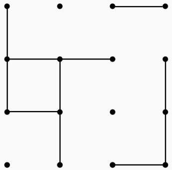
So far Q can only be integer, $Q \geq 2$. There is another formulation of the Potts model that allows to take Q real.

The Potts model: random cluster formulation

Map to **random cluster model** [Fortuin, Kasteleyn '69]

$$Z = \sum_{\{s_i\}} \exp\left(\beta \sum_{\langle i,j \rangle} \delta_{s_i,s_j}\right) = \ldots = \sum_{\text{graph } G} v^{B(G)} Q^{C(G)}$$

with $v \equiv e^{\beta} - 1$, $B(G)$ number of bonds in graph G and $C(G)$ number of clusters in G



For $Q \in \mathbb{N},\ Q \geq 2$ they are the same (high temperature expansion).
Random cluster formulation allows to take $Q \in \mathbb{R}^+$.

Phase transition of the Potts model

Now we can take $Q \in \mathbb{R}^+$. The model has a phase transition which is

$$2^{\text{nd}} \text{ order } \text{ if } Q \leq Q_c(d)$$

$$1^{\text{st}} \text{ order } \text{ if } Q > Q_c(d).$$

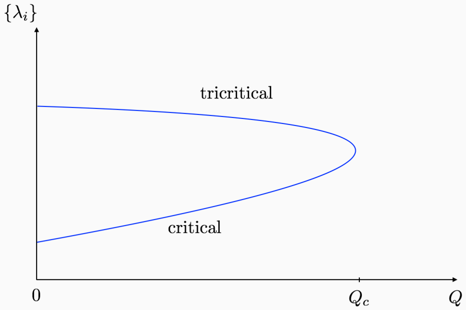
In two dimension exact result $Q_c(2) = 4$. [Baxter '73]
Approximate result $Q_c(3) \simeq 2.45$. [Lee, Kosterlitz '91]
$Q_c(4) = 2$.

What happens to the fixed point at $Q = Q_c$?

RG of Potts

Potts has another fixed point: **tricritical point**. UV fixed point (one more relevant parameter than the critical point).

To reach it from the microscopic model need to add vacancies to the Potts model, *dilute Potts model* [Nienhuis, Berker, Riedel, Schick '79]. In $d = 2$:



Weakly first order phase transition

For $Q \gtrsim Q_c$, the transition is weakly first order: large separation of scales.

In $d = 2$

Q	5	6	7	8	9	10
ξ/a	2512.2	158.9	48.1	23.9	14.9	10.6

In $d = 3$ for $Q = 3$, $\xi/a \sim 10$.

RG for walking?

Same mechanism below the conformal window of QCD and in the Potts model for $Q > Q_c$.

- Two fixed points with a flow connecting them;
- They annihilate for some value of some parameter;
- After annihilation, large separation of scales between UV and IR.

Differences: number of relevant parameters.

RG taught us a lot about 2^{nd} order phase transitions, can it teach us something about weakly 1^{st} order phase transitions?

Understanding walking behavior: complex CFTs

Complex CFTs

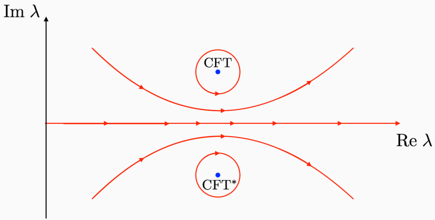
One operator becomes marginal when the two points annihilate. λ is its coupling and it has a beta function of the kind

$$\beta(\lambda) = -\epsilon - B\lambda^2 \qquad B > 0, 0 < \epsilon \ll 1$$

No real zeros but **two complex zeros**

$$\lambda^* = \pm i\sqrt{\frac{\epsilon}{B}}$$

Interpret these as complex CFTs close to the real axis

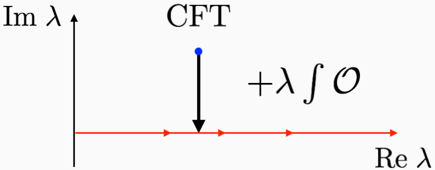


Walking theory is passing close to the complex CFTs

Real theory from complex CFTs

No extra fine tuning necessary: if we start at real coupling theory has to pass in between the complex CFTs.

Idea: take CFT and add the operator with coupling λ to go back on real axis of coupling.



An aside: what does "complex" mean?

Normally divide CFTs in unitary and non-unitary. For our purposes, it's better to divide them in **real** and **complex**.

Intuitively (for bosons): real QFT has real couplings, complex QFT has complex couplings.

Real QFT : closed under conjugation. If operator $\mathcal{O}$ is in the theory, also $\mathcal{O}^\dagger$ is.

$$\mathcal{O}(\tau, x)^\dagger = \mathcal{O}^\dagger(-\tau, x)$$

$$\langle \mathcal{O}_1(\tau_1, x_1) \ldots \mathcal{O}_n(\tau_n, x_n) \rangle^\dagger = \langle \mathcal{O}_1^\dagger(-\tau_1, x_1) \ldots \mathcal{O}_n^\dagger(-\tau_n, x_n) \rangle$$

Complex QFT: operator $\mathcal{O}^\dagger$ may not be part of the theory.

Unitary: real + reflection positivity.

Examples of real CFTs

Real CFT: if an operator $\mathcal{O}$ has complex dimension $\Delta_{\mathcal{O}}$, operator $\mathcal{O}^\dagger$ with dimension $\Delta_{\mathcal{O}}^*$ is also part of the spectrum. Examples:

Any unitary CFT

Wilson - Fisher fixed point in $d = 4 - \epsilon$ [Wilson, Fisher '72; Hogervorst, Rychkov, van Rees '15].

$O(n)$ model with $n \in \mathbb{R}$, Potts model with $Q \in \mathbb{R}, Q \leq Q_c$

Examples of complex CFTs

Complex CFT: no such requirement. Examples:

$\mathcal{N} = 4$ with complex coupling

Fishnet theory at large N [Gurdogan, Kazakov '15]

Potts CFT with $Q > Q_c$ (today)

In our case: CFT and CFT* are complex theories. Add the close to marginal operator $\mathcal{O}$ with some coupling to go back to real coupling: walking theory is real.

A calculable example: 2d Potts model

Critical and tricritical Potts model in two dimensions

Critical and tricritical fixed points for $Q < 4$, colliding at $Q = 4$.

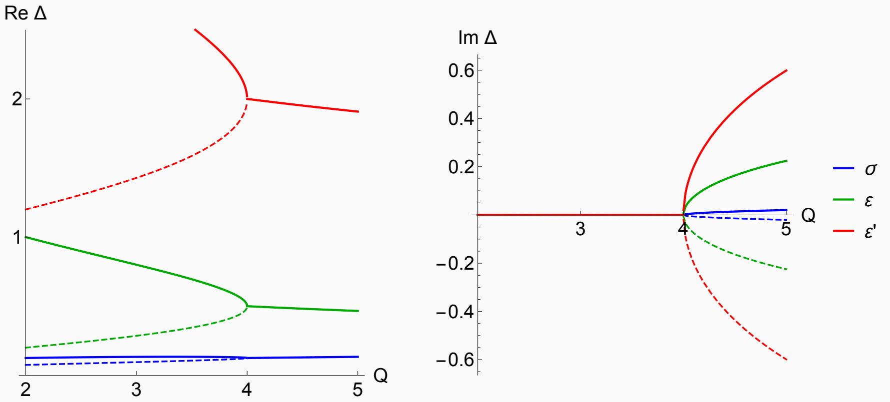
Coulomb gas formalism: free boson with special boundary conditions. Gives us spectrum of the theory at the fixed points for real Q. [di Francesco, Saleur, Zuber '87]

$Q = 2$: minimal models $\mathcal{M}_{3,4}$ (critical) and $\mathcal{M}_{4,5}$ (tricritical)
$Q = 3$: minimal models $\mathcal{M}_{5,6}$ and $\mathcal{M}_{6,7}$
$Q = 4$: limit $m \to \infty$ of $\mathcal{M}_{m,m+1}$.

Analitically continue it to $Q > 4$.

Spectrum of Potts

The CFTs are complex: $\mathcal{O}$ is in the spectrum but $\mathcal{O}^\dagger$ is not.



Thick lines $\rightarrow$ critical point. Dashed lines $\rightarrow$ tricritical point.

Conformal perturbation theory

Only one almost marginal operator singlet under S_Q, ε'.
For $Q < 4$, drives the flow from tricritical to critical fixed point.
For $Q \gtrsim 4$, dimension $\Delta = 2 + i\gamma$, $\gamma = \frac{2\sqrt{Q-4}}{\pi} \ll 1$.

We can do computations in a $Q-4$ expansion when the two CFTs are close to the real axis.

Deform CFT by

$$S_{\mathrm{CFT}} + \lambda \int \mathrm{d}^2 x \, \varepsilon'(x)$$

Gives us a β function

$$\beta(\lambda) = i\gamma\lambda + \pi C_{\varepsilon'\varepsilon'\varepsilon'} \lambda^2 + \dots$$

$C_{\varepsilon'\varepsilon'\varepsilon'}$ is the OPE coefficient $\sim \langle \varepsilon'\varepsilon'\varepsilon' \rangle$. First order perturbation theory: enough to take OPE coefficients at $Q = 4$.

Conformal perturbation theory

Two zeros:

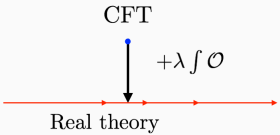
$\lambda = 0$: our starting CFT. $[\varepsilon'] = 2 + i\gamma$;

$\lambda = i\lambda_{\rm FP} = -\frac{i\gamma}{\pi C_{\varepsilon'\varepsilon'\varepsilon'}}$: CFT*. $[\varepsilon'] = 2 + \beta'(i\lambda_{\rm FP}) = 2 - i\gamma$.

Go halfway between the two CFTs:

$$\beta\left(\frac{i\lambda_{\rm FP}}{2} + u\right) = \frac{\gamma^2}{4\pi C_{\varepsilon'\varepsilon'\varepsilon'}} + \pi C_{\varepsilon'\varepsilon'\varepsilon'} u^2$$

walking beta function!



If u is real, beta function remains real: from complex theory I recover real theory (unitary only if Q is integer).

Correlation function of the Potts model

When the theories are close to the real axis, large hierarchy. On the real axis:

$$\beta\left(\frac{i\lambda_{\mathrm{FP}}}{2}+u\right)=\frac{\gamma^2}{4\pi C_{\varepsilon'\varepsilon'\varepsilon'}}+\pi C_{\varepsilon'\varepsilon'\varepsilon'}u^2$$

In our case, $\gamma = 2\frac{\sqrt{Q-4}}{\pi}$, therefore

$$\frac{\xi}{a}=\exp\left(\int_{u_{\mathrm{IR}}}^{u_{\mathrm{UV}}}\frac{du}{\beta\left(\frac{i\lambda_{\mathrm{FP}}}{2}+u\right)}\right)\approx\exp\frac{\pi^2}{\sqrt{Q-4}}.$$

Leading behavior agrees with exact result for the square lattice from Bethe ansatz [Buddenoir, Wallon '93].

Other operators

In 1st order perturbation theory, for operator driving the flow ε', dimension at CFT* is trivially the complex conjugate of the dimension at CFT. For other operators this is non trivial.

$$[\phi]_{\mathrm{CFT}^*} = [\phi]_{\mathrm{CFT}} + 2\pi C_{\phi\phi\varepsilon'} i\lambda_{\mathrm{FP}} + \dots$$

Non-trivial requirement

$$\frac{\mathrm{Im}\Delta_\phi}{C_{\phi\phi\varepsilon'}} = \frac{\mathrm{Im}\Delta_{\varepsilon'}}{C_{\varepsilon'\varepsilon'\varepsilon'}}$$

Satisfied for all operators we checked.

This also implies that the dimension of the operator for the walking theory is real.

Drifting critical exponent

Two point function in walking region are not exactly power laws but have small corrections.

Can integrate Callan-Symanzik equation. Start flow between the two CFTs at $i\lambda^*/2$ at some length scale r_0

$$\langle \sigma(r)\sigma(0)\rangle = \frac{1}{r^{2\Delta_0}} \exp\left(2\int_0^{\bar{u}(r)} du \frac{\gamma_\sigma(u + i\lambda_{\mathrm{FP}}/2)}{\beta(u + i\lambda_{\mathrm{FP}}/2)}\right)$$

γ_σ is the anomalous dimension of σ.

Get

$$\langle \sigma(r)\sigma(0)\rangle = \frac{1}{r^{1/4}} \left(\cos \frac{\sqrt{Q-4}\log r/r_0}{\pi}\right)^{-\frac{1}{8}}.$$

Other checks

Two loop beta function: besides OPE coefficient need the four point function $\langle \varepsilon' \varepsilon' \varepsilon' \varepsilon' \rangle$. Now the fact that ε' at CFT* is the complex conjugate of ε' at CFT is not trivial. It turns out it is.

Check passed also by energy operator ε.

Our expansion parameter is $\sim \frac{Q-4}{\pi^2}$. Explains why correlation length is still largish at the transition even for $Q = 10$.

To sum it up

Two fixed points collide and go to the complex plane;

Walking is a consequence of being close to the complex fixed points;

Since walking theory is approximated by complex CFTs, start from one of the complex CFTs and add a perturbation in order to obtain the walking theory. Ingredients: conformal data of complex CFTs;

Obtain prediction on observables of the walking theories: correlation length, two point functions, ...

Open questions

Other systems showing this behavior? One example, $O(n)$ model in two dimensions [in progress]. Néel/VBS transition?

How to study the spectrum of the complex CFTs?

How natural is the relation between OPE coefficients and imaginary part of scaling dimensions? Does it follow from crossing symmetry?

First modern bootstrap paper ruled out walking technicolor but it assumed unitary CFT. Any room for walking technicolor?

Backup slides

The Potts model: random cluster formulation

High temperature expansion

$$Z = \sum_{\{s_i\}} \exp\left(\beta \sum_{\langle i,j \rangle} \delta_{s_i,s_j}\right) = \sum_{\{s_i\}} \prod_{\langle i,j \rangle} \exp\left(\beta \delta_{s_i,s_j}\right) =$$

$$= \sum_{\{s_i\}} \prod_{\langle i,j \rangle} \left(v\delta_{s_i,s_j} + 1\right) = \sum_{\{s_i\}} \sum_{\text{graph G}} \prod_{\text{bond} \in G} v\delta_{s_i,s_j} =$$

$$= \sum_{\text{graph G}} v^{B(G)} \sum_{\{s_i\}} \prod_{\text{bond} \in G} \delta_{s_i,s_j} = \sum_{\text{graph G}} v^{B(G)} Q^{C(G)}$$

with $v \equiv e^{\beta} - 1$, $B(G)$ number of bonds in graph G and $C(G)$ number of clusters in G