Scattering Amplitudes from Geometries

Song He

Institute of Theoretical Physics, CAS

with N. Arkani-Hamed, Y. Bai, G. Yan 1711.09102

+ work in progress

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Motivations

Search for "holographic" S-matrix theory: fascinating geometric structures underlying scattering amplitudes, in some auxiliary space

- $\mathcal{M}_{g,n}$: **perturbative string** amps = correlators of worldsheet CFT $\to$ **twistor strings & scattering equations**, same worldsheet but without stringy excitations [Witten; CHY; Mason, Skinner; Berkovits...]
- $G_+(k,n)$: **amplituhedron for** $\mathcal{N}=4$ **SYM** [Arkani-Hamed et al.]

Both geometries have "factorizing" boundary structures: locality and unitarity naturally emerge (without referring to the bulk)

What questions to ask, directly in the "kinematic space", to generate local, unitary dynamics? Avatar of these geometries?

Amplitudes as Forms

Scattering amps as differential forms on kinematic space $\to$ a new picture for amplituhedron [Arkani-Hamed, Thomas, Trnka] & more!

Forms on momentum-twistor space = superamp in $\mathcal{N}=4$ SYM: $\eta_i \to dZ_i \implies \Omega_n^{(4k)}$ for N^{k}MHV tree; similarly $\Omega^{(2n-4)}(\lambda, \tilde{\lambda})$ [w. Zhang]

(tree) Amplituhedron ="positive" region $\cap$ $4k$-dim subspaces
$\Omega_n^{(4k)}|_{\text{subspace}}$ = **canonical form** of **positive geometry** [Arkani-Hamed, Bai, Lam]

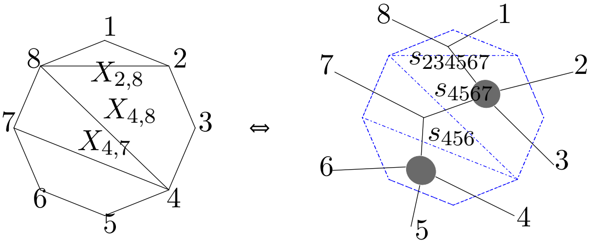
This talk: identical structure for wide variety of theories in any dim:

- Bi-adjoint ϕ^3 from kinematic and worldsheet associahedra
- YM/NLSM: "geometrizing" color & its duality to kinematics
- Real and complex integrals of forms, double-copy & strings?

Kinematic Space

The kinematic space, $\mathcal{K}_n$, for n massless momenta p_i is spanned by Mandelstam variables $s_{i\,j}$ with $\sum_{j \neq i} s_{i\,j} = 0\,;\quad \dim \mathcal{K}_n = \frac{n(n-3)}{2}$.

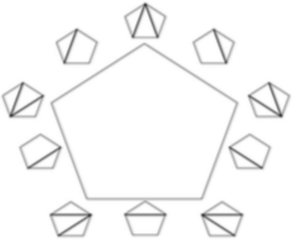
Given an ordering $(12 \cdots n)$, planar variables $X_{i,j} := s_{i,i+1,\cdots,j-1}$ dual to $\frac{n(n-3)}{2}$ diagonals of a n-gon form a basis of $\mathcal{K}_n$

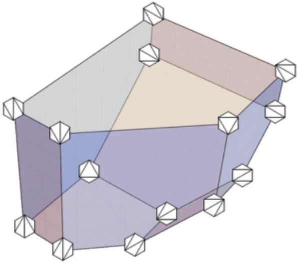


A planar cubic tree graph consists of $n-3$ compatible planar variables as poles, and it is dual to a full triangulation of the n-gon.

The Associahedron

The **associahedron** polytope encodes combinatorial "factorization": each co-dim d face represent a triangulation with d diagonals or planar tree with d propagators (vertices $\leftrightarrow$ planar cubic trees)

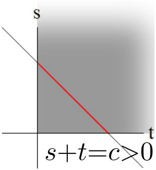




Universal factorization structures of any massless tree amps (in particular ϕ^3), but how to realize it directly in kinematic space?

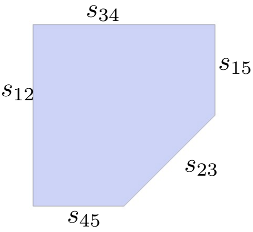
Kinematic Associahedron

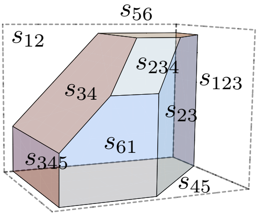
Δ_n: all $X_{i,j} \geq 0$ (top-dim cone); H_n: $(n-3)$-dim subspace defined by $X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} (= -s_{ij}) = c_{i,j}$ as *positive constants*, for all non-adjacent $1 \leq i, j < n$. Then $\mathcal{A}_n := \Delta_n \cap H_n$ [ABHY]



$$e.g.\ \mathcal{A}_4 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$$

$$\mathcal{A}_5 = \{s_{12}, \cdots, s_{51} > 0\} \cap \{-s_{13}, -s_{14}, -s_{24} = \text{const} > 0\}$$



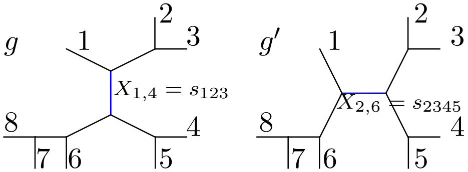


Planar Scattering Forms

The *planar scattering form* for ordering $(12\cdots n)$ is a sum of rank-$(n-3)$ $d\log$ forms for $\mathbf{Cat}_{n-2}$ planar cubic graphs with $\mathrm{sign}(g)=\pm 1$:

$$\Omega_n^{(n-3)} := \sum_{\text{planar } g} \mathrm{sign}(g) \bigwedge_{a=1}^{n-3} d\log X_{i_a,j_a}$$

with $\mathrm{sign}(g) = -\mathrm{sign}(g')$ for any g, g' related by a *mutation*



Sign-flip rule fixed by projectivity: invariant under *local* GL(1) transf. $X_{i,j} \to \Lambda(X) X_{i,j}$ (well-defined in a projectivized space)

Projectivity is equivalent to requiring that the form only depends on *ratios* of variables, *e.g.* $\Omega_4^{(1)} = \frac{ds}{s} - \frac{dt}{t} = d\log\frac{s}{t}$ and

$$\begin{aligned}\Omega_5^{(2)} = &\quad \frac{ds_{12}}{s_{12}} \wedge \frac{ds_{34}}{s_{34}} + \frac{ds_{23}}{s_{23}} \wedge \frac{ds_{45}}{s_{45}} + \cdots + \frac{ds_{51}}{s_{51}} \wedge \frac{ds_{23}}{s_{23}} \\ = &\quad d\log\frac{X_{1,3}}{X_{2,4}} \wedge d\log\frac{X_{1,3}}{X_{1\,4}} + d\log\frac{X_{1,3}}{X_{2,5}} \wedge d\log\frac{X_{3,5}}{X_{2,4}} \\ \Omega_6^{(2)} = &\quad \textstyle\sum_{g=1}^{14} \pm \wedge (d\log X)^3 = \sum \pm\, d\log \mathrm{ratio}'s\end{aligned}$$

It follows immediately that $\Omega^{(n-3)}$ is cyclically invariant up to a sign $i \to i{+}1$: $\Omega_n^{(n-3)} \to (-1)^{n-3}\,\Omega_n^{(n-3)}$, and it factorizes correctly *e.g.*

$$X_{1,m} = s_{1,\cdots,m-1} \to 0: \qquad \Omega_n \to \Omega_m \wedge d\log X_{1,m} \wedge \Omega_{n-m+2}$$

Projectivity is a remarkable property of $\Omega_n^{(n-3)}$, not true for each diagram or any proper subset of planar Feynman diagrams.

Canonical Form of $\mathcal{A}_n$

Unique form of any positive geometry= "volume" of the dual: $\Omega(A)$ has $d\log$ singularities on all boundaries ∂A with $\mathrm{Res} = \Omega(\partial A)$

For simple polytopes: $\sum_v \pm \wedge\, d\log F$ for faces $F=0$ adjacent to v

Canonical form of $\mathcal{A}_n$ = Pullback of Ω_n to $H_n \propto$ planar ϕ^3 amplitude!

$$e.g. \qquad \Omega(\mathcal{A}_4) = \Omega_4^{(1)}|_{H_4} = \left(\tfrac{ds}{s} - \tfrac{dt}{t}\right)\Big|_{-u=c>0} = \left(\tfrac{1}{s} + \tfrac{1}{t}\right)\, ds$$

$$\Omega(\mathcal{A}_5) = \Omega_5^{(2)}|_{H_5} = \left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right) ds_{12} \wedge ds_{34}$$

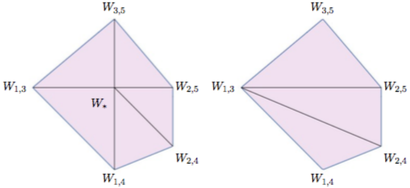
$$\Omega(\mathcal{A}_n) = \sum \mathrm{sgn}(g) \bigwedge_{a=1}^{n-3} d\log X_{i_a,j_a} = d^{n-3}X \; m(12\cdots n|12\cdots n)$$

Similarly for $m(\alpha|\beta)$: "volume" of degenerate $\mathcal{A}_n$ (faces at infinity)

Triangulations & Recursion Relations for ϕ^3 Amps

Geometric picture: Feynman-diagram expansion = triangulation of the dual into $\mathbf{Cat}_{n-2}$ simplices by introducing the point at "infinity"

Triangulate the dual or itself in other ways $\to$ new rep. of ϕ^3 amps



$$\Omega(\mathcal{A}_5) = d^2\mathbf{s}\ \left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right)$$

$$= d^2\mathbf{s}\ \left(\frac{s_{12}+s_{51}}{s_{12}s_{34}s_{51}} + \frac{s_{12}+s_{51}}{s_{12}s_{51}s_{23}} + \frac{s_{12}-s_{45}+s_{23}}{s_{12}s_{23}s_{45}}\right)$$

= sum of 3 triangles of $\mathcal{A}_5$ itself

(each with spurious poles $\sim s^2$)

Similar to "local" or "BCFW" triangulations of the amplituhedron: manifest new prop. of ϕ^3 theory obscured by Feynman diagrams!

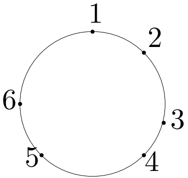
Worldsheet Associahedron & Scattering Equations

A well-known associahedron: minimal blow-up of the open-string worldsheet $\mathcal{M}^+_{0,n} := \{\sigma_1 < \sigma_2 < \cdots < \sigma_n\}/\mathrm{SL}(2,\mathbb{R})$ [Deligne, Mumford]

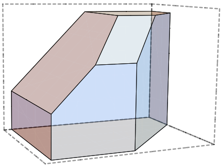
This is non-trivial in σ's but becomes manifest *e.g.* using cross ratios

The *canonical form* of $\overline{\mathcal{M}}^+_{0,n}$ is the "Parke-Taylor" form

$$\omega_n^{\mathrm{WS}} := \frac{1}{\mathrm{vol}\ [\mathrm{SL}(2)]} \prod_{a=1}^{n} \frac{d\sigma_a}{\sigma_a - \sigma_{a+1}} := \mathrm{PT}(1,2,\cdots,n)\ d\mu_n$$



$$\xrightarrow[\text{as a map from } \overline{\mathcal{M}}^+_{0,n} \text{ to } \mathcal{A}_n]{\text{scattering equations}}$$



Pushforward from the wolrdsheet

On H_n, scattering eqs provide a diffeomorphism from $\overline{\mathcal{M}}_{0,n}^{+}$ to $\mathcal{A}_n$:

$$s_{a,a+1} = \sigma_{a,a+1} \sum_{1<i+1\leq a\leq j<n} \frac{c_{i,j}}{\sigma_{i,j}} \qquad \text{for } a = 1, \ldots, n-3 \ (\sigma_n \to \infty)$$

Diff $A \to B \implies$ pushforward $\Omega(A) \to \Omega(B)$ [Arkani-Hamed, Bai, Lam]

$$y = f(x) \qquad \implies \qquad \Omega(B)_y = \sum_{x=f^{-1}(y)} \Omega(A)_x$$

Canonical form of $\mathcal{A}_n$, $\Omega_n^{\phi^3}$ is the pushforward of ω_n^{WS} by summing over $(n-3)!$ sol. of scattering eqs (geometric origin of CHY)

$$\sum_{\text{sol.}} d\mu_n \, \mathrm{PT}(\alpha)|_{H(\alpha)} = m(\alpha|\alpha) \, d^{n-3}\mathbf{s}$$

Projective Scattering Forms

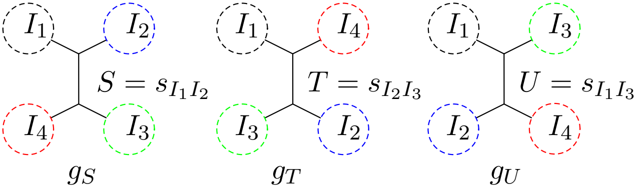
General scattering forms: sum over all cubic graphs with numerators

$$\Omega[N] = \sum_g N(g) \bigwedge_{I=1}^{n-3} d\log s_I\,, \quad e.g.\ N_s\, d\log s + N_t\, d\log t + N_u\, d\log u$$

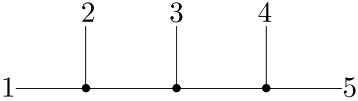
Projectivity: require $\Omega[N]$ to be *i.e.* covariant under $s_I \to \Lambda(s) s_I$

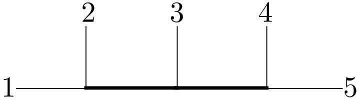
$\Longrightarrow$ kinematic numerators can be chosen to satisfy Jacobi identities

$$N(g_S) + N(g_T) + N(g_U) = 0\,, \quad e.g.\ N_s + N_t + N_u = 0$$



Color is Kinematics





$$f^{a_1 a_2 b} f^{b a_3 c} f^{c a_4 a_5} \quad \leftrightarrow \quad \mathrm{d}s_{12} \wedge \mathrm{d}s_{45}$$

Duality between *color factors* and *differential forms on* $\mathcal{K}_n$ for cubic graphs: $C(g)$ and $W(g)$ satisfy the same algebra.

$$\textbf{Claim}: \; W(g) := \pm \bigwedge_{I=1}^{n-3} ds_I \implies W(g_S) + W(g_T) + W(g_U) = 0$$

Scattering forms are color-dressed amps without color factors.
For U(N), partial amps are pullbacks to subspaces (as bi-adjoint ϕ^3).

Uniqueness of YM and NLSM Forms

Remarkably rigid objects encoding full amps in YM & NLSM

Gauge invariance: $\Omega^{\rm YM}$ invariant under every shift $\epsilon_i^\mu \to \epsilon_i^\mu + \alpha p_i^\mu$

Adler zero: $\Omega^{\rm NLSM}$ vanishes under every soft limit $p_i^\mu \to 0$

Key: forms are projective $\implies$ unique $\Omega^{\rm YM}$ and $\Omega^{\rm NLSM}$!

Implies the amp "uniqueness theorem" [Arkani-Hamed, Rodina, Trnka]: $(n{-}1)!$ parameters for amp vs. unique form up to an overall const.

Alternatively they are pushforward of rigid worldsheet objects

$$\Omega_n^{\rm YM} = \sum_{\rm sol.} d\mu_n {\rm Pf}'\Psi_n\,, \quad \Omega_n^{\rm NLSM} = \sum_{\rm sol.} d\mu_n \,{\det}' A_n$$

Integrals of Canonical Forms

Natural to integrate Ω_P in P provided regulators for all facets W_a's

$$\mathcal{I}_P \equiv \epsilon^d \int_P \Omega_P(Y) \prod_{\text{facets}} (Y \cdot W_a)^{\epsilon X_a}, \; e.g.\; \epsilon \int_0^1 \frac{dx}{x(1-x)} x^{\epsilon A} (1-x)^{\epsilon B} \,,$$

The regulators ensure no log divergences, and the leading term in ϵ comes from vertices, *e.g.* for simple polytopes (d facets for each):

$$\lim_{\epsilon \to 0} \mathcal{I}_P(\{X\}) = \sum_{\text{vertex } i} \prod_{a}^{d} \frac{1}{X_a} \,, \quad e.g.\; \lim_{\epsilon \to 0} \mathcal{I}_{[0,1]} = \frac{1}{A} + \frac{1}{B}$$

$$\lim_{\epsilon \to 0} \epsilon^2 \int \frac{dx dy}{x(y-x)(1-y)} x^{\epsilon A} (y-x)^{\epsilon B} (1-y)^{\epsilon C} = \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA} \,.$$

Looks like *canonical function* of some polytope in X space.

Integral vs. Pushforward

The geometry Q can be obtained via a map from P to X space: "scattering equations" =saddle-point of "Koba-Nielson" factors

$$\sum_a X_a \, d\log(Y \cdot W_a) = 0\,, \quad \text{or} \ \sum_a \frac{X_a}{Y \cdot W_a} W_a^I = 0 \ (I = 1, \cdots, d)$$

Conjecture: Leading term of the integral $\mathcal{I}_P$ equals canonical function of Q, $F_Q(X)$, which is given by pushforward of $\Omega(P)$:

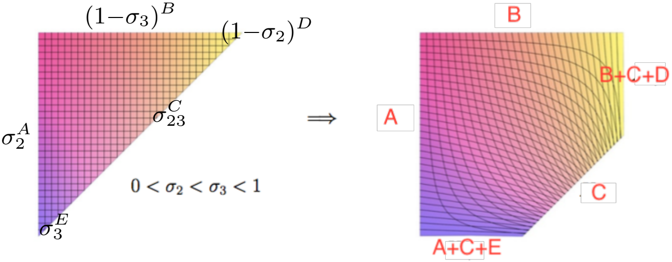
$$\lim_{\epsilon \to 0} \int_P \Omega(P) \prod_a (Y \cdot W_a)^{\epsilon X_a} = F_Q(X)\,, \ \text{where}$$

$$d^d X \, F_Q(X) = \sum_{\text{sol.}} \Omega(P)|_H\,, e.g.\ \Omega(Q) = \sum_{\text{vertex } i} \pm \prod_a d\log X_a\,,$$

We believe this to be the general mechanism behind CHY: field-theory limit of string integral = pushforward via scattering eqs

Same for integrals with extra walls passing through vertices of P: Q can be thought of as "blowup" of P at those boundaries, *e.g.*

$$\lim_{\epsilon\to 0}\int \frac{d\sigma_2 d\sigma_3}{\sigma_2\sigma_{23}(1-\sigma_3)}\text{“KN”} = \frac{1}{AB}+\left(\frac{1}{A}+\frac{1}{C}\right)\frac{1}{(A+C+E)}+\left(\frac{1}{B}+\frac{1}{C}\right)\frac{1}{(B+C+D)}$$



The leading term, and remarkably the pushforward, only depends on the combinatorics, not any details of the extra walls.

Complex Integrals vs. CHY formula

Another natural integral: $|\Omega_P(Y)|^2$ on $\mathbb{C}^d$. Same limit as the real one:

$$\mathcal{I}_P^{\mathbb{C}} := (\frac{\epsilon}{2\pi i})^d \int_{\mathbb{C}^d} |\Omega_P(Y)|^2 \prod_{\text{facets}} |Y \cdot W_a|^{\epsilon X_a}, \quad \lim_{\epsilon \to 0} \mathcal{I}_P^{\mathbb{C}} = \lim_{\epsilon \to 0} \mathcal{I}_P \,,$$

Key: the limit again comes from every vertex, which gives the same residue as the real case: $\lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{dz d\bar{z}}{|z|^2} |z|^{\epsilon A} = \lim_{\epsilon \to 0} \int \frac{r dr}{r^2} r^{\epsilon A} = \frac{1}{A}$.

The same holds for complex integrals with extra walls, where the limit equals to "CHY formula" (self-intersecting number [Mizera])

$$\lim_{\epsilon \to 0} \mathcal{I}_P^{\mathbb{C}} = \frac{1}{(2\pi i)^d} \oint_{|E^I| = \varepsilon} \frac{\Omega_P(Y)\, \hat{\Omega}_P(Y)}{\prod_{I=1}^d E^I} \,, \quad E^I := \sum_a \frac{X_a W_a^I}{Y \cdot W_a} \,.$$

Integrals of General Forms

These integrals extract rational functions in X space from forms, and we can apply them to general (non-$d\log$) forms like $\Omega^{\rm YM/NLSM}$.

Consider pullback of $\Omega^{\rm YM}(x)=\sum_g N_g^{\rm YM}(\epsilon,k)\wedge_{i,j} d\log x_{ij}$ to H_n: the regulated integral in $\mathcal{A}_n$ (all $x_{ij}>0$) gives the partial amplitude!

$$\lim_{\epsilon\to 0}\epsilon^{n-3}\int_{x_{ij}>0}\Omega^{\rm YM}(x)|_H \prod_{i,j} x_{ij}^{\epsilon X_{ij}} = \sum_g \frac{{\rm Res}_g \Omega^{\rm YM}}{\prod X_{ij}^{(g)}} = M_n^{\rm YM}\,,$$

$$e.g. \qquad \lim_{\epsilon\to 0}\int_{0<x<c}(N_s d\log x + N_t d\log(c-x))x^{\epsilon s}(c-x)^{\epsilon t} = \frac{N_s}{s}-\frac{N_t}{t}\,.$$

Each vertex gives $n-3$ planar poles $X_{i\,j}$, times the residue. Here X's are given by the kinematic data defining residues, $N_g^{\rm YM}(\epsilon,p)$.

Complex Integrals and Gravity Amplitude

Complex integral of general $|\Omega|^2$: pullback to *any* generic subspace and put walls for *all* poles of Ω, and we get residue squared

$$\lim_{\epsilon\to 0}\int_{\mathbb{C}^d} |\Omega^{(d)}(Y)|^2 \prod_a |Y\cdot W_a|^{\epsilon X_a} = \sum_{\text{vertex } i} \frac{|\mathrm{Res}_i\ \Omega^{(d)}|^2}{\prod_a^d X_a}\,,$$

If we take Ω_n^{YM} in BCJ form, it gives exactly the gravity amplitude!

$$\lim_{\epsilon\to 0}\int_{\mathbb{C}^d} |\Omega_n^{\mathrm{YM}}(x)|^2 \prod_I^{2^{n-2}-1} |x_I|^{\epsilon s_I} = \sum_g \frac{|\mathrm{Res}_g \Omega_n^{\mathrm{YM}}|^2}{\prod s_I^{(g)}} = M_n^{\mathrm{GR}}\,, e.g.$$

$$\lim_{\epsilon\to 0}\int_{\mathbb{C}} |\frac{N_s dx}{x} + \frac{N_t dy}{y} + \frac{N_u dz}{z}|^2 |x|^{\epsilon s}|y|^{\epsilon t}|z|^{\epsilon u} = \frac{|N_s|^2}{s} + \frac{|N_t|^2}{t} + \frac{|N_u|^2}{u}\,.$$

Projectivity Needed for Double Copy

It is crucial to start with a *projective form*, otherwise the integral also has non-vanishing residue at infinity $\sim N_s + N_t + N_u$. In general, projectivity ensures the absence of pole at infinity along any direction!

Projectivity $\leftrightarrow$ No pole at infinity $\leftrightarrow$ Double copy from $|\Omega|^2$

Different ways to represent $\Omega^{\mathrm{YM/NLSM}} \to$ the integral always gives the same limit. Also asymmetric: $\lim_{\epsilon\to 0} \int \Omega_n^{\mathrm{YM}} (\Omega_n^{\mathrm{NLSM}})^* = M_n^{\mathrm{BI}}$

Any projective form admits $\Omega = \sum_\alpha N_\alpha \Omega^{\phi^3}(\alpha)$, and integral gives

$$\lim_{\epsilon\to 0} \int_{\mathbb{C}^{n-3}} \Omega_L \, \Omega_R^* \prod_I |x_I|^{\epsilon s_I} = \sum_{\alpha,\beta} N_L(\alpha) N_R(\beta) \, m(\alpha|\beta) \, .$$

which in particular implies KLT if expanded in a $(n{-}3)!$ basis.

Outlook

- How to α'-deform canonical forms? String amps from $\mathcal{A}_n$? Generalizations to general cluster polytopes *etc.*
- Loops: halohedra *etc.* at one loop [Salvatori] picture similar to amplituhedron? connections to ambitwistor strings?
- Four Dimensions: "amplituhedron" in momentum space; forms combining helicity amps & pushforward from twistor string
- A unified geometric picture for amplitudes & more?

Thank you !